MODULI FOR SINGULARITIES

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Introduction

The aim of this article is to report on the authors' recent methods and results about moduli spaces for curve singularities and for modules over the local ring of a fixed curve singularity. We emphasize especially the general concept which lies behind these constructions. Therefore, the article might be useful to the reader who wishes to have the leading ideas and the main steps of the proofs explained without going into all the details. We also calculate explicit examples (for singularities and for modules) which illustrate the general theorems.

The construction of moduli spaces for certain objects means a **geometric classification** of the objects with respect to some equivalence relation. This is adequate, in particular, when it becomes too complicated to give a complete classification through (parametrized) normal forms. The basic concept, which stems from Mumford's Geometric Invariant Theory, is that of a **coarse moduli space**. Such a coarse moduli space for singularities respectively for modules over the local ring of a fixed singularity consists of a complex space or an algebraic variety M such that the points of M correspond in a unique way to equivalence classes of singularities or modules (with certain invariants fixed). Moreover, it is required that (flat) families of singularities or modules correspond to subvarieties of M. This means that the structure of M reflects neighbouring relations between objects which are given by small deformations. There exists also the concept of a fine moduli space M (where it is required that a universal family over M exists, from which any other family can be induced by a unique base change), but such a fine moduli space does, in general, not exist without imposing some extra rigidifying structure on the objects to be classified. Nevertheless, in many cases of interest one has first to find some fine moduli space from which the coarse one can be constructed.

The construction of moduli spaces for objects of projective geometry has a long and interesting tradition and is still a very active field of research. In his famous paper on Abelian Functions [Ri], Riemann states: "Die allgemeine Fläche vom Geschlecht g hängt von $3g-3$ stetig veränderlichen Größen ab, welche die Moduln dieser Klasse genannt werden sollen." This means that $3g-3$ is the dimension of the moduli space of Riemann surfaces of genus g. Note that it makes perfect sense to talk about dimensions, even without having a precise definition of a moduli space.

Precise algebraic definitions of different concepts of moduli spaces were given 1965 by Mumford in his celebrated book "Geometric Invariant Theory" [MuF], where he also applied this to Riemann surfaces. According to Mumford, one has to fix the functor, for which a moduli space is to be constructed:

– First of all fix the equivalence relation. For subvarieties of a fixed $\mathbb{P}^n$ it could be just equality (leading to the Hilbert scheme) or isomorphism of varieties. For hypersurface singularities it could be right equivalence of the defining function or contact equivalence, that is isomorphism of the germ defined by the function (which turns out to be much more complicated than right equivalence and which will be our concern). For modules, it will be isomorphism of modules.

- Define the notion of a (flat) family of objects over some category of base spaces, together with the notion of base change.
- The equivalence relation has to be extended to families. Note that, in general, there exist different possible extensions, which may lead to different concepts of a moduli space.

We shall not discuss the last two items in this paper, but would just like to mention the following. If one wants to classify singularities with respect to isomorphism of their complete local rings, so–called algebroid singularities, or modules over complete local rings, one can consider families over complex germs respectively over the spectrum of a complete local ring, or over base spaces, which are algebraic varieties of finite type over the ground field. In the latter case, one has to be careful with base change, since the fibres are still formal objects along a section. The necessary foundational material for such families of modules was developed in [GP 2].

After fixing the functor, a general method for constructing a moduli space is the following:

1. Fix some rough invariants (for Riemann surfaces, for instance, the genus, for isolated singularities, for instance, the Tjurina number τ and the Milnor number μ).
2. Construct an algebraic family $X \to T$ with finite dimensional base space T, which contains all isomorphism classes to be classified. This is sometimes called a parametrizing **bounded family** of the problem. This family should be locally versal. That is, for any family $Y \to S$ and $s \in S$ there is a neighbourhood U of s in S such that $Y|U \to U$ can be induced from $X \to T$ by some base change. In singularity theory this can be the versal deformation of the "worst" object (if such a worst object exists). This will be our approach. Of course, one then has to prove that indeed all objects which we want to classify, and not only those which are close to the worst object, are contained in T.
3. In general T will contain many equivalent objects and one tries to interpret the equivalence classes as orbits of the action of a Lie group or an algebraic group G acting on T. If one starts with a versal deformation $X \to T$, the kernel of the Kodaira–Spencer map is a (infinite dimensional) Lie algebra $\mathcal{L}$ and the integral manifolds of $\mathcal{L}$ give rise

to trivial subfamilies. In our cases, we can show that each equivalence class coincides with an integral manifold. Moreover, it is possible to find a finite dimensional Lie subalgebra $L \subset \mathcal{L}$ with the same integral manifolds. $G = exp\ L$ is the algebraic group with the desired properties. The topological space T/G may be considered as a classifying space but, in general, it is not an algebraic variety or a complex space.

4. (a) Describe a big open subset $T_0 \subset T$ such that T_0/G exists as complex space or algebraic variety. The objects of T_0 may be called generic, or stable.

   (b) Better, find a stratification $T = \cup T_i$ of T into locally closed G–invariant subvarieties T_i such that the closure $\bar{T_i}$ is the union of strata and such that T_i/G exists.

5. Find fine invariants of the objects to be classified, such that at least the reduction of each stratum is characterized exactly by fixing these invariants.

So far, this somewhat schematic description applies to moduli problems in global algebraic geometry as well as to singularities. Roughly speaking, one has the following analogy between the local and global situation:

– Moduli spaces for singularities (local) versus moduli spaces for projective varieties (global).

– Moduli spaces for modules over the local ring of a fixed singularity (local) versus moduli spaces for vector bundles on a fixed projective variety (global).

But the basic fundamental difference between the local and the global situation is that the group G occurring in 3), for the global case, is mostly reductive and hence has a finitely generated ring of invariant algebraic functions. Moreover, Mumford's theory of stable points [MuF] gives a powerful criterion (the Hilbert–Mumford criterion) for the existence of a geometric quotient. In the local case, the groups are almost never reductive, but either solvable or even unipotent, or the product of such a group with a reductive group. Consequently, the ring of invariants can be infinitely generated. It seems that the first really practicable criterion for the existence of a geometric quotient for actions of unipotent groups was developed in [GP 1]. At least for us, this criterion plays the same basic role as the Hilbert–Mumford criterion in the global case. Note, however, that the criterion of [GP 1] gives a complete stratification, whilst the Hilbert–Mumford criterion gives only the big open stratum of stable points.

The construction of moduli spaces for singularities has been carried out only for few cases. The first paper aiming at this goal seems to be that of

Ebey [Eb], who carried out step 1) to 3) above for reduced curve singularities. But, in his lecture notes, Zariski [Za] was the first to actually construct moduli spaces. Fixing the equisingularity type of a plane curve singularity, Zariski studied the problem of moduli from the parametric point of view as well as from the point of view of deformations. Many basic ideas about moduli for singularities go back to him. He was able to compute several special examples just using power series methods, in particular, he carried out step 1) to 4a) and 5) for singularities with equation $x^m + y^{m+1}$. Laudal and Pfister [LaP] developed a quite general frame for the construction of moduli spaces for arbitrary singularities. The starting point is the semiuniversal deformation of the "worst" object to be classified and the existence of a moduli stratum (before considered by Palamodov [Pa] in the complex analytic category) in the base space of the semiuniversal deformation. They proved that it has an algebraic representative under certain assumptions, which hold, for instance, for isolated hypersurface singularities. In this case, the moduli stratum is just the τ–constant stratum. Moreover, they solve steps 1) — 4a) and 5) for plane curve singularities of the form $x^p + y^q$ by this approach (with respect to contact equivalence). An explicit formula for the dimension of this moduli space is given in [BGM].

Note that the τ–constant stratum is, in general, not a moduli space, not even for families over complex germs respectively over the spectrum of a complete local ring. Usually there is a finite group which has to be divided out. For more details see [LaP], further references are given in [LaP] and in the survey article [Gr].

In this paper we shall discuss three special moduli problems where all steps 1) — 5) have been carried out.

First we explain in some detail the classification of irreducible plane curve singularities with semigroup $\langle p, q\rangle$. In this case we extend the results of [LaP] from the generic case to a complete stratification of the semiuniversal μ–constant deformation by fixing a certain Hilbert function of the Tjurina algebra. This extension to lower dimensional strata was possible because of the results of [GP 1]. Subsequently, the authors were able to extend these results to semi–Brieskorn singularities of arbitrary dimension [GP 3]. We report on the results of [LuP] on plane curve singularities with semigroup $\langle 2p, 2q, 2pq + d\rangle$, which is somewhat exceptional since no further stratification was necessary. Moreover, we state some results of [GP 2] on torsion free modules of rank 1 over the local ring of an irreducible curve singularity. The original approach in [GP 2] was different, via "sandwiching" of modules, but we show that this case also fits into the general method via deformation theory of the "worst" object. As to all this a basic ingredient is the criterion for the existence of a geometric quotient of a unipotent group action which has to be discussed first.

We give an outline of the arguments in all three cases and explain the main steps of the constructions. For complete proofs we refer to [LaP], [LuP], [GP 1] and [GP 2].

1 Geometric quotients of unipotent group actions

Let K be a field of characteristic 0.
Let G be an algebraic group acting algebraically on an algebraic variety X. If Y is an algebraic variety and $\pi : X \to Y$ a morphism then $\pi : X \to Y$ is called a **geometric quotient**, if

1. π is surjective and open
2. $(\pi_* \mathcal{O}_X)^G = \mathcal{O}_Y$
3. π is an orbit map, i.e. the fibres of π are orbits of G.

If a geometric quotient exists it is uniquely determined and we just say that X/G exists.

By a general result of Rosenlicht which holds for arbitrary algebraic groups there exists an open dense G-stable subset $U \subset X$ such that U/G exists if X is reduced. But U is not uniquely determined and it is not at all clear how to construct such an open subset.

If G is reductive and $X = Spec\, A,\ A$ of finite type over K, then A^G is of finite type over K and $X \to Spec\, A^G$ is a geometric quotient iff all orbits are closed and have the same dimension. The Hilbert Mumford criterion for "stable" points ([MuF]) is the basic tool for the construction of moduli spaces in global algebraic geometry.

In singularity theory the groups are almost never reductive. In our applications the groups are unipotent. Furthermore it may happen that the ring of invariants A^G is not of finite type. But even if A^G is of finite type and if all orbits have the same dimension (they are closed since G is unipotent) it may happen that X/G does not exist.

