ON THE MOND'S CLASSIFICATION OF SIMPLE MAP GERMS FROM $\mathbb{C}^2$ TO $\mathbb{C}^3$

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Abstract. Mond gave the classification of simple map germs $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ with respect to $\mathcal{A}$-equivalence. The aim of this article is to characterize Mond's classification in terms of certain invariants. On the basis of this characterization we present an algorithm to compute the type of the simple map germs $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ without computing the normal form and also give its implementation in the computer algebra system Singular.

Key words: Simple Map Germ, $\mathcal{A}$-equivalence, Codimensionn
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1. Introduction

In local singularity theory, classification and recognition of singularities of map germs with respect to some equivalence relation are two fundamental problems. Classification of map germs with respect to equivalence relation, means finding a list of map germs and showing that all map germs are equivalent to a map germ in the list. Recognition means finding some criteria which describe, how to find an equivalent map germ in the list. There are many classification results for singularities of map germs, however there are much less studies for the recognition problem.

Let $\mathbb{C}$ be the field of complex numbers. Two analytic maps $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ belong to the same germ $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ if they coincide in a neighborhood of 0 in $\mathbb{C}^2$. Let $\mathcal{M}$ is the set of all analytic map germs $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$. Let $\mathcal{A} = Aut_{\mathbb{C}}(\mathbb{C}^2, 0) \times Aut_{\mathbb{C}}(\mathbb{C}^3, 0)$. Then we have a canonical action of $\mathcal{A}$ on $\mathcal{M}$ defined by

$$\mathcal{A} \times \mathcal{M} \to \mathcal{M}$$

such that

$$((\varphi, \psi), f) \mapsto \psi \circ f \circ \varphi^{-1}.$$

The map germs $f, g \in \mathcal{M}$ are called $\mathcal{A}$-equivalent ($f \sim_{\mathcal{A}} g$) if they are in the same orbit under the action of $\mathcal{A}$. In the classification of map germs with respect to the action of the group $\mathcal{A}$ the tangent spaces to the orbit under the action of this group and their codimension play an important role (cf. [4]). Given $f \in \mathcal{M}$ the orbit map $\theta_f : \mathcal{A} \to \mathcal{M}$ is defined by $\theta_f(\varphi, \psi) = \psi \circ f \circ \varphi^{-1}$. The corresponding tangent map has as image the tangent space to the orbit at $f = (f_1, f_2, f_3)$:

$$T_{\theta_f, id} =:< x, y >< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} >_{\mathbb{C}[[x,y]]} + < f_1, f_2, f_3 > \mathbb{C}[[f_1, f_2, f_3]]^2.$$

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The $\mathcal{A}$-modality of a map germ f is the smallest integer m such that a sufficiently small neighbourhood of f can be covered by a finite number of m-parameter families of orbits under the action of $\mathcal{A}$ on f. A map germ f is called $\mathcal{A}$-simple if the $\mathcal{A}$-modality of the germ is 0.

The classification and recognition problems for singularities of map germs from the plane into the plane was studied by several authors (see [6], [5], [7], [8], [9], [13], [15], [16]). Rieger classified map germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ of corank atmost one and $\mathcal{A}$-codimension ≤ 6. He also classified $\mathcal{A}$-simple and $\mathcal{A}$-unimodal singularities of map germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$. Moreover Mond [14] gave the classification of $\mathcal{A}$-simple singularities of map germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$. The aim of this article is to study the recognition problem for the Mond's classification of simple map germs.

Let $f \in \mathcal{M}$ be a map germ then the codimension of the tangent space, $\mathrm{codim}(f) = dim_{\mathbb{C}} \frac{<x,y>\mathbb{C}[[x,y]]}{T_{\theta_f, id}}$ and the multiplicity of f, $m(f) = dim_{\mathbb{C}} \frac{\mathbb{C}[[x,y]]}{<I_f>}$, where I_f is the ideal generated by components of f, are the invariants used in the classification. An other important invariant is the number of cross-caps of f, which is define as follows:

Definition 1.1. Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be an analytic map. The singular algebra of f at 0 is defined by $\frac{\mathbb{C}[[x,y]]}{J_f}$, where J_f is the ideal generated by the 2-minors of the Jacobian matrix df of f. Then $c(f) = dim_{\mathbb{C}} \frac{\mathbb{C}[[x,y]]}{J_f}$ is called the number of cross-caps [1].

