A Family of Cohen–Macaulay Modules over Singularities of Type $\mathbf{X^t + Y^3}$

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Abstract

Let K be a field with $\mathrm{char}K \neq 3$ and i, t two positive integers such that $1 \leq i < t/2$, $t \neq 3i$. The classification problem for maximal Cohen–Macaulay modules over $K[[X,Y]]/(X^t + Y^3)$ is complicated if $t \geq 6$, because there exist parameter families of non–isomorphic maximal Cohen–Macaulay modules (see [Sc], or [GK], [Yo, Ch. 9] and [DG]). Here we describe parameter families of such modules N, such that N/YN is a direct sum of copies of $K[[X]]/(X^i)$, $K[[X]]/(X^{t-i})$.

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Introduction

An important theorem from the general course of algebra states that each abelian finitely generated group is a direct sum of groups of type $\mathbb{Z}$, $\mathbb{Z}/p^n\mathbb{Z}$, $p, n \in \mathbb{N}$, p being a prime number. In general, given even an Artinian local ring A, it is not easy to describe the finitely generated A–modules. If A is a complete hypersurface ring of type $A = K[[X]]/(f)$, $X = (X_1, \ldots, X_n)$, $0 \neq f \in (X)$, K being a field, it could be sufficient to describe only the maximal Cohen–Macaulay (in short MCM) A–modules, because modulo a part of the free resolution, every finitely generated A–module can be identified with a MCM one.

Let K be a field, with char $K \neq 3$, $R = K[[X]]/(X^t)$ $t \geq 2$, $R' = K[[X, Y]]/(X^t + Y^3)$, $\widetilde{R} = R[Y]/(Y^2) \cong R'/(Y^2)$.

The following theorem extends a well–known result of Knörrer [Kn] and it is a particular case of [Po], [HP, 2.8].

Theorem 0.1. *Every indecomposable MCM R–module N is a direct summand of the first syzygy $\Omega^1_{R'}(N/Y^2N)$ of N/Y^2N over R'.*

Note that the image of the functor $\widetilde{R} \otimes_{R'} - : \mathrm{MCM}(R') \longrightarrow \mathrm{Mod}\,\widetilde{R}$, $N \longrightarrow N/Y^2N$ is enclosed in the category $\mathcal{L}$ of infinitesimal deformations of MCM R–modules, which are liftable to $R'/(Y^4)$ (see [ADS], [HP], or here, Section 1). Using the above theorem, we see that $\widetilde{R} \otimes_{R'} -$ defines an an "almost" bijection between the set of isomorphism classes of indecomposable MCM R'–modules and the set of isomorphism classes of indecomposable infinitesimal deformations of finitely generated R–modules, which are liftable to $R'/(Y^4)$ (see [HP, 3.1] and [EPP, 1.6]). "Almost" injection means here that if $N/Y^2N \cong N'/Y^2N'$ then either $N \cong N'$ or $N \cong \Omega^1_{R'}(N')$ and "almost" surjection means that given an indecomposable infinitesimal deformation T of a certain finitely generated R–module, which is liftable to $R'/(Y^4)$, then there exists an indecomposable MCM R'–module N, such that either $N/Y^2N \cong T$, or $N/Y^2N \cong T \oplus \Omega^1_{\widetilde{R}}(T)$. Thus the description of $\mathrm{MCM}(R')$ is almost complete if we are able to describe the infinitesimal deformations of finitely generated R–modules, which are liftable to $R'/(Y^4)$.

A MCM R'–module N is called isolated, if there exists just a finite set of non–isomorphic MCM R'–modules N', such that $N'/YN' \cong N/YN$. If $t \leq 5$, $K = \mathbb{C}$, then the singularity $X^t + Y^3$ is simple and every MCM R'–module is isolated (see [Kn], [BGS]). If $t \geq 6$, then R' is not of finite Cohen–Macaulay representation type and by the first Brauer–Thrall Conjecture (see [Yo, Ch. 6]) there exist non–isolated MCM R'–modules. using the above procedure, one can usually describe the indecomposable isolated MCM R'–modules (see [HP, 4.2], [EPP, 3.1], [MPP, 5.2, 5.4], [En, 2.1]).

However, the description of non–isolated MCM R'–modules is more complicated, and even when we are able to do it, it is almost impossible to decide which are the indecomposable ones.

Theorem 0.2 ([MPP, 5.3]). *Let $d', d'' \geq 0$ be two integers, $P_i = k[[X]]/(X^i)$ and suppose $t > 6$ and K algebraically closed, char $K \neq 3$. Then*

1) there exists a MCM R'–module N such that $N/YN \cong P_2^{d'} \oplus P_{t-2}^{d''}$ if and only if $d' = d''$,