An analysis of "bad" examples suggested the following definition of stability which we proposed in [GP 1].

Definition: Let G be a unipotent algebraic group, $Z = Spec\, A$ an affine G-variety and $X \subset Z$ open and G-stable. Let $\pi : X \to Y := Spec\, A^G$ be the canonical map. A point $x \in X$ is called **stable** under the action of G **with respect to** A (**or with respect to** Z) if the following holds:
There exists an $f \in A^G$ such that $x \in X_f = \{y \in X, f(y) \neq 0\}$ and $\pi : X_f \to Y_f = Spec\, A_f^G$ is open and an orbit map.
If $X = Z = Spec\, A$ we call a point stable with respect to A just **stable**.

Let $X^s(A)$ denote the set of stable points of X (under G with respect to A).

Proposition 1.1. *1. $X^s(A)$ is open and G-stable*

2. *$X^s(A)/G$ exists and is a quasiaffine algebraic variety*

3. *If $V \subset Spec\, A^G$ is open, $U = \pi^{-1}(V)$ and $\pi : U \to V$ is a geometric quotient then $U \subset X^s(A)$*

4. *If X is reduced then $X^s(A)$ is dense in X.*

The aim of this chapter is to describe effective criteria for stability, i.e. to give sufficient conditions for the existence of a geometric quotient in terms of given coordinates of X and a given representation of G in $Aut(X)$. These criteria are easier to formulate in terms of the Lie algebra of G.
Let $L =$ Lie (G). If G is unipotent then L is nilpotent and the representation of $G \to Aut_K(A)$ induces a commutative diagramme:

$$\begin{array}{ccc} G & \to & Aut_K(A) \\ exp \uparrow & & \uparrow \; exp \\ L & \to & Der_K^{nil}(A) \end{array}$$

Here $Der_K^{nil}(A)$ is the set of K-linear nilpotent derivations δ of A (δ is nilpotent if for any $a \in A$ there is an $n(a)$ such that $\delta^{n(a)}(a) = 0$).

The best results are obtained for free actions or for abelian L. In these cases we obtain necessary and sufficient conditions for the existence of a locally trivial geometric quotient.

Definition: A geometric quotient $\pi : X \to Y$ is **locally trivial** if an open covering $\{V_i\}_{i \in I}$ of Y and $n_i \geq 0$ exist, such that $\pi^{-1}(V_i) \cong V_i \times A_K^{n_i}$ over V_i where A_k^n denotes the n-dimensional affine space over K.

We use the following notations:
Let $L \subseteq Der_K^{nil}(A)$ be a nilpotent Lie-algebra and $d : A \to Hom_K(L, A)$ the differential defined by $da(\delta) = \delta(a)$. If $B \subseteq A$ is a subalgebra then $\int B := \{a \in A \mid \delta(a) \in B \text{ for all } \delta \in L\}$.

Theorem 1.2. *Let A be a reduced noetherian K-algebra and $L \subseteq Der_K^{nil}(A)$ be a finite dimensional abelian Lie algebra. The following conditions are equivalent:*

1. *There exists an open subset $U \subset Spec\, A^L$ such that $Spec\, A \to U$ is a locally trivial geometric quotient.*

2. *$AdA = Ad \int A^L$ and $Hom_K(L, A)/AdA$ is flat over A.*

2'. There are $x_1, \dots, x_n \in A, \delta_1, \dots, \delta_m \in L$ such that

- *rank $(\delta_i(x_j))$ is locally constant and equal to the orbit dimension of the action of L.*
- *$d\delta_i(x_j) = 0$ for all i, j.*

3. There is a filtration $F^\bullet(A)$ such that

- *$0 = F^{-1}(A) \subset F^0(A) \subset F^1(A) \subset \dots$ and $A = \cup_{i \in \mathbb{Z}} F^i(A)$,*
- *$\delta F^i(A) \subseteq F^{i-1}(A)$ for all $i \in \mathbb{Z}$ and $\delta \in L$,*
- *$Hom_K(L, A)/AdF^i(A)$ is flat over A for all i.*

3'. There are $x_1, \dots, x_n \in A, \delta_1, \dots, \delta_m \in L$ and $i_1, \dots, i_k \in \{1, \dots, n\}$ such that

- *$1 \leq i_1 < i_2 < \dots < i_k = n$*
- *$E(s) := rank(\delta_i(x_j))_{j \leq i_s}$ is locally constant and $E(k)$ is the orbit dimension of the action of L*
- *$d\delta_i(x_\ell) \in \sum_{\nu \leq i_{r-1}} Adx_\nu$ for all i and $\ell \leq i_r$.*

4. $Spec\, A = \cup_{f \in S} D(f), S \subseteq A^L$ and for $f \in S$ there is a sub-Lie algebra $L^{(f)} \subseteq L$ such that

- *$L^{(f)} \otimes_K A_f = L \otimes_K A_f$*
- *$H^1(L^{(f)}, A_f) = 0$*

Proof: (1) implies (2) and (4) is proved in [GP 1] Theorem 4.1.
(2') resp. (3') is the same as (2) resp. (3) expressed in coordinates.
(2) implies (3) using the filtration defined by $F^0(A) = A^L, F^i(A) = \int F^{i-1}(A)$.
(4) implies (1) by theorem 3.10 of [GP 1].
(3) implies (1) follows from theorem 4.7 in [GP 1].

Corollary 1.3. *Let A and L be as in the theorem and let $F^\bullet(A)$ be a filtration of A such that $\delta F^i(A) \subseteq F^{i-1}(A)$ for all i and all $\delta \in L$. Let $Spec\, A = \cup U_\alpha$ be the flattening stratification of $Spec\, A$ defined by the A-modules $Hom_K(L, A)/AdF^i(A)$. Then U_α is invariant under the action of L and $U_\alpha \to U_\alpha/L$ is a geometric quotient.*

If the Lie algebra L is nilpotent but no longer abelian we need some extra conditions on a central series of L to obtain a stratification as in the corollary: So, let A be a noetherian K-algebra and $L \subseteq Der_K^{nil} A$ a finite dimensional nilpotent Lie algebra. Suppose that $A = \cup_{i \in \mathbf{Z}} F^i(A)$ has a filtration

$$F^\bullet : 0 = F^{-1}(A) \subset F^0(A) \subset F^1(A) \subset \dots$$

by subvector spaces $F^i(A)$ such that

$$(F) \qquad \delta F^i(A) \subseteq F^{i-1}(A) \text{ for all } i \in \mathbb{Z} \text{ and all } \delta \in L.$$

Assume, furthermore, that

$$Z_\bullet : L = Z_0(L) \supseteq Z_1(L) \supseteq \ldots \supseteq Z_\ell(L) \supseteq Z_{\ell+1}(L) = 0$$

is filtered by sub Lie algebras $Z_j(L)$ such that

$$(Z) \qquad [L, Z_j(L)] \supseteq Z_{j+1}(L) \text{for all } \ j \in \mathbb{Z}$$

The filtration $Z_\bullet$ of L induces projections

$$\pi_j : Hom_K(L, A) \to Hom_K(Z_j(L), A).$$

For a point $t \in Spec\, A$ with residue field $\kappa(t)$ let

$$r_i(t) := dim_{\kappa(t)} AdF^i(A) \otimes_A \kappa(t) \qquad i = 1, \ldots, k,$$

$$k \ \text{ minimal such that } \ AdF^k(A) = AdA$$

$$s_j(t) := dim_{\kappa(t)} \pi_j(AdA) \otimes_A \kappa(t) \qquad j = 1, \ldots, \ell$$

($s_j(t)$ is the orbit dimension of $Z_j(L)$ at t).

Let $Spec\, A = \cup U_\alpha$ be the flattening stratification of the modules

$$Hom_K(L, A)/AdF^i(A), \qquad i = 1, \ldots, k$$

and

$$Hom_K(Z_j(L), A)/\pi_j(AdA), \qquad j = 1, \ldots, \ell.$$

Theorem 1.4. *U_α is invariant and admits a locally trivial geometric quotient with respect to the action of L.*

Remarks:

1. The functions $r_i(t)$ and $s_i(t)$ are constant along U_α.

2. Let $x_1, \ldots, x_n \in A, \ \delta_1, \ldots, \delta_m \in L$ satisfying the following properties:

   - there are $\nu_1, \ldots \nu_k, \ 0 \leq \nu_1 < \ldots < \nu_k = n$, such that $dx_1, \ldots, dx_{\nu_i}$ generate the A-module $AdF^i(A)$;
   - there are $\mu_o, \ldots, \mu_\ell, \ 1 = \mu_o < \mu_1 < \ldots < \mu_\ell$ such that $\delta_{\mu_j}, \ldots, \delta_m \in Z_j(L)$ and $Z_j(L) \subseteq \sum\limits_{i \geq \mu_j} A\delta_i$. Then

$$rank(\delta_\alpha(x_\beta)(t))_{\beta \leq \nu_i} = r_i(t), \ i = 1, \ldots, k$$

$$rank(\delta_\alpha(x_\beta)(t))_{\alpha \geq \mu_j} = s_j(t), \ j = 1, \ldots, \ell.$$

Hence the U_α are defined set theoretically by fixing $rank(\delta_\alpha(x_\beta)(t))_{\beta\leq\nu_i}, i = 1,\dots,k$ and $rank(\delta_\alpha(x_\beta)(t))_{\alpha\geq\mu_j}, j = 1,\dots,\ell$. But notice that the U_α carry a unique, not necessarily reduced, analytic structure with respect to the flattening property and which is defined by the corresponding subminors.

The key lemma to prove these theorems is the following:

Proposition 1.5. *Let A be a commutative K-algebra and $\delta_1,\dots,\delta_n \in Der_K^{nil}(A)$ and $x_1,\dots,x_n \in A$ satisfying the following properties:*

1. $[\delta_i,\delta_j] \in \sum\limits_{\nu=1}^{n} A\delta_\nu$
2. *$det(\delta_i(x_j))$ is a unit in A*
3. *For any $k = 1,\dots,n$ and any k-minor M of the first k columns of $(\delta_i(x_j))$ we have*
$$\underline{\delta}(M) \in \sum_{\nu<k} A\underline{\delta}(x_\nu)$$
(with the conventions $x_0 = 0$ and $\underline{\delta} = \begin{pmatrix}\delta_1\\ \vdots\\ \delta_n\end{pmatrix}$).