2. Characterization of simple map germs from $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$

In this section we give the characterization of simple map germs of corank ≤ 1 in terms of certain invariants. Table 1 contains all simple map germs from $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ of corank ≤ 1 of Mond's classification.

Table 1

Type	Normal form	Conditions
S	$(x, y, 0)$	$-$
S_0	(x, y^2, xy)	$-$
S_k	$(x, y^2, y^3 + x^{k+1}y)$	$k \geq 1$
B_k	$(x, y^2, x^2y + y^{2k+1})$	$k \geq 2$
C_k	$(x, y^2, xy^3 + x^k y)$	$k \geq 3$
F_4	$(x, y^2, x^3y + y^5$	$-$
H_k	$(x, xy + y^{3k-1}, y^3)$	$k \geq 2$

Table 2 contains the invariants, we used to characterize the Mond's classification.

Lemma 2.1. *Let $f(x, y)$ be a map germ from $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$.*

(1) *The corank of f is zero if and only if $m(f) = 1$. In this case $f \sim_{\mathcal{A}} (x, y, 0)$.*
(2) *If the corank of f is 1 then $f \sim_{\mathcal{A}} (x, p(x, y), q(x, y))$ with $p(x, 0) = 0 = q(x, 0)$.*

Proof. This is obvious. $\qquad \square$

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[1] Under a generic perturbation the singular point of f at 0 splits into $c(f)$ cross-caps.

Table 2

Type	codim(f)	$c(f)$	$m(f)$
S	0	0	1
S_0	2	1	2
S_k	$k+2$	$k+1$	2
B_k	$k+2$	2	2
C_k	$k+2$	k	2
F_4	6	3	2
H_k	$k+2$	2	3

Let $f(x,y)$ be a map germ from $(\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$ of corank ≤ 1 then one can use Algorithm 1 to find a map germ of the form $(x,p(x,y),q(x,y))$ such that $f \sim_{\mathcal{A}} (x,p(x,y),q(x,y))$.

Algorithm 1 Normal Form of a Map (`norMap`)

Input: A map germ $f(x,y) = (f_1,f_2,f_3)$ from $(\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$ of corank ≤ 1.
Output: A map germ of the form $(x,p(x,y),q(x,y))$ such that
$\quad f \sim_{\mathcal{A}} (x,p(x,y),q(x,y))$.
1: Permute the generators of f such that $\mathrm{ord}(f_1) = 1$.
2: Compute $d =$determinacy(f).
3: Compute a map φ such that $\varphi(f_1) = x$ mod $< x,y >^{d+1}$.
4: Define $g(x,y) = (g_1,g_2,g_3) =$jet$((x,\varphi(f_2),\varphi(f_3)),d)$.
5: **return** (g).

2.1. **Singular Example.**

`ring R=0,(x,y),ds;`

In the following example we have as an input the map $f(x,y) = (f_1,f_2,f_3)$, where

$$f_1 = x + 2y + x^2 y + 3xy^2 + x^2 y^2, f_2 = x^2 + xy + 3xy^2 + xy^3 + 5y^4,$$
$$f_3 = x^4 - 3x^3 y + x^5 - 15x^4 y + 90x^3 y^2 - 270x^2 y^3 + 405xy^4 - 243y^5.$$

which in SINGULAR is written as:

```
> poly f1=x+2y+x2y+3xy2+x2y2;
> poly f2=x2+xy+3xy2+xy3+5y4;
> poly f3=x4-3x3y+x5-15x4y+90x3y2-270x2y3+405xy4-243y5;
> ideal f=f1,f2,f3;
```

If you want to find the 5-jet of a map germ of the form $(x,p(x,y),q(x,y))$ $\mathcal{A}$-equivalent to f, you can use the following procedure:

```
> norMap(f,5);
```

we obtain the following output:

```
_[1]=x
_[2]=x2-3xy+2y2+3xy2-6y3-2x3y+5x2y2+2xy3-3y4-2x3y2+8x2y3-17xy4+18y5
_[3]=x4-11x3y+42x2y2-68xy3+40y4+x5-25x4y+250x3y2-1250x2y3+3125xy4-3125y5
```

i.e.

$$p(x,y) = x^2 - 3xy + 2y^2 + 3xy^2 - 6y^3 - 2x^3 y + 5x^2 y^2 + 2xy^3 - 3y^4 - 2x^3 y^2$$

$$+8x^2y^3 - 17xy^4 + 18y^5,$$
$$q(x,y) = x^4 - 11x^3y + 42x^2y^2 - 68xy^3 + 40y^4 + x^5 - 25x^4y + 250x^3y^2$$
$$-1250x^2y^3 + 3125xy^4 - 3125y^5.$$

Now in the following sequence of propositions, we give a characterization of Mond's classification of simple map germs in terms of the codimension of tangent space, the multiplicity and the number of cross-caps of the map germ f.