Let $L \subseteq \sum\limits_{\nu=1}^{n} A\delta_\nu$ be any K-Lie algebra such that $\delta_1,\dots,\delta_n \in L$, then $A^L[x_1,\dots,x_n] = A$ and $x_1,\dots,x_n$ are algebraically independent over A^L.

The proposition implies that $Spec\,A \to Spec\,A^L$ is a (trivial) geometric quotient with fibre K^n. In particular, $A^L \cong A/(x_1,\dots,x_n)$ is of finite type over K if A is of finite type and every point of $Spec\,A$ is stable.
The proof of this proposition is done by induction on n. The condition (3) guaranties that the elements of the first column of $(\delta_i(x_j))$ are already in A^L and furthermore this property can be kept during the induction.

Condition (3) is satisfied in our application, even a stronger one:

3'. $\quad \underline{\delta}(\delta_j(x_k)) \in \sum_{\nu<k} A\underline{\delta}(x_\nu) \qquad k = 1,\dots,n$

i.e. the derivative-vector of each element of the matrix $(\delta_i(x_j))$ is an A-linear combination of earlier columns.

We conjecture that in case of a free action the condition (3) of the proposition can be omitted.
Conjecture: Let $L \subseteq Der_K^{nil}(A)$ be a nilpotent Lie algebra of dimension n and $\delta_1\dots,\delta_n \in L, x_1,\dots,x_n \in A$ such that $det(\delta_i(x_j))$ is a unit. Then there

are $y_1, \ldots, y_n \in A$ such that $A = A^L[y_1, \ldots, y_n]$ (equivalently $H^1(L, A) = 0$).

Remark: If we would require in the conjecture $x_i = y_i, i = 1, \ldots, n$ then this conjecture is equivalent to the Jacobian Umkehrproblem.

2 A moduli space for plane curve singularities with semigroup $\langle p, q\rangle$

We assume that $K = \mathbb{C}$. We follow the advice of the introduction:

1. The "worst" object is the singularity defined by $x^p + y^q$. A versal deformation of $x^p + y^q$ fixing the semigroup $\langle p, q\rangle, p < q$ and $gcd(p, q) = 1$, is given by

$$F(x, y, \underline{T}) = x^p + y^q + \sum_{(i,j)\in B} T_{iq+jp-pq}x^i y^j,$$

where

$$B = \{(i, j) \mid iq + jp > pq, i \leq p - 2, j \leq q - 2\}.$$

Let $\underline{T} = \{T_{iq+jp-pq}\}_{(i,j)\in B}$, $X = Spec\,\mathbb{C}[\underline{T}][[x, y]]/F$, and $F_t \in \mathbb{C}\ [[x, y]]$ given by $F_t(x, y) = F(x, y, t)$, $t \in T = Spec\,\mathbb{C}[\underline{T}]$. Then the family $X \to T$ has the following properties:

   1.1 $X \to T$ is a versal deformation of $Spec\,\mathbb{C}[[x, y]]/x^p + y^q$ fixing the semigroup $\langle p, q\rangle$.

   1.2 Every plane curve singularity with semigroup $\langle p, q\rangle$ is represented in this family, i.e. there is a $t \in T$ such that the given singularity is isomorphic to $X_t = Spec\,\mathbb{C}[[x, y]]/(F_t)$.

2. The Kodaira-Spencer map of the family $X \to T$ is given by

$$\rho : Der_{\mathbf{C}}\,\mathbb{C}[\underline{T}] \longrightarrow \mathbb{C}[\underline{T}][[x, y]]/\left(F, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right)$$

$$\rho(\delta) = \text{ class } (\delta F) = \text{ class } \left(\sum_{(i,j)\in B} \delta(T_{iq+jp-pq})x^i y^j\right).$$

The kernel of the Kodaira-Spencer map is a Lie algebra $\mathcal{L}$ which is a finitely generated $\mathbb{C}[\underline{T}]$-module and has the following property:

   2.1 For $t, t' \in T$ the singularities X_t and $X_{t'}$ are isomorphic iff t and t' are in the same integral manifold of $\mathcal{L}$, i.e. $T/\mathcal{L}$ is a classifying space for all singularities with semigroup $\langle p, q\rangle$.
   T has a natural $\mathbb{C}^*$-action defined by $deg\,T_\alpha = -\alpha$. This $\mathbb{C}^*$-action is induced by the $\mathbb{C}^*$-action on $\mathbb{C}\,[[x, y]]/x^p + y^q$ given by $deg\,x = q$ and $deg\,y = p$ in order to keep F homogeneous. The induced grading of $\mathcal{L} \subseteq Der_{\mathbb{C}}\mathbb{C}[\underline{T}]$ is defined by $deg\frac{\partial}{\partial T_\alpha} = \alpha$. One can show that $\mathcal{L}$ is generated as $\mathbb{C}[\underline{T}]$-module by homogeneous vector fields $\{\delta_\alpha\}$ with the following properties:

2.2 There are homogeneous vector fields $\delta_{aq+bp} \in \mathcal{L}$ for $(a,b) \in B^{\checkmark} := \{(p-2-i, q-2-j) \mid (i,j) \in B\}$ such that

- $\{\delta_{aq+bp}\}_{(a,b)\in B^{\checkmark}}$ generate $\mathcal{L}$ as $\mathbb{C}[\underline{T}]$-module
- $deg\, \delta_\alpha = \alpha$
- $\delta_\alpha(T_\beta) = \delta_{\beta^{\checkmark}}(T_{\alpha^{\checkmark}})$, where $a^{\checkmark} := pq - 2p - 2q - a$ for $a \in \mathbb{Z}$
- $[\delta_\alpha, \delta_\beta] \in \sum\limits_{\nu \geq \alpha+\beta} \mathbb{C}[\underline{T}]\delta_\nu$

Remark: $\mathbb{C}[\underline{T}]\,[[x,y]]/(\frac{\partial F}{\partial x}/\frac{\partial F}{\partial y})$ is a free $\mathbb{C}[\underline{T}]$-module of rank μ. The multiplication by F defines an endomorphism of this module. The module admits a basis $\{u_\alpha\}$ represented by quasihomogeneous polynomials of degree α such that

$$u_\alpha F = \sum_\beta \delta_\alpha(T_\beta) \cdot u_{\beta+pq}.$$

This determines the Lie algebra L_0 generated by $\delta_\alpha = \sum\limits_\beta \delta_\alpha(T_\beta)\partial/\partial T_\beta$ and the action of L_0 on T. For concrete examples, one has to compute the matrix of the endomorphism of $\mathbb{C}[\underline{T}]\,[[x,y]]/(\frac{\partial F}{\partial x}/\frac{\partial F}{\partial y})$ given by multiplication with F (with respect to the basis $\{u_\alpha\}$). For this purpose there exists a fast algorithm which has also been implemented (cf. [LaP], appendix with B. Martin).
Let L_0 be the Lie algebra generated by $\{\delta_{aq+bp}\}_{(a,b)\in B^{\checkmark}}$ as Lie algebra and $L = [L_0, L_0]$, then L_0 is finite dimensional and solvable and L is nilpotent $L_0/L \cong \mathbb{C}\delta_0, \delta_0$ is the Euler vector field. Because the $\mathbb{C}[\underline{T}]$-generators $\{\delta_\alpha\}$ of $\mathcal{L}$ are in L_0, L_0 and $\mathcal{L}$ have the same integral manifolds which are the orbits of the action of L_0. This implies that the action of the kernel of the Kodaira-Spencer map $\mathcal{L}$ is induced by the action of the algebraic group $G_0 := exp\, L_0$ with the same quotient ring of invariants $\mathbb{C}[I]^{\mathcal{L}} = \mathbb{C}[I]^{G_0}$.

3. The grading of $\mathbb{C}[\underline{T}]$ induces for each $a \geq 0$ a filtration $F_a^\bullet(\mathbb{C}[\underline{T}])$, where $F_a^i(\mathbb{C}[\underline{T}])$ is the $\mathbb{C}$-vector space generated by all quasihomogeneous polynomials of degree $\geq -(a+ip), p \geq a \geq 0$. Similarly, we get filtrations $H_a^\bullet = H_a^\bullet(\mathbb{C}\,[[x,y]])$ on $\mathbb{C}\,[[x,y]]$ by defining H_a^n to be the ideal generated by all quasihomogeneous polynomials of degree $\geq a + np$. The **Hilbertfunction of the Tjurina algebra** of $X_t, \mathbb{C}\,[[x,y]]/(F_t, \partial F_t/\partial x, \partial F_t/\partial y)$, with respect to $H_a^\bullet$ is by definition the function $\tau_a^\bullet(t)$,

$$n \mapsto \tau_a^n(t) := dim_{\mathbb{C}}\mathbb{C}\,[[x,y]]/(F_t, \partial F_t/\partial x, \partial F_t/\partial y, H_a^n)$$

Notice that $\tau_a^n(t) = \tau(X_t)$, the Tjurina number of X_t if n is big and $\tau_a^n(t) = dim_{\mathbb{C}}\mathbb{C}\,[[x,y]]/H_a^n$ (hence independent of t) if n is small.

Remark: We introduced the filtrations $F_a^\bullet$ and $H_a^\bullet$ for different a because the general theory works for arbitrary a but in some cases a good choice of a gives bigger strata (cf. Theorem 2.1(3) and the examples at the end of this section).
There is only a finite range of n such that $\tau_a^n(t)$ can vary with t. We usually identify $\tau_a^\bullet$ with the finite tuple of values which might vary with t. Moreover, if $\mu \in \mathbb{N}$, we also write μ for the constant function on $\mathbb{N}$.

If $\delta \in L$ is a homogeneous vector field then $deg\,\delta \geq p$. This implies that $\delta\, F_a^i \subseteq F_a^{i-1}$ for all $\delta \in L$.
Let k be minimal such that $dF_a^k(\mathbb{C}[\underline{T}])$ generates $\mathbb{C}[\underline{T}]d\mathbb{C}[\underline{T}]$ over $\mathbb{C}[\underline{T}]$ and consider the following filtration of L induced by the filtration of $\mathbb{C}[\underline{T}]$:

$$L = Z_1^a(L) \supset Z_2^a(L) \supset \ldots \supset Z_k^a(L) \supset Z_{k+1}^a(L) = \{0\},$$

$Z_i^a(L) :=$ the Lie algebra generated by $\{\delta_\alpha\}_{\alpha \in S_i}$, where $S_i = \{\alpha \mid T_{\alpha^\checkmark} \in F_a^{k-i}, \alpha \neq 0\}$. Since $deg\,\delta_\alpha = \alpha \geq p$ if $\alpha \neq 0$ we obtain $[L, Z_i^a(L)] \subseteq Z_{i+1}^a(L)$.

Some pictures might be helpful:

The monomials x^iy^j, (i,j) any point of the rectangle $\{i \leq p-2, j \leq q-2\}$, are a $\mathbb{C}$-basis of the Tjurina algebra $\mathbb{C}[[x,y]]/(x^p+y^q)$ of X_0. The monomials in the shaded region $B = \{(i,j) \mid iq+jp > pq, i \leq p-2, j \leq q-2\}$ correspond to deformations of X_0 with fixed semigroup $\langle p,q \rangle$. They occur as coefficients of the parameters T_α of the versal base space T for such deformations. The T_α are indexed by $\alpha = ip+jp-pq$ if x^iy^j is the coefficient of T_α (this is unique since $gcd(p,q)=1$). Hence the (increasing) filtration of $\mathbb{C}[\underline{T}]$, $F_a^\bullet$, is generated by those T_α such that the coefficients x^iy^j belong to a strip as indicated above. Note that the degree of the T_α decreases at most by p $(= min\,\{p,q\})$ if we go from F_a^i to F_a^{i+1}, the different choices of $a, 0 \leq a \leq p$, means just a shift of the starting point.
$B^{\checkmark} = \{(p-2-i, q-2-j) \mid (i,j) \in B\}$ is just the mirror of B at the centre of the rectangle. The vector fields δ_α which generate the Lie algebra L are indexed by the weights $\alpha = aq+bp$ of the points $(a,b) \in B^{\checkmark}$. The (decreasing) filtration $Z_\bullet^a$ of L is given by dual strips, indexed in a complementary manner: $\delta_\alpha \in Z_i^a \Longleftrightarrow T_{\alpha^\checkmark} \in F_a^{k-i}, \alpha^\checkmark = pq-2p-2q-\alpha$, i.e. Z_{k-i}^a is the mirror image of F_a^i.

The second picture shows the (decreasing) filtration $H_a^\bullet$ of $\mathbb{C}[[x,y]]$, H_a^i is generated by all monomials above and on the dotted line. Hence $\tau_a^i(t)$ is the number of monomials in the shaded region which are linear independent modulo $(F_t, \partial F_t/\partial x,$
$\partial F_t/\partial y, H_a^i)$, for $t=0$ these are all monomials.

Let $\{U_\alpha^a\}$ now be the flattening stratification on $T = Spec\,\mathbb{C}[\underline{T}]$ corresponding to $F_a^\bullet$ and $Z_\bullet^a$. Notice that each U_α^a is a locally closed, not necessarily reduced subvariety of T.

Now we can apply Theorem 1.4 and obtain that $U_\alpha^a \to U_\alpha^a/L$ is a geometric quotient. Moreover, $L_0/L \cong \mathbb{C}^*$ acts on U_α^a/L and $U_\alpha^a/\mathcal{L} = U_\alpha^a/L_0$ is a geometric quotient of U_α^a by L_0. For $t \in T$ define $\underline{e}_{(t)}^a = (e_0^a(t), \ldots, e_k^a(t)) \in \mathbb{N}^{k+1}$ by

$$e_i^a(t) = rank(\delta_\alpha(T_\beta)(t))_{\substack{\beta \leq a+ip \\ \alpha \geq 0}}, i = 0, \ldots, k.$$

Theorem 2.1. *Let T be the base space of the versal deformation with fixed semigroup of $Spec\,\mathbb{C}[[x,y]]/(x^p+y^q)$ and $\{U_\alpha^a\}_\alpha$ the stratification of T defined above. The following holds:*

1. *$\underline{e}^a$ is constant on U_α^a and takes different values for different α. The scheme structure of U_α^a is defined by the corresponding minors of $(\delta_\alpha(T_\beta))$. Moreover, $e_i^a(t) = \mu(X_t) - \tau_a^{q+i}(t)$. In particular, $e_k(t) = \mu(X_t) - \tau(X_t)$, where $\mu(X_t) = \mu = (p-1)(q-1)$ is the Milnor number and $\tau(X_t)$ the Tjurina number of the curve singularity $X_t =$ Spec $\mathbb{C}[[x,y]]/(F_t)$.*

2. *Let $\underline{e} = (e_1, \ldots, e_k) \in \mathbb{N}^{k+1}$ and let $U_{\underline{e}}^a$ denote the unique stratum such $\underline{e}^a(t) = \underline{e}$ for $t \in U_{\underline{e}}^a$ and assume that $U_{\underline{e}}^a$ is not empty. The geometric*

quotient $U^a_{\underline{e}}/\mathcal{L}$ is quasiaffine and of finite type over $\mathbb{C}$. It is a coarse moduli space for the functor which associates to any complex space germ S the set of isomorphism classes of flat families (with section) over S of plane curve singularities with fixed semigroup $\langle p,q\rangle$ and fixed Hilbert function $\tau^\bullet_a(t)=\mu-\underline{e}$ of the Tjurina algebra.

3. *Let $T_{\tau_{min}}$ be the open dense subset of T defined by singularities with minimal Tjurina number τ_{min}. Then there exists an a such that $T_{\tau_{min}} = U^a_\alpha$ for a suitable α. In particular, the geometric quotient $T_{\tau_{min}}/\mathcal{L}$ exists and is a coarse moduli space for curves with semigroup $\langle p,q\rangle$ and Tjurinanumber τ_{min}. $T_{\tau_{min}}/\mathcal{L}$ is locally isomorphic to an open subset of a weighted projective space.*

Proof:

1. $t\in U^a_\alpha$ iff $rank\,(\delta_\alpha(T_\beta)(t))_{\beta\leq a+ip \atop \alpha>0} =: r_i(t)$ ($\alpha=0$ excluded) and $rank\,(\delta_\alpha(T_\beta)(t))_{\alpha\in S_i} =: s_i(t)$ are constant (remark 2 after Theorem 1.4). But since $\delta_\alpha(T_\beta)=\delta_{\beta^\checkmark}(T^\checkmark_\alpha)$ ($\alpha=0$ included) we have $s_k=e_0,\ldots,s_1=e_{k-1}$ and $r_i=max\{0,\,e_i-1\}$. It is also clear that the scheme structure required by the flattening property is given by the minors. For $t\in U^a_\alpha$ consider the induced $\mathbb{C}$-base $\{u_i(t)\}$ of $\mathbb{C}[[x,y]]/(\frac{\partial F_t}{\partial x},\frac{\partial F_t}{\partial y}), u_i(t)F(x,y,t) = \sum\limits_j \delta_i(T_j)(t)u_{j+pq}(t)$ (as in the remark 2.3). This implies by definition of $\tau^\bullet_a(t), e_i(t)=\mu(X_t)-\tau^{q+i}_a(t)$.
2. This follows from the fact that $U^a_{\underline{e}}$ is locally a versal family for singularities with fixed semigroup and Hilbert function and that $U^a_{\underline{e}}/\mathcal{L}$ is a geometric quotient.
3. Is proved in [LaP].

Remark: We can actually enlarge the functor for which $\mathcal{U}^a_{\underline{e}}/\mathcal{L}$ is a coarse moduli space. Namely, we can extend the category of base spaces from germs to complex spaces S for which $H^1(S,\mathbb{Z}/d\mathbb{Z})=0$ (compare [GP 3]).

Example: $p=5, q=11$

1. The versal deformation of x^5+y^{11} fixing the semigroup $\langle 5,11\rangle$ is given by

$$\begin{aligned}&F(x,y,\underline{T}) =\\ &\quad x^5+y^{11}+T_1\,x\,y^9+T_2\,x^2\,y^7+T_3\,x^3\,y^5+T_7\,x^2\,y^8+\\ &\quad T_8\,x^3\,y^6+T_{12}\,x^2\,y^9+T_{13}\,x^3\,y^7+T_{18}\,x^3\,y^8+T_{23}\,x^3\,y^9\end{aligned}$$

$$B=\{(1,9),(2,7),(3,5),(2,8),(3,6),(2,9),(3,7),(3,8),(3,9)\}$$

$$B^\checkmark=\{(0,0),(0,1),(0,2),(1,0),(0,3),(1,1),(0,4),(1,2),(2,0)\}$$

2. The following vector fields generate the kernel $\mathcal{L}$ of the Kodaira-Spencer map (these can be computed using the algorithm given in [LaP]).

$$\begin{aligned}
\delta_0 &= T_1\frac{\partial}{\partial T_1} + 2T_2\frac{\partial}{\partial T_2} + 3T_3\frac{\partial}{\partial T_3} + \ldots + 23T_{23}\frac{\partial}{\partial T_{23}} \\
\delta_5 &= A\frac{\partial}{\partial T_7} + B\frac{\partial}{\partial T_8} + C\frac{\partial}{\partial T_{12}} + D\frac{\partial}{\partial T_{13}} E\frac{\partial}{\partial T_{18}} + 18T_{18}\frac{\partial}{\partial T_{23}} \\
\delta_{10} &= 2T_2\frac{\partial}{\partial T_{12}} + (3T_3 + T_1T_2)\frac{\partial}{\partial T_{13}} + D\frac{\partial}{\partial T_{18}} + 13T_{13}\frac{\partial}{\partial T_{23}} \\
\delta_{11} &= T_1\frac{\partial}{\partial T_{12}} + 2T_2\frac{\partial}{\partial T_{13}} + C\frac{\partial}{\partial T_{18}} + 12T_{12}\frac{\partial}{\partial_{23}} \\
\delta_{15} &= B\frac{\partial}{\partial T_{18}} + 8T_8\frac{\partial}{\partial T_{23}} \\
\delta_{16} &= A\frac{\partial}{\partial T_{18}} + 7T_7\frac{\partial}{\partial T_{23}} \\
\delta_{20} &= 3T_3\frac{\partial}{\partial T_{23}} \\
\delta_{21} &= 2T_2\frac{\partial}{\partial T_{23}} \\
\delta_{22} &= T_1\frac{\partial}{\partial T_{23}}
\end{aligned}$$