Proposition 2.2. *Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be an analytic map of corank 1.*

  (1) *If f is simple then $m(f) = 2$ or 3 and $codim(f) < \infty$.*
  (2) *If $m(f) = 2$ then $f \sim_{\mathcal{A}} (x, y^2, yp(x, y^2))$. Moreover if $c(f) = 2$ then $f \sim_{\mathcal{A}}$ (x, y^2, xy).*
  (3) *If $m(f) = 3$ then f is simple if and only if $codim(f) = k < \infty$ and $c(f) = 2$. In this case $f \sim_{\mathcal{A}} (x, xy + y^{3k-7}, y^3)$.*

Proof. (1) is a consequence of Theorem $1 : 1$ in [14]. To prove (2) we use Theorem 4.2 of [14] to obtain that $f \sim_{\mathcal{A}} g$ with 2-jet of g in $\{(x, y^2, xy), (x, y^2, 0), (x, xy, 0), (x, 0, 0)\}$. If the 2-jet is (x, y^2, xy) or $(x, y^2, 0)$ then $m(f) = 2$ and if the 2-jet is $(x, xy, 0)$ or $(x, 0, 0)$ then $m(f) \geq 3$. We may assume that the 2-jet of f is (x, y^2, xy) or $(x, y^2, 0)$. Proposition $4.1 : 1$ of [14] implies that $f \sim_{\mathcal{A}} (x, y^2, yp(x, y^2))$. If $c(f) = 2$ then the 2-jet of f is (x, y^2, xy). Theorem $4 : 3$ of [14] implies that $f \sim_{\mathcal{A}} (x, y^2, xy)$.
To prove (3) assume that $m(f) = 3$ and f is simple then it is a consequence of Theorem $1 : 1$ of [14] that $codim(f) < \infty$ and $c(f) = 2$. Now assume that $codim(f) = k < \infty$ and $c(f) = 2$ then 2-jet of f is $(x, xy, 0)$ or $(x, 0, 0)$. The second case implies that $c(f) \geq 4$. This implies that 2-jet of f is $(x, xy, 0)$. Theorem $4.1 : 2$ of [14] implies that $c(f) = 2$ is only possible if the 3-jet of f is equivalent to (x, xy, y^3). Then Theorem $4.2.1 : 2$ of [14] implies that $f \sim_{\mathcal{A}} (x, xy + y^{3k-7}, y^3)$. $\square$

The $\mathcal{A}$-classification of map germs whose 2-jet is equivalent to $(x, y^2, 0)$ reduces to a classification of germs of functions $(x, y^2, yp(x, y^2))$.

Definition 2.3. Let $f, g \in \mathbb{C}[[x, y^2]]$. $f \sim_{\mathcal{K}^T} g$ if there exist a unit $u \in \mathbb{C}[[x, y^2]]$ and an automorphism $\varphi : \mathbb{C}[[x, y]] \to \mathbb{C}[[x, y]]$ such that

  (1) $\varphi(x) \in \mathbb{C}[[x, y^2]]$ and $\varphi(y) \in y\mathbb{C}[[x, y^2]]$,
  (2) $u.f = \varphi(g)$.

The following proposition is a consequence of Proposition $4.1 : 1$ from [14].

Proposition 2.4. *Let $f_i := (x, y^2, yp_i(x, y^2))$, $i = 1, 2$ be two map germs. Then $f_1 \sim_{\mathcal{A}} f_2$ if and only if $p_1(x, y^2) \sim_{\mathcal{K}^T} p_2(x, y^2)$. Moreover f_i is $\mathcal{A}$-simple if and only if $p_i(x, y^2)$ is $\mathcal{K}^T$-simple.*

The $\mathcal{K}^T$-simple germs are classified by the following proposition.