with $A = 2T_2 - \frac{9}{11}T_1^2$, $B = 3T_3 - \frac{7}{11}T_1T_2$, $C = 7T_7 + \frac{3}{11}T_1T_3^2$, $D = 8T_8 - \frac{8}{11}T_1T_7 + \frac{2}{11}T_1^2T_3^2$, $E = 13T_{,3} - \frac{117}{11}T_1T_{12} + \frac{3}{11}T_1^2T_3T_8 + \frac{55}{11^2}T_1T_2T_3T_7 + \frac{7}{5\cdot 11^3}T_1^3T_2^2T_3^2$.
For the filtration F_a^i of $\mathbb{C}[\underline{T}]$ we choose $a = 5$, i.e. $F_a^i = \langle T_\alpha \mid \alpha \leq 5(1+i)\rangle$ (we omit the index a) $F^0 = \langle T_1, T_2, T_3\rangle$, $F^1 = \langle T_1, T_2, T_3, T_7, T_8\rangle$, $F^2 = \langle T_1, T_2, T_3, T_7, T_8, T_{12}, T_{13}\rangle$, $F^3 = \langle T_1, T_2, T_3, T_7, T_8, T_{12}, T_{13}, T_{18}\rangle$, $F^4 = \langle T_1, \ldots, T_{23}\rangle$.
The minimial i such that dF^i generates $\mathbb{C}[\underline{T}]\, d\,\mathbb{C}[\underline{T}]$ is 4, hence $k = 4$. The stratification $\{U_\alpha\}$ is given by fixing the rank of $(\delta_\alpha(T_\beta))_{\substack{\beta\leq 5(i+1)\\ \alpha\geq 0}}$ for $i = 0, \ldots, 4$. Calculation shows:

$$
\begin{array}{lcl}
U_1 & = & \{t \in Spec\, \mathbb{C}[\underline{T}] \mid rank(\delta_\alpha(T_\beta)(t)) = 6\} = \{t \mid \tau(X_t) = 34\} \\
& = & \{t \mid \underline{e}(t) = (1,2,4,5,6)\} = \{t \mid 4t_2^2 - 3t_1t_3 + t_1^2 t_2 \neq 0\} \\
& & \text{with } e_i(t) = \mu(x_t) - \tau_a^{q+i}(t). \text{We have } U_1 = T_{\tau_{min}}. \\
U_2 & = & \{t \mid \underline{e}(t) = (1,2,3,4,5)\} \\
& = & \{t \mid 4t_2^2 - 3t_1t_3 - t_1^2t_2 = 0 \text{ and } A(t) \neq 0 \text{ or } B(t) \neq 0\} \\
U_3 & = & \{t \mid \underline{e}(t) = (1,1,3,4,5)\} \\
& = & \{t \mid 4t_2^2 - 3t_1t_3 - t_1^2t_2 = A(t) = B(t) = 0 \text{ and} \\
& & D(t)(2t_2C(t) - t_1D(t) - C(t)(C(t)(3t_3 + t_1t_2) - 2t_2D(t)) \neq 0\} \\
U_4 & = & \{t \mid \underline{e}(t) = (1,1,2,3,4)\} \\
& = & \{t \mid A(t) = B(t) = t_1(9t_1C(t) - 11D(t)) = 0 \text{ and} \\
& & C(t)^2 - t_1E(t) \neq 0 \text{ or } D(t)^2 - (\frac{9}{11})^2 t_1^3 E(t) \neq 0\} \\
U_5 & = & \{t \mid \underline{e}(t) = (1,1,2,2,3)\} \\
& = & \{t \mid A(t) = B(t) = C(t)^2 - t_1E(t) = D(t) - \frac{9}{11}t_1C(t) = 0 \text{ and } t_1 \neq 0\} \\
U_6 & = & \{t \mid \underline{e}(t) = (0,0,1,2,3)\} \\
& = & \{t \mid t_1 = t_2 = t_3 = t_7 = t_8 = 0, t_{13} \neq 0\} \\
U_7 & = & \{t \mid \underline{e}(t) = (0,0,1,1,2)\} \\
& = & \{t \mid t_1 = \ldots = t_8 = 0, t_{13} = 0 \text{ and } t_{12} \neq 0\} \\
U_8 & = & \{t \mid \underline{e}(t) = (0,0,0,1,2)\} \\
& = & \{t \mid t_1 = \ldots = t_{13} = 0 \text{ and } t_{18} \neq 0\} \\
U_9 & = & \{t \mid \underline{e}(t) = (0,0,0,0,1)\} \\
& = & \{t \mid t_1 = \ldots = t_{18} = 0 \text{ and } t_{23} \neq 0\} \\
U_{10} & = & \{t \mid \underline{e}(t) = (0,0,0,0,0)\} \\
& = & \{0\} \\
U_1/\mathcal{L} & = & D(2T_2A - T_1B) \subseteq \text{ Proj } \mathbb{C}[T_1T_2T_3, y] = \mathbb{P}^3_{(1:2:3:10)},\ y = AT_8 - BT_7
\end{array}
$$

Conclusion: The space of plane curves with semigroup $\langle 5, 11\rangle$ ist stratified into ten strata $U_1, \ldots, U_{10}$, corresponding to the different values of the Hilbert function $\tau_5^\bullet$ of the Tjurina algebra; U_1 is the τ_{min}- and U_{10} the τ_{max}-stratum. The quotients $U_i/\mathcal{L}$ exist and are a coarse moduli space for such singularities with corresponding fixed value of $\tau_5^\bullet$.

Remarks:

1. It is not always possible to choose $a = p$ for the filtration to obtain $T_{\tau_{min}}$ as one stratum in the corresponding stratification. In the case $p = 13, q = 36$ we have to choose $a = 9$ (cf. [LaP]).
2. In the $< 5, 11 >$-example we have $U_2 \cup U_3 = \{t \mid rank(\delta_\alpha(T_\beta))(t) = 5\} = \{t \mid \tau(X_t) = 35\}$.

The geometric quotient $U_2 \cup U_3/\mathcal{L}$ does not exist (cf. [LaP]), i.e. fixing τ is not enough, it is necessary to work with a finer stratification.

3 A moduli space for irreducible plane curve singularities with semigroup $\langle 2p, 2q, 2pq+d\rangle$

1. The "worst" object is the singularity defined by $(x^p+y^q)^2+x^\alpha y^\beta, \alpha q+\beta p = 2pq+d, \alpha < p$. A versal deformation of $(x^p+y^q)^2+x^\alpha y^\beta$, fixing the semigroup $\langle 2p, 2q, 2pq+d\rangle, p < q, gcd(p,q) = 1$ and $d\, odd$, is given by

$$\begin{array}{rcll} F(x,y,\underline{H},\underline{W}) & = & (x^p+y^q & + \sum_{(i,j)\in B_0} H_{iq+jp-pq}x^iy^j)^2 + x^\alpha y^\beta \\ & & & + \sum_{(i,j)\in B_1} W_{iq+jp-2pq}x^iy^j, \end{array}$$

$$B_0 = \{(i,j), iq+jq > pq, i \le p-2, j \le q-2\},$$

$$B_1 = \{(i,j), iq+jp > 2pq+d, i < p, j < \delta\}$$

$$\cup \{(i,j), iq+jp > 2pq+d, i < \gamma, j < \delta+q\}$$

with γ, δ defined by $\gamma < p, \gamma q+\delta p = 3pq-q-p+d$. Let $\underline{H} = \{H_{iq+jp-qp}\}_{(i,j)\in B_0}$,
$\underline{W} = \{W_{iq+jp-2pq}\}_{(i,j)\in B_1}$, $T := Spec\, \mathbb{C}[\underline{H},\underline{W}], X := Spec\, \mathbb{C}[\underline{H},\underline{W}]\,[[x,y]]/F$.
The family $X \to T$ has the following properties:

   1.1 $X \to T$ is a versal deformation of $Spec\, \mathbb{C}\,[[x,y]]/(x^p+y^q)^2+x^\alpha y^\beta$ fixing the semigroup $\langle 2p, 2q, 2pq+d\rangle$.

   1.2 Every plane curve singularity with semigroup $\langle 2p, 2q, 2pq+d\rangle$ is represented in this family.

   1.3 The group μ_d of $d-th$ roots of unity acts on T via $F(\lambda^q x, \lambda^p y, \underline{h}, \underline{w}) = \lambda^{2pq}F(\lambda, y, \lambda\circ\underline{h}, \lambda\circ\underline{w})$ for $\lambda \in \mu_d$.

   1.4 $dim\, T = 2(p-1)(q-1)-p-q+2+\left[\frac{q}{p}\right]$.

2. The Kodaira-Spencer map of the family $X \to T$ is given by
$\rho : Der_{\mathbb{C}}\mathbb{C}\,[\underline{H},\underline{W}] \longrightarrow \mathbb{C}\,[\underline{H},\underline{W}]\,[[x,y]]/(F, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y})$
$\rho(\delta) = class\,(\delta F)$
$= class\,(2(x^p+y^q+\sum\limits_{(i,j)\in B_0} H_{iq+jp-pq}x^iy^j) \sum\limits_{(i,j)\in B_0} \delta H_{iq+jp-pq}x^iy^j$
$+ \sum\limits_{(i,j)\in B_1} \delta W_{iq+jp-2pq}x^iy^j)$.

   The kernel of the Kodaira-Spencer map is a Lie algebra $\mathcal{L}$ which is a finitely generated $\mathbb{C}\,[\underline{H},\underline{W}]$-module and has the following property:

   2.1 For t, t' the singularities X_t and $X_{t'}$ are isomorphic iff for a suitable $\lambda \in \mu_d$ $\lambda\circ t$ and t' are in the same integral manifold of $\mathcal{L}$, i.e. $T/\mathcal{L}/\mu_d$ is a classifiying space for all singularities with semigroup $\langle 2p, 2q, 2pq+d\rangle$.

2.2 It is always possible for $(\alpha, b) \neq (0,0)$ to obtain a unique decomposition

$$x^a y^b F = (x^p + y^q + \sum_{(i,j)\in B_0} H_{iq+jp-pq} x^i y^j) \sum_{(i,j)\in B_0} E^{qa+pb}_{iq+jp-pq} x^i y^j$$

$$+ \sum_{(i,j)\in B_1} D^{qa+pb}_{iq+jp-2pq} x^i y^j mod(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}).$$

This defines vector fields δ_{aq+pb} of the kernel of the Kodaira-Spencer map by $\delta_{qa+pb}(H_s) = \frac{1}{2}E_s^{qa+pb}$ and $\delta_{qa+pb}(W_s) = D_s^{qa+pb}$. $\mathcal{L}$ is generated as $\mathbb{C}[\underline{H}, \underline{W}]$-module by the vector fields $\{\delta_s\}$. These vector fields have the following properties:

- $[\delta_\ell, \delta_m] \in \sum_{s>l+m} \mathbb{C}[\underline{H}, \underline{W}]\delta_s$
- $\delta_\ell = 0$ if $l > 2pq - 2p - 2q$
- $\delta_\ell(W_m) = 0$ if $m < l + d$
- if $(\alpha{+}a, \beta{+}b)$ or $(\alpha{+}a{-}p, \beta{+}b{-}q) \in B_1$ then $\delta_{aq+bp}(W_{d+aq+bp}) = -\frac{d}{2pq}$.

Now it is not difficult to see that

- $aq{+}bp < pq{-}q$ implies $(\alpha{+}a, \beta{+}b)$ or $(\alpha{+}a{-}p, \beta{+}b{+}q) \in B_1$
- $(i,j) \in B_1$ and $iq{+}jp \geq 3pq{+}d{-}q$ implies $(i,j) = (\alpha{+}a, \beta{+}b)$ or $(i,j) = (\alpha + a - p, \beta + b + q)$ for a suitable (a,b).

From these properties we deduce that already

$\{\delta_\ell\}_{\ell \in L}, L = \{\ell = aq{+}bp, (\alpha{+}a, \beta{+}b) \text{ or } (\alpha{+}a{-}p, \beta{+}b{-}q) \in B_1\}$

generates $\mathcal{L}$ as $\mathbb{C}[\underline{H}, \underline{W}]$-module. Now for $\ell \in L$ we know that $\delta_\ell(W_{\ell+d}) = -\frac{d}{2pq}$ and $\delta_\ell(W_m) = 0$ if $m < \ell + d$. This implies that $\{\delta_\ell\}_{\ell\in L}$ and $\{W_{\ell+d}\}_{\ell\in L}$ satisfy the properties of proposition 1.5. This implies that $T \to T/\mathcal{L}$ is a geometric quotient and in particular $T/\mathcal{L} \cong Spec\,\mathbb{C}[\underline{H}, \underline{W'}]$ with $\underline{W'} = \{W_{iq+jp-2pq}\}_{(i,j)\in B_2}, B_2 = \{(i,j) \in B_1, iq + jp - 2pq - d \notin L\}$.
Notice that $dim\,T/\mathcal{L} = (p-2)(q-2) + [\frac{q}{p}] - 1$. We obtain as in section 2:

Theorem 3.1. *$\mathbb{C}^{(p-2)(q-2)+[\frac{q}{p}]-1}/\mu_d$ is a coarse moduli space for families of plane curve singularities with semi-group $\langle 2p, 2q, 2pq{+}d\rangle$. The Tjurina number of all these singularities is constant and equal to $\tau = \mu - (p-1)(q-1)$, where $\mu = (2p-1)(2q-1) + d$.*

For details of the proof see [LuP]. Note that we did not need to stratify in this case. The remark after Theorem 2.1 applies also to Theorem 3.1.

4 A moduli space for torsion free modules of rank 1 over the local ring of an irreducible curve singularity

Let R be the local ring of an irreducible curve singularity and $\bar{R} = \mathbb{C}[[t]]$ the normalization of R. Let $Mod\,(R)$ be the category of torsion free rank 1 R-modules, c the conductor of $R, \delta = \delta(R) = dim_{\mathbb{C}}\bar{R}/R$ be the δ-invariant, and $\bar{M} = M \otimes_R \bar{R}/\text{torsion}$.

Lemma 4.1. *1. Any $M \in Mod\,(R)$ is isomorphic to some fractional ideal M' such that $R \subset M' \subset \bar{R}$ and $dim_{\mathbb{C}}\bar{R}/M' = dim_{\mathbb{C}}\bar{M}/M$.*

2. Let $M, M' \subset \bar{R}$ be two fractional ideals such that $dim_{\mathbb{C}}\bar{R}/M = dim_{\mathbb{C}}\bar{R}/M'$. Then $M \simeq M'$ iff there is $u \in \bar{R}^*$ such that $uM = M'$.

3. For any $M \in Mod\,(R), M \subset \bar{R}$ and $dim\,\bar{R}/M = d$ we have $t^{d+\delta}\bar{R} \subseteq M$.

For a proof cf. [GP 2]; it is easy, use $M \subset M \otimes_R Quot(R) \cong \mathbb{C}[[t]][t^{-1}]$.

Definition: Let $M \in Mod(R)$ and $R \subset M' \subset \bar{R}$ such that $M \simeq M'$. We define $\Gamma(M) := \{v(m'), m' \in M'\}, (v$ the valuation of $\bar{R})$ and $\delta(M)$ as the number of gaps in $\Gamma(M)$.

Remark: $\Gamma(M)$ does not depend on the choice of M'. $\Gamma(M)$ is a $\Gamma(R)$ set. $\delta(M) = dim_{\mathbb{C}}\,\bar{M}/M$. In analogy to singularities, we may consider $\Gamma(M)$ as the "topological type" of M.

The aim is now to classify all torsion free rank 1 modules with fixed value set Γ. Let $R_c = \mathbb{C}[[t^c, t^{c+1}, \ldots]]$ and $M \in Mod(R)$ then $M \in Mod(R_c)$. We first solve the problem for $Mod(R_c)$. It is easy to see that R^* acts on the space which parametrizes objects of $Mod(R_c)$. The fixed point scheme will then be the solution for $Mod(R)$ since these points correspond to R_c modules which are also R-modules.

Let $\Gamma = \{\gamma_0, \gamma_1, \ldots, \gamma_k, c, c+1, \ldots\}, 0 = \gamma_0 < \gamma_1 < \ldots < \gamma_k < c$.

1. The "worst" object in $Mod_\Gamma(R_c)$ is the monomial module $M_0 = \sum\limits_{i=0}^{k} t^{\gamma_i}R_c + t^c\bar{R}$. A versal deformation of M_0 fixing Γ is the $\mathbb{C}[\underline{\lambda}][[t^c, t^{c+1}, \ldots]]$-module

$$\mathcal{M}_\Gamma = \sum_{i=0}^{k} m_i\mathbb{C}[\underline{\lambda}][[t^c, t^{c+1}, \ldots]] + t^c\mathbb{C}[\underline{\lambda}][[t]] \subset \mathbb{C}[\underline{\lambda}][[t]],$$

with
$$\begin{aligned}
\underline{\lambda} &= \{\lambda_{ij}\}_{(i,j)\in I}\\
I &= \{(i,j), 0\le i\le k, j>0, j+\gamma_i\notin\Gamma\}\\
m_i &= t^{\gamma_i}+\textstyle\sum_{j+\gamma_i\notin\Gamma}\lambda_{ij}t^{j+\gamma_i}
\end{aligned}$$

Let $T = Spec\,\mathbb{C}[\underline{\lambda}]$ and $\mathfrak{X} = Spec\,\mathbb{C}[\underline{\lambda}][[t^c, t^{c+1}, \ldots]]$ be the trivial deformation of R_c. $\mathcal{M}_\Gamma$ is a coherent $\mathcal{O}_{\mathfrak{X}}$-module, which is flat over T and its restriction M_t to the fibre $\mathfrak{X}_t = Spec\,R_c, t\in T$, is an element of $Mod\,(R_c)$. One shows:

1.1 $\mathcal{M}_\Gamma$ is a versal deformation of M_0 fixing the value set Γ.

1.2 Every R_c-module with value set Γ is represented in this family, i.e. there is a $t\in T$ such that M_t is isomorphic to the given module.

2. The Kodaira-Spencer map of the family $\mathcal{M}_\Gamma$ is given by

$$\rho : Der_{\mathbb{C}}\mathbb{C}[\underline{\lambda}] \to Ext^1_{\mathbb{C}[\underline{\lambda}][[t^c,t^{c+1},\ldots]]}(\mathcal{M}_\Gamma, \mathcal{M}_\Gamma).$$

Because $\mathcal{M}_\Gamma\subset\mathbb{C}[\underline{\lambda}][[t]]$ is embedded it factors through the Kodaira-Spencer map of the embedded family:

$$Der_{\mathbb{C}}\mathbb{C}\,[\underline{\lambda}] \to Hom_{\mathbb{C}[\underline{\lambda}][[t^c,t^{c+1},\ldots]]}(\mathcal{M}_\Gamma, \mathbb{C}\,[\underline{\lambda}][[t]]/\mathcal{M}_\Gamma),\ \delta\mapsto\varphi_\delta$$

and $\varphi_\delta(m)$ =class of $\delta(m)$, where δ is lifted to $\mathbb{C}\,[\underline{\lambda}][[t]]$ by $\delta(t)=0$.

2.1 Now one can prove that the kernel of the Kodaira-Spencer map $\mathcal{L}$ is the Lie algebra generated as $\mathbb{C}\,[\underline{\lambda}]$-module by the vectorfields $\{\delta_\ell\}_{1\le\ell\le c-1}$
$\delta_\ell = \sum\limits_{(i,j)\in I} h_{\ell,i,j}\frac{\partial}{\partial\lambda_{ij}}$ with $h_{\ell,i,j} = \lambda_{i,j-\ell} - \sum\limits_{\nu=i+1}^{k}\lambda_{i,\gamma_\nu-\gamma_i-\ell}\lambda_{\nu,j+\gamma_i-\gamma_\nu}$
(with the convention $\lambda_{ij}=0$ if $j<0, \lambda_{i,0}=1$).

Remark:

(1) $h_{\ell,i,j}=0$ if $\ell>j$

(2) $h_{j,i,j}=1$ if $(i,j)\in I$.

Furthermore $L := \sum\limits_{\ell=1}^{c-1}\mathbb{C}\delta_\ell$ is an abelian Lie algebra.

2.2 It is not difficult to see that for $t,t'\in T$ the modules M_t and $M_{t'}$ are isomorphic iff they are in the same integral manifold of $\mathcal{L}$, i.e. in the same orbit under the action of L.