Proposition 2.5. *Every $\mathcal{K}^T$-simple germ in $\mathbb{C}[[x, y^2]]$ is equivalent to a germ given in Table 3.*

Proof. For a proof see Theorem $4.1 : 19$ in [14]. $\square$

Finally we need a characterization of the simple $\mathcal{A}$-germs by using suitable invariants.

Proposition 2.6. *Let $f = (x, y^2, yp(x, y^2))$, $codim(f) = k < \infty$, $c = c(f)$ and μ the Milnor number of p.*

Table 3

Type	Normal form	Codimension
S_k	$x^{k+1} + y^2, (k \geq 1)$	k
T_k	$x^2 + y^{2k}, (k \geq 2)$	k
D_k	$x^k + xy^2, (k \geq 3)$	k
E_6	$x^3 + y^4$	4

(1) *If* $\mathrm{ord}(p) = 2$ *and* $\mu = k - 2$ *then* $f \sim_{\mathcal{A}} (x, y^2, y^3 + x^{k-1}y)$.
(2) *If* $\mathrm{ord}(p) = 2$ *and* $\mu = 2k - 5$ *then* $f \sim_{\mathcal{A}} (x, y^2, x^2y + y^{2k-3})$.
(3) *If* $\mathrm{ord}(p) = 3$, $\mu = 6$ *and* $k = c + 3$ *then* $f \sim_{\mathcal{A}} (x, y^2, x^3y + y^5)$.
(4) *If* $\mathrm{ord}(p) = 3$ *and* $k = c + 2$ *then* $f \sim_{\mathcal{A}} (x, y^2, xy^3 + x^{k-2}y)$.

f is simple if and only if one of the cases (1)-(4) *hold.*

Proof. Theorem $1:1$ from [14] implies that f is simple if one of the cases (1)-(4) hold. To prove the other direction assume first that $\mathrm{ord}(p) = 2$. Similarly to the classification of Arnold this implies that $p \sim_{\mathcal{K}^T} y^2 + x^l$ or $x^2 + y^{2s}$ for suitable l respectively s. But $\mathrm{codim}(x, y^2, y^3 + x^l y) = l + 1$ and $\mathrm{codim}(x, y^2, x^2y + y^{2s+1}) = s + 2$. This implies that $k = l + 1$ respectively $k = s + 2$. On the other hand $\mu(y^2 + x^l) = l - 1$ and $\mu(x^2 + y^{2s}) = 2s - 1$. This implies that we have the cases (1) or (2) and f is simple. Now assume that $\mathrm{ord}(p) = 3$. If $\mu = 6$ then $p \sim_{\mathcal{K}^T} x^3 + y^4$ or $xy^2 + x^5$. In the first case we have $c = 3$ and in the second case $c = 5$. In the first case we have $k = 6$ and in the second case $k = 7$. This implies that $f \sim_{\mathcal{A}} (x, y^2, x^3y + y^5)$ if $k = c + 3 = 6$ and $f \sim_{\mathcal{A}} (x, y^2, xy^3 + x^5y)$ if $k = c + 2 = 7$. Now $c = c(f) = \dim_{\mathbb{C}} \frac{\mathbb{C}[[x,y]]}{<y, \frac{\partial(yp)}{\partial y}>} = \dim_{\mathbb{C}} \frac{\mathbb{C}[[x]]}{<p(x,0)>}$. If $\mathrm{jet}(p, 3) = x^3$ then $c = 3$. We may assume that $\mathrm{jet}(p, 3) = xy^2$ then $p \sim_{\mathcal{K}^T} xy^2 + x^c$. This implies that $f \sim_{\mathcal{A}} (x, y^2, xy^3 + x^{k-2}y)$. $\qquad\square$

Now we are ready to give the algorithm to classify the simple map germs.

Algorithm 2 Simple Map Germs (`class`)

Input: A germ $f(x,y) = (f_1(x,y), f_2(x,y), f_3(x,y))$ from $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ of corank ≤ 1 .

Output: $(x, g(x,y), h(x,y))$, the normal form over $\mathbb{C}$ or 0 if f is not simple.

1: Compute $\mathrm{codim}(f) = k$, the codimension of f, d, the determinacy of f, m, the multiplicity of f and c, the number of cross-caps of f;
2: **if** $k = \infty$ **then**
3:     **return** (0);
4: **if** $\mathrm{ord}(f_i) > 1$ for all i **then**
5:     **return** (0);
6: **if** $m \neq 1, 2, 3$ **then**
7:     **return** (0);
8: **if** $m = 1$ **then**
9:     **return** $(x, y, 0)$.
10: **if** $c = 1$ **then**
11:     **return** (x, y^2, xy);
12: **if** $c = 2$ and $m = 3$ **then**
13:     **return** $(x, xy + y^{3k-7}, y^3)$;
14: By using Algorithm 1 transform f into a germ $\mathcal{A}$-equivalent to $(x, g_2(x,y), g_3(x,y))$.
15: Remove the pure powers of x from g_2, g_3.
16: **if** $\mathrm{jet}((g_2, g_3), 2) = (0, 0)$ **then**
17:     **return** (0);
18: Permute g_2 and g_3 such that $\mathrm{jet}(g_2, 2) \neq 0$.
19: Define: $g_3 = g_3 - \alpha g_2$, where α is a constant such that $\mathrm{jet}(g_3, 2) = 0$;
20: Choose a map φ such that $\varphi(\mathrm{jet}(g_2, 2)) = y^2$ and $\varphi(x) = x$.
21: Define $(x, g_2, g_3) = (x, \varphi(g_2), \varphi(g_3))$.
22: Remove the pure powers of x from g_2, g_3.
23: Compute a map φ such that $\varphi(x) = x$, $\varphi(g_2) = y^2 \mod < x, y >^{d+1}$.
24: Define $(x, g_2, g_3) = (x, y^2, \mathrm{jet}(\varphi(g_3), d))$.
25: Kill in g_3 all terms of the type $x^i y^{2j}$.
26: Define: $p = f_3/y$.
27: Compute μ the Milnor number of p.
28: **if** $\mathrm{ord}(p) = 2$ and $\mu = k - 2$ **then**
29:     **return** $(x, y^2, y^3 + x^{k-1}y)$;
30: **if** $\mathrm{ord}(p) = 2$ and $\mu = 2k - 5$ **then**
31:     **return** $(x, y^2, x^2y + y^{2k-3})$;
32: **if** $\mathrm{ord}(p) = 3$, $\mu = 6$ and $k = c + 3$ **then**
33:     **return** $(x, y^2, x^3y + y^5)$;
34: **if** $\mathrm{ord}(p) = 3$ and $k = c + 2$ **then**
35:     **return** $(x, y^2, xy^3 + x^{k-2}y)$;
36: **return** (0).

---

3. Singular Examples

We have implemented the Algorithm in the computer algebra system Singular [11]. Code can be downloaded from https://www.mathcity.org/files/ahsan/Mond-Procedure.txt. We give some examples.

```
ring R=0,(x,y),ds;
```

In the first example we have as an input the map $f(x,y) = (f_1, f_2, f_3)$, where

$f_1 = 9x^2+6xy+y^2+6x^2y^2+2xy^3+x^2y^4$, $f_2 = x+2y+9x^2+13xy+y^2+6x^2y^2+2xy^3+x^2y^4$,

$f_3 = 27x^3+27x^2y+9xy^2+y^3+27x^3y^2+18x^2y^3+3xy^4+9x^3y^4+3x^2y^5+3x^8+43x^7y+266x^6y^2$
$+924x^5y^3+1960x^4y^4+2576x^3y^5+2016x^2y^6+832xy^7+128y^8+147x^8y+1813x^7y^2+9408x^6y^3$
$+26460x^5y^4+43120x^4y^5+39985x^3y^6+18816x^2y^7+3136xy^8+3088x^8y^2+31913x^7y^3$
$+133854x^6y^4+288400x^5y^5+329840x^4y^6+181776x^3y^7+33376x^2y^8+128xy^9+36064x^8y^3$
$+300713x^7y^4+963340x^6y^5+1448440x^5y^6+972160x^4y^7+201488x^3y^8+3136x^2y^9+253134x^8y^4$
$+1606955x^7y^5+3570630x^6y^6+3107580x^5y^7+754600x^4y^8+32928x^3y^9+1070846x^8y^5$
$+4684351x^7y^6+5935272x^6y^7+1795948x^5y^8+192080x^4y^9+2554664x^8y^6+6269011x^7y^7$
$+2655506x^6y^8+672280x^5y^9+2823576x^8y^7+2235331x^7y^8+1411788x^6y^9+823543x^8y^8$
$$+1647086x^7y^9 + 823543x^8y^9.$$