2.3 $\mathbb{C}[\underline{\lambda}]$ admits a $\mathbb{C}^*$-action defined by $deg\, \lambda_{ij} = j$. The vector fields δ_ℓ are homogeneous of degree - ℓ. Let $\Gamma_0 \subset \Gamma$ be the maximal semigroup acting on Γ and $a = mult(\Gamma_0 \setminus \{0\})$. For the sub Lie algebras $L^{(0)} := \sum_{i \geq a} \mathbb{C}\delta_i$ and $L^{(1)} := \sum_{i<a} \mathbb{C}\delta_i$, we have $H^1(L^{(1)}, \mathbb{C}[\underline{\lambda}]) = 0$ and $H^1(L^{(1)}, \mathbb{C}[\underline{\lambda}]^{L^{(0)}}) = 0$.

3. Consider now the filtration $F^i(\mathbb{C}[\underline{\lambda}]) :=$ the $\mathbb{C}$-vector space generated by all quasihomogeneous polynomials of degree less than $(i+1)a$.
If $\delta \in L^{(0)}$ then $\delta F^i \subseteq F^{i-1}$. Let $T = Spec\, \mathbb{C}[\underline{\lambda}] = \cup U_\alpha$ be the flattening stratification of the $\mathbb{C}[\underline{\lambda}]$-modules $Hom_{\mathbb{C}}(L^{(0)}, \mathbb{C}[\underline{\lambda}])/\mathbb{C}[\underline{\lambda}]dF^i\mathbb{C}[\underline{\lambda}]$.
We may apply corollary 1.3 and obtain that $U_\alpha \to U_\alpha/L^{(0)}$ is a geometric quotient. Since $H^1(L^{(0)}, \mathbb{C}[\underline{\lambda}]^{L^{(0)}}) = 0$ we obtain that $U_\alpha \to U_\alpha/L$ is a geometric quotient.

Remark: For $t \in T$ let $E(t)(n) := rank\,(\delta_\ell(\lambda_{ij})(t))_{j<n}$, then $E(t)(\nu a)$ is constant along U_α for all ν. More precisely, let d be maximal such that $d \notin \Gamma$ and $\underline{r} = (r_1, \ldots, r_{[\frac{d}{a}]})$ such that $E(t_0)(\nu a) = r_{\nu-1}, \nu = 2, \ldots, [\frac{d}{a}] + 1$ for some $t_0 \in U_\alpha$ then

$$t \in U_\alpha \quad \Longleftrightarrow \quad E(t)(\nu a) = r_{\nu-1}, \quad \nu = 2, \ldots, \left[\frac{d}{a}\right] + 1$$

Let $\underline{E}(t) := (E(t)(2a), \ldots E(t)([\frac{d}{a}] + 1)a))$. We also write $U_{\underline{r}}$ instead of U_α. The invariants $\underline{E}(t)$ which describe the stratification $\{U_{\underline{r}}\}$ can be interpreted as follows:

Remark: For $M \in Mod(R)$, the ring $End_R(M)$ dominates R, we have $R \subset End_R(M) \subset \bar{R}$, and M is an $End_R(M)$-module. We assume that $M \subset \bar{R}$ and define a filtration $End^\bullet(M)$ of $\bar{R}$ by

$$End^n(M) := \{g \in \bar{R} \mid g\, M_{n,i} \subset M_{n,i} \text{ for all } i\}$$

where $M_{n,i} = M \cap \mathbb{C}[[t^i, t^{i+1}, \ldots,]] + \mathbb{C}[[t^{n+i}, t^{n+i+1}, \ldots]]$. $End^n(M)$ is independent of the embedding of M into $\bar{R}$. The function $E(t)$ has the following interpretation:

$$E(t)(n) = dim_{\mathbb{C}} \bar{R}/End^n(M_t),$$

where M_t is the module corresponding to $t \in T$, i.e. $E(t)$ is the Hilbert function of $\bar{R}/End_R(M)$ with respect to the filtration $End^\bullet(M)$.

Theorem 4.2. *Let $R_c = \mathbb{C}[[t^c, t^{c+1}, \ldots]], \underline{r} \in \mathbb{Z}^k, k = [\frac{d}{a}]$, and $U_{\underline{r}}$ the stratum such that $\underline{E}(t) = \underline{r}$ for $t \in U_{\underline{r}}$. The geometric quotient $U_{\underline{r}}/L$ exists, is a quasiaffine algebraic scheme and a coarse moduli space for flat families of torsion free R_c-modules with fixed value set Γ and fixed Hilbert function of $\bar{R}_c/End(M)$ with respect to the filtration $End^\bullet(M)$. The same holds for the* *local ring R of an arbitrary irreducible curve singularity if we replace $U_{\underline{r}}$ by the fixed point scheme $U_{\underline{r}}^{R^*}$ of $U_{\underline{r}}$ with respect to the natural action of R^*.*

Proof: For the proof that the geometric quotient $U_{\underline{R}}/L$ exists, we use the criteria of chapter 1. To see that $U_{\underline{R}}/L$ indeed is a coarse moduli space, also for families over non-reduced base spaces, many details have to be checked and one has to be very careful about the functors defining families of such modules. First of all, one has to generalize the notion of Γ- and E-constant families to non-reduced base spaces. This can finally be achieved by requiring that certain sheaves are flat over the base. Then one has to prove that the strata $U_{\underline{r}}$, together with the family of modules which can be constructed over it, is, in each of its points, a representative of a versal Γ- and E-constant deformation of the corresponding module. Last, but not least, some foundational work has to be done on families of objects which are algebraic along the base but formal resp. analytic along the fibres.

For a complete proof see [GP 2].

Example: We construct in detail the stratification $\{U_{\underline{r}}\}$ of the space of torsion free R_c-modules of rank 1 with fixed value set Γ, and for each stratum we determine the quotient for the example

$$R = R_c = \mathbb{C}[[t^8, t^9, \ldots]], c = 8,$$
$$\Gamma = \Gamma_0 = \{0, 2, 4, 6, 8, 9, \ldots\}, k = 3.$$

Recall that Γ_0 is the maximal semigroup contained in the set Γ, $a =$ smallest non-gap ($\neq 0$) of Γ_0, $d =$ biggest gap of Γ and $k = [\frac{d}{a}]$.

We have:

$$I = \{(0,1), (1,1, (2,1), (3,1), (0,3), (1,3), (2,3), (0,5), (1,5), (0,7)\}.$$

The matrix $H(\underline{\lambda}) = (h_{\ell,i,j}(\underline{\lambda}))$ of the coefficients of the vector fields $\delta_1, \ldots, \delta_7$ is:

$$\begin{pmatrix}
1\ldots 1 & -\lambda_{11}\lambda_{01} & -\lambda_{21}\lambda_{11} & -\lambda_{31}\lambda_{21} & -\lambda_{21}\lambda_{03} - \lambda_{12}\lambda_{01} & -\lambda_{31}\lambda_{13} - \lambda_{23}\lambda_{11} & * \\
 & \lambda_{01} - \lambda_{11} & \lambda_{11} - \lambda_{21} & \lambda_{21} - \lambda_{31} & \lambda_{03} - \lambda_{13} & \lambda_{13} - \lambda_{23} & * \\
 & 1 & 1 & 1 & -\lambda_{21}\lambda_{01} & -\lambda_{31}\lambda_{11} & * \\
 & & & & \lambda_{01} - \lambda_{21} & \lambda_{11} - \lambda_{31} & * \\
 & & & & 1 & 1 & \lambda_{01} - \lambda_{31} \\
 & & & & & & 1
\end{pmatrix}$$

We have $a=2, d=7$.

Let $T = Spec\,\mathbb{C}[\underline{\lambda}] = \cup U_{\underline{r}}$ be the stratification constructed before. Then for $\underline{r} \in \{(3,5,6),(2,4,5),(3,4,5),(2,3,4)\}$ we have $U_{\underline{r}} \neq \emptyset$.

$$\begin{aligned}
U_{\Gamma,(3,5,6)} &= \{\lambda, \lambda_{01}-\lambda_{11}-\lambda_{21}+\lambda_{31} \neq 0\} \\
U_{\Gamma,(3,4,5)} &= \{\lambda, \lambda_{01}-\lambda_{11}-\lambda_{21}+\lambda_{31} = 0, 2\lambda_{11}-\lambda_{01}-\lambda_{21} \neq 0\} \\
U_{\Gamma,(2,4,5)} &= \{\lambda, \lambda_{01}-\lambda_{11}-\lambda_{21}+\lambda_{31} = 2\lambda_{11}-\lambda_{01}-\lambda_{21} = 0, \\
&\qquad 2\lambda_{31}-\lambda_{03}-\lambda_{23}+(\lambda_{01}-\lambda_{11})(\lambda_{11}\lambda_{31}-\lambda_{01}\lambda_{21}) \neq 0\} \\
U_{\Gamma,(2,3,4)} &= \{\lambda, \lambda_{01}-\lambda_{11}-\lambda_{21}+\lambda_{31} = 2\lambda_{11}-\lambda_{01}-\lambda_{21} = 0, \\
&\qquad 2\lambda_{13}-\lambda_{03}-\lambda_{23}+(\lambda_{01}-\lambda_{11})(\lambda_{11}\lambda_{31}-\lambda_{01}\lambda_{21}) = 0\}
\end{aligned}$$

$$\begin{aligned}
\text{let } L_1 &:= \mathbb{C}\delta_1 + \mathbb{C}\delta_3 + \mathbb{C}\delta_5 + \mathbb{C}\delta_7, \ \text{ then} \\
\mathbb{C}[\underline{\lambda}] &= \mathbb{C}[\underline{\lambda}]^{L_1}[\lambda_{01},\lambda_{03},\lambda_{05},\lambda_{07}] \\
\text{and} & \\
\mathbb{C}[\underline{\lambda}]^{L_1} &= \mathbb{C}[\bar\lambda_{11},\bar\lambda_{21},\bar\lambda_{31},\bar\lambda_{13},\bar\lambda_{23},\bar\lambda_{15}] \\
\bar\lambda_{11} &= \lambda_{11}-\lambda_{01} \\
\bar\lambda_{21} &= \lambda_{21}-\lambda_{01} \\
\bar\lambda_{31} &= \lambda_{31}-\lambda_{01} \\
\bar\lambda_{13} &= \lambda_{13}-\lambda_{03}-\lambda_{01}(\lambda_{11}\lambda_{01}-\lambda_{11}\lambda_{21}) + \tfrac{1}{2}\lambda_{01}^2(\lambda_{01}-\lambda_{21}) \\
\bar\lambda_{23} &= \lambda_{23}-\lambda_{03}-\lambda_{01}(\lambda_{11}\lambda_{01}-\lambda_{21}\lambda_{31}) + \tfrac{1}{2}\lambda_{01}^2(\lambda_{11}+\lambda_{01}-\lambda_{21}-\lambda_{31}) \\
\bar\lambda_{15} &= \lambda_{15}-\lambda_{05}-(\lambda_{01}\lambda_{21}-\lambda_{11}\lambda_{31})\lambda_{03}+\lambda_{01}(\lambda_{31}-\lambda_{01})(\lambda_{13}-\lambda_{03}) \\
&\quad +\lambda_{01}\lambda_{11}(\lambda_{23}-\lambda_{03}) - \tfrac{1}{2}\lambda_{01}^2(\lambda_{23}-\lambda_{03}) \\
&\quad + \text{polynomial in } \lambda_{01},\lambda_{11},\lambda_{21},\lambda_{31}.
\end{aligned}$$