In Singular this can be written as:

```
ideal I=9x2+6xy+y2+6x2y2+2xy3+x2y4,x+2y+9x2+13xy+y2+6x2y2+2xy3+x2y4,
        27x3+27x2y+9xy2+y3+27x3y2+18x2y3+3xy4+9x3y4+3x2y5+3x8+43x7y
        +266x6y2+924x5y3+1960x4y4+2576x3y5+2016x2y6+832xy7+128y8
        +147x8y+1813x7y2+9408x6y3+26460x5y4+43120x4y5+39985x3y6
        +18816x2y7+3136xy8+3088x8y2+31913x7y3+133854x6y4+288400x5y5
        +329840x4y6+181776x3y7+33376x2y8+128xy9+36064x8y3+300713x7y4
        +963340x6y5+1448440x5y6+972160x4y7+201488x3y8+3136x2y9+253134x8y4
        +1606955x7y5+3570630x6y6+3107580x5y7+754600x4y8+32928x3y9
        +1070846x8y5+4684351x7y6+5935272x6y7+1795948x5y8+192080x4y9
        +2554664x8y6+6269011x7y7+2655506x6y8+672280x5y9+2823576x8y7
        +2235331x7y8+1411788x6y9+823543x8y8+1647086x7y9+823543x8y9
```

To compute the required normal form we use the procedure:

```
> class(I);
```

and we obtain the output:

```
_[1]=x
_[2]=y2
_[3]=y3+x7y
```

i.e.
$$f(x,y) \sim_{\mathcal{A}} (x, y^2, y^3 + x^7y).$$

In the second example we have as an input the map $f(x,y) = (f_1, f_2, f_3)$, where

$f_1 = 9x^2+6xy+y^2+6x^2y^2+2xy^3+x^2y^4$, $f_2 = x+2y+9x^2+13xy+y^2+6x^2y^2+2xy^3+x^2y^4$,

$f_3 = 3x^3+13x^2y+16xy^2+4y^3+42x^3y+98x^2y^2+28xy^3+243x^5+405x^4y+418x^3y^2$
$+143x^2y^3+19xy^4+y^5+14x^3y^3+28x^2y^4+405x^5y^2+540x^4y^3+319x^3y^4+60x^2y^5$
$+5xy^6+270x^5y^4+270x^4y^5+90x^3y^6+10x^2y^7+90x^5y^6+60x^4y^7+10x^3y^8+15x^5y^8$
$$+5x^4y^9 + x^5y^{10}.$$

In Singular this can be written as:

```
ideal J=9x2+6xy+y2+6x2y2+2xy3+x2y4,x+2y+9x2+13xy+y2+6x2y2+2xy3+x2y4,
        3x3+13x2y+16xy2+4y3+42x3y+98x2y2+28xy3+243x5+405x4y+418x3y2
        +143x2y3+19xy4+y5+14x3y3+28x2y4+405x5y2+540x4y3+319x3y4+60x2y5
        +5xy6+270x5y4+270x4y5+90x3y6+10x2y7+90x5y6+60x4y7+10x3y8+15x5y8
        +5x4y9+x5y10
```

To compute the required normal form we use the procedure:

```
> class(J);
```

and we obtain the output:

```
_[1]=x
_[2]=y2
_[3]=x2y+y5
```

i.e.

$$f(x,y) \sim_{\mathcal{A}} (x, y^2, x^2y + y^5).$$

In the third example we have as an input the map $f(x,y) = (f_1, f_2, f_3)$, where

$$f_1 = 3x^2 + 7xy + 2y^2 + 21x^2y + 7xy^2 + x^2y^2 + 2xy^3 + 7x^2y^3 + 6561x^8 + 17496x^7y$$
$$+20412x^6y^2 + 13608x^5y^3 + 5670x^4y^4 + 1512x^3y^5 + 252x^2y^6 + 24xy^7 + y^8$$
$$+17496x^8y^2 + 40824x^7y^3 + 40824x^6y^4 + 22680x^5y^5 + 7560x^4y^6 + 1512x^3y^7$$
$$+168x^2y^8 + 8xy^9 + 20412x^8y^4 + 40824x^7y^5 + 34020x^6y^6 + 15120x^5y^7$$
$$+3780x^4y^8 + 504x^3y^9 + 28x^2y^10 + 13608x^8y^6 + 22680x^7y^7 + 15120x^6y^8$$
$$+5040x^5y^9 + 840x^4y^{10} + 56x^3y^{11} + 5670x^8y^8 + 7560x^7y^9 + 3780x^6y^{10}$$
$$+840x^5y^{11} + 70x^4y^{12} + 1512x^8y^{10} + 1512x^7y^{11} + 504x^6y^{12} + 56x^5y^{13}$$
$$+252x^8y^{12} + 168x^7y^{13} + 28x^6y^{14} + 24x^8y^{14} + 8x^7y^{15} + x^8y^{16},$$

$$f_2 = x+2y+3x^2+14xy+2y^2+21x^2y+7xy^2+x^2y^2+2xy^3+7x^2y^3+6561x^8+17496x^7y$$
$$+20412x^6y^2+13608x^5y^3+5670x^4y^4+1512x^3y^5+252x^2y^6+24xy^7+y^8+17496x^8y^2$$
$$+40824x^7y^3+40824x^6y^4+22680x^5y^5+7560x^4y^6+1512x^3y^7+168x^2y^8+8xy^9$$
$$+20412x^8y^4+40824x^7y^5+34020x^6y^6+15120x^5y^7+3780x^4y^8+504x^3y^9+28x^2y^{10}$$
$$+13608x^8y^6+22680x^7y^7+15120x^6y^8+5040x^5y^9+840x^4y^{10}+56x^3y^{11}+5670x^8y^8$$
$$+7560x^7y^9+3780x^6y^{10}+840x^5y^{11}+70x^4y^{12}+1512x^8y^{10}+1512x^7y^{11}$$
$$+504x^6y^{12}+56x^5y^{13}+252x^8y^{12}+168x^7y^{13}+28x^6y^{14}$$
$$+24x^8y^{14}+8x^7y^{15}+x^8y^{16},$$

$$f_3 = 27x^3 + 27x^2y + 9xy^2 + y^3 + 27x^3y^2 + 18x^2y^3 + 3xy^4 + 9x^3y^4 + 3x^2y^5 + x^3y^6.$$

In SINGULAR this can be written as:

```
ideal K=3x2+7xy+2y2+21x2y+7xy2+x2y2+2xy3+7x2y3+6561x8+17496x7y
        +20412x6y2+13608x5y3+5670x4y4+1512x3y5+252x2y6+24xy7+y8
        +17496x8y2+40824x7y3+40824x6y4+22680x5y5+7560x4y6+1512x3y7
        +168x2y8+8xy9+20412x8y4+40824x7y5+34020x6y6+15120x5y7+3780x4y8
        +504x3y9+28x2y10+13608x8y6+22680x7y7+15120x6y8+5040x5y9+840x4y10
        +56x3y11+5670x8y8+7560x7y9+3780x6y10+840x5y11+70x4y12+1512x8y10
        +1512x7y11+504x6y12+56x5y13+252x8y12+168x7y13+28x6y14+24x8y14
        +8x7y15+x8y16,x+2y+3x2+14xy+2y2+21x2y+7xy2+x2y2+2xy3+7x2y3
        +6561x8+17496x7y+20412x6y2+13608x5y3+5670x4y4+1512x3y5+252x2y6
        +24xy7+y8+17496x8y2+40824x7y3+40824x6y4+22680x5y5+7560x4y6
        +1512x3y7+168x2y8+8xy9+20412x8y4+40824x7y5+34020x6y6+15120x5y7
        +3780x4y8+504x3y9+28x2y10+13608x8y6+22680x7y7+15120x6y8+5040x5y9
```

```
        +840x4y10+56x3y11+5670x8y8+7560x7y9+3780x6y10+840x5y11+70x4y12
        +1512x8y10+1512x7y11+504x6y12+56x5y13+252x8y12+168x7y13+28x6y14
        +24x8y14+8x7y15+x8y16,27x3+27x2y+9xy2+y3+27x3y2+18x2y3+3xy4
        +9x3y4+3x2y5+x3y6
```

To compute the required normal form we use the procedure:

```
> class(K);
```

and we obtain the output:

```
_[1]=x
_[2]=xy+y8
_[3]=y3
```

i.e.

$$f(x,y) \sim_{\mathcal{A}} (x, xy + y^8, y^3).$$

Acknowledgments

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