We have

$$\begin{aligned}
\delta_2 \mid \mathbb{C}[\underline{\lambda}]^{L_1} &= (2\bar\lambda_{11}-\bar\lambda_{21})\tfrac{\partial}{\partial\bar\lambda_{13}} + (\bar\lambda_{11}-\bar\lambda_{31}+\bar\lambda_{21})\tfrac{\partial}{\partial\bar\lambda_{23}} + (2\bar\lambda_{13}-\bar\lambda_{23}-\bar\lambda_{11}^2\bar\lambda_{31})\tfrac{\partial}{\partial\bar\lambda_{15}} \\
\delta_4 \mid \mathbb{C}[\underline{\lambda}]^{L_1} &= (\bar\lambda_{11}-\bar\lambda_{31}+\bar\lambda_{21})\tfrac{\partial}{\partial\bar\lambda_{15}} \\
\delta_6 \mid \mathbb{C}[\underline{\lambda}]^{L_1} &= 0
\end{aligned}$$

Let $\bar U_{\Gamma,\underline{r}} = \varphi(U_{\Gamma,\underline{r}}), \varphi : U_\Gamma \to U_\Gamma/L_1 = Spec\,\mathbb{C}[\bar{\underline{\lambda}}]^{L_1}$ be the quotient map and $L_0 = K\delta_2 + K\delta_4$. Then

$$
\begin{aligned}
\bar{U}_{\Gamma,(3,5,6)} &= \{\bar{\lambda}, \bar{\lambda}_{11} - \bar{\lambda}_{31} + \bar{\lambda}_{21} \neq 0\} \\
\bar{U}_{\Gamma,(3,4,5)} &= \{\bar{\lambda}, \bar{\lambda}_{11} - \bar{\lambda}_{31} + \bar{\lambda}_{21} = 0, 2\bar{\lambda}_{11} - \bar{\lambda}_{21} \neq 0\} \\
\bar{U}_{\Gamma,(2,4,5)} &= \{\bar{\lambda}, \bar{\lambda}_{11} - \bar{\lambda}_{31} + \bar{\lambda}_{21} = 2\bar{\lambda}_{11} - \bar{\lambda}_{21} = 0, 2\bar{\lambda}_{13} - \bar{\lambda}_{23} - \bar{\lambda}_{11}^2\bar{\lambda}_{31} \neq 0\} \\
\bar{U}_{\Gamma,(2,3,4)} &= \{\bar{\lambda}, \bar{\lambda}_{11} - \bar{\lambda}_{31} + \bar{\lambda}_{21} = 2\bar{\lambda}_{11} - \bar{\lambda}_{21} = 2\bar{\lambda}_{31} - \bar{\lambda}_{23} - \bar{\lambda}_{11}^2\bar{\lambda}_{31} = 0\} \\
U_{\Gamma,(3,5,6)}/L &= \bar{U}_{\Gamma,(3,5,6)}/L_0 \\
&= Spec\,\mathbb{C}[\bar{\lambda}_{11}, \bar{\lambda}_{21}, \bar{\lambda}_{31}, \bar{\lambda}_{13}(\bar{\lambda}_{11} - \bar{\lambda}_{31} + \bar{\lambda}_{21}) - \bar{\lambda}_{23}(2\bar{\lambda}_{11} - \bar{\lambda}_{21})]_g \\
g &:= \bar{\lambda}_{11} - \bar{\lambda}_{31} + \bar{\lambda}_{21} \\
U_{\Gamma,(3,4,5)}/L &= \bar{U}_{\Gamma,(3,4,5)}/L_0, \\
&= Spec\,\mathbb{C}[\bar{\lambda}_{11}, \bar{\lambda}_{21}, \bar{\lambda}_{23}, \bar{\lambda}_{15}(2\bar{\lambda}_{11} - \bar{\lambda}_{21}) - \bar{\lambda}_{13}(2\bar{\lambda}_{13} - \bar{\lambda}_{23} - \bar{\lambda}_{11}^2 \\
&\quad (\bar{\lambda}_{11} + \bar{\lambda}_{21})) + \bar{\lambda}_{13}^2]_h \\
h &= 2\bar{\lambda}_{11} - \bar{\lambda}_{21} \\
U_{\Gamma,(2,4,5)}/L &= \bar{U}_{\Gamma,(2,4,5)}/L_0 \\
&= Spec\,\mathbb{C}[\bar{\lambda}_{21}, \bar{\lambda}_{13}, \bar{\lambda}_{23}]_{2\bar{\lambda}_{13} - \bar{\lambda}_{23} - 3\bar{\lambda}_{21}^3} \\
U_{\Gamma,(2,3,4)}/L &= \bar{U}_{\Gamma,(2,3,4)} = Spec\,\mathbb{C}[\bar{\lambda}_{11}, \bar{\lambda}_{13}, \bar{\lambda}_{15}]
\end{aligned}
$$

Remark: Let $V := U_{\Gamma,(2,4,5)} \cup U_{\Gamma,(3,4,5)}$. Then $V = \{\underline{\lambda} \in U_\Gamma \mid$ orbit dimension at λ is $5\}$ and even $\Gamma(End_R(M_\lambda)) = \{0, 2, 4, 8, 9, \ldots\}$ is constant on V. It can be shown that the geometric quotient V/L does not exist, neither in the algebraic nor in the analytic category.

Conclusion: The space of torsion free R-modules, $R = \mathbb{C}[[t^8, t^9, \ldots]]$ is stratified into four strata, corresponding to the four different values of the Hilbert function of $\bar{R}/End_R(M)$. The quotients of these strata by L are coarse moduli spaces for R-modules (torsion free, rank 1) with value set Γ and Hilbert function the corresponding value. On the union of two of these strata the orbit dimension of L is constant but the quotient does not exist.

References

[BGM] Briançon, J.; Granger, M.; Maisonobe, Ph.: Le nombre de modules du germe de courbe plane $x^a + y^b = 0$. Math. Ann. **279**, 535–551 (1988).

[Ca] Casas, E.: Moduli of algebroid plane curves. Lecture Notes in Math., Vol. 961, 32–83 (1983).

[De] Delorme, C.: Sur les modules des singularités des courbes planes. Bull. Soc. Math. France **106**, 417–446 (1978).

[DG] Demazure, M.; Gabriel, P.: "Groupes Algébriques". Masson & Cie, North Holland, Amsterdam 1970.

[DR] Dixmier, J.; Raynaud, M.: Sur le quotient d'une variété algébrique par un groupe algébrique. Adv. Math. Suppl. Stud. **7A**, 327–344 (1981).

[Eb] Ebey, S.; The classification of singular points of algebraic curves. Trans. Amer. Math. Soc. **118**, 454–471 (1965).

[GP 1] Greuel, G.-M.; Pfister, G.: Geometric quotients of unipotent group actions. To be published in Proc. Lond. Math. Soc. (1993).

[GP 2] Greuel, G.-M.; Pfister, G.: Moduli spaces for torsion free modules on curve singularities I. J. Algebraic Geometry **2** 81–135 (1993).

[GP 3] Greuel, G.M.; Pfister, G.: On moduli spaces of semiquasihomogeneous singularities. Preprint, Kaiserslautern 1993.

[Gr] Greuel, G.-M.: Deformation und Klassifikation von Singularitäten und Moduln. Jber. d. Dt. Math.–Verein., Jubiläumstagung 1990, 177-238, B.G. Teubner (1992).

[Gra] Granger, M.: Sur une espace de modules de germe du courbe plane. Bull. Soc. Math. France 2è sér. **103** (1979).

[LaP] Laudal, O.A.; Pfister, G.: "Local Moduli and Singularities". Lecture Notes in Math., Vol. 1310 (1988).

[LMP] Laudal, O.A.; Martin, B.; Pfister, G. Moduli of plane curve singularities with C^*–action. Banach Center Publ. **40** (1988).

[LuP] Luengo, I.; Pfister, G.: Normal forms and moduli spaces of curve singularities with semigroup $\langle 2p, 2q, 2pq + d\rangle$. Compositio Math. **76**, 247-264 (1990).

[Ma] Martin, B.: Moduli space of binary semi–homogeneous functions over the complex field. Preprint (1990).

[MuF] Mumford, D.; Fogarty, J.: “Geometric Invariant Theory”. (Second, enlarged edition.) Ergeb. Math. Grenzgeb., Bd. 34. Berlin-Heidelberg-New York: Springer 1982.

[Pa] Palamodov, V.P.; Moduli in versal deformations of complex spaces. In: “Variétés Analytiques Complexes”, Nice 1977 (Eds.: Hervier, Y.; Hirschowitz, A.). Lecture Notes in Math., Vol. 683, pp. 74–115. Berlin–Heidelberg–New York: Springer 1978.

[PS] Pfister, G.; Steenbrink, J.H.M.: Reduced Hilbert scheme for irreducible curve singularities. Preprint, Nijmegen 1990.

[Ri] Riemann, B.: Theorie der Abel'schen Funktionen. J. reine angew. Math. **54**, 115-155 (1857).

[Ro] Rosenlicht, M.: A remark on quotient spaces. An. Acad. Brasil. Ciênc. **35**, 487–489 (1963).

[Sch] Schönemann, H.: Normalformen und Modulräume von Kurvensingularitäten mit der Halbgruppe $np, nq, npq + d$. Dissertation, Berlin 1990.

[Za] Zariski, O.: Le problème des modules pour les branches planes (redigè par F. Kmety et M. Merle avec un appendice de B. Teissier). Pub. du Centre de Mathématiques de l'Ecole Polytechnique, Paris 1976.