2) every MCM R'–module N such that $N/YN \cong (P_2 \oplus P_{t-2})^{d'}$ is a direct sum of copies of the ideals (X^2, Y), (X^{t-2}, Y) and of the modules $N_{\lambda r}$ given by $(\eta_{\lambda r}, \nu_{\lambda r})$, $\lambda \in K^*$, $r \in \mathbb{N}$, where

$$\begin{aligned}
\eta_{\lambda r} &= \left(\begin{array}{c|c} X^2 I_r + Y A_{\lambda r} & XY I_r \\ \hline XY A_{\lambda r}^{-1} - X^{t-5} Y A_{\lambda r}^2 & X^{t-2} I_r - X^{t-4} Y A_{\lambda r} + Y^2 A_{\lambda r}^{-1} \end{array}\right), \\
\nu_{\lambda r} &= \left(\begin{array}{c|c} X^{t-2} I_r - X^{t-4} Y A_{\lambda r} + Y^2 A_{\lambda r}^{-1} & -XY I_r \\ \hline -XY A_{\lambda r}^{-1} + X^{t-5} Y A_{\lambda r}^2 & X^2 I_r + Y A_{\lambda r} \end{array}\right),
\end{aligned}$$

I_r being the unit $r \times r$–matrix and $A_{\lambda r}$ being an $r \times r$ Jordan cell,

$$A_{\lambda r} = \begin{pmatrix} \lambda & 0 & \ldots\ldots & & 0 \\ 1 & \ddots & & & \\ 0 & \ddots & \ddots & & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & \ldots & 0 & 1 & \lambda \end{pmatrix}$$

Note that $(N_{\lambda_r})_{\lambda \in K^*}$ is a family of non–isomorphic MCM R'–modules such that $N_{\lambda r}/YN_{\lambda r} \cong (P_2 \oplus P_{t-2})^{r'}$, $N_{\lambda r} \cong \Omega^1_{R'}(N_{\lambda r})$ and $\operatorname{rank}_{R'} N_{\lambda r} = r$. In particular $N_{\lambda r}$ is non–isolated.

The purpose of this paper is to extend Theorem 0.2, describing the non–isolated MCM R'–modules N such that $N/YN \cong P_i^{d'} \oplus P_{t-i}^{d''}$ for $i > 2$, $t \neq 3i$ (if $i = 1$) there exist only isolated MCM modules (see [MPP, 5.2]). Our Theorem 2.1 reduces the description of those MCM R'–modules to a matrix problem. Roughly speaking, it states that the set of isomorphism classes of those modules N for which (X^i, Y), (X^{t-i}, Y) are not direct summands, has a stratification, each stratum being in bijection with the set of conjugacy classes of an algebraic group of matrices.

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1 Infinitesimal deformations

Let M be a finitely generated R–module, which is a direct sum of $(P_i)_{1\le i\le t}$. An $\widetilde{R}$–module T is an infinitesimal deformation of M if $T/YT \cong M$ and the complex $T \xrightarrow{Y} T \xrightarrow{Y} T$ is exact (the last condition states that $\mathrm{Tor}_j^{\widetilde{R}}(R,T)=0, j\ge 2$), that is, T is a lifting of M to $\widetilde{R}$ according to [ADS]. T is liftable to $R'/(Y^4)$ if there exists an $R'/(Y^4)$–module E such that $E/Y^2E \cong T$ and the complex $E \xrightarrow{Y} E \xrightarrow{Y^3} E \xrightarrow{Y} E$ is exact.

From now on M is a direct sum of d' copies of P_i and d'' copies of P_{t-i}, $1\le i<t/2$.

A matrix factorization (φ,ψ) of M is given by

$$\varphi=\left(\begin{array}{c|c} X^iI_{d'} & 0 \\ \hline 0 & X^{t-i}I_{d''}\end{array}\right),\quad \psi=\left(\begin{array}{c|c} X^{t-i}I_{d'} & 0 \\ \hline 0 & X^{i}I_{d''}\end{array}\right)$$

and an infinitesimal deformation T of M to $\widetilde{R}$ is defined by a matrix factorization over $K[[X,Y]]/(Y^2)$ of the form $(\varphi+Y\alpha,\ \psi-Y\beta)$, where $\alpha,\beta\in S_d:=M(d\times d,\ K[[X]])$, $d=d'+d''$ satisfy $\alpha\psi=\varphi\beta$ (see [Ei], [PP], [HP], Section 3). By [HP, 3.4], T is liftable to $R'/(Y^4)$ if and only if there exist $\gamma,\gamma',\delta,\delta'\in S_d$ such that

$$(*)\qquad \begin{cases}\alpha\beta &= \gamma\psi+\varphi\delta\\ I_d &= -\gamma\beta+\gamma'\psi+\alpha\delta+\varphi\delta'.\end{cases}$$

In particular, α,β have the form

$$\alpha=\left(\begin{array}{c|c} \alpha_{ii} & \alpha_{i,t-i} \\ \hline \alpha_{t-i,i} & X^{t-2i}\beta_{t-i,t-i}\end{array}\right),\quad \beta=\left(\begin{array}{c|c} X^{t-2i}\alpha_{ii} & \alpha_{i,t-i} \\ \hline \alpha_{t-i,i} & \beta_{t-i,t-i}\end{array}\right)$$

for a $d'\times d'$ block α_{ii}, a $d'\times d''$ block $\alpha_{i,t-i}$, a $d''\times d'$ block $\alpha_{t-i,i}$, a $d''\times d''$ block $\beta_{t-i,t-i}$ (see [MPP, 1.2]) and satisfy

$$(1)\qquad X^{t-2i}\alpha_{ii}^2+\alpha_{i,t-i}\alpha_{t-i,i}\equiv 0 \mod X^i;$$

$$(2)\qquad X^{t-2i}\beta_{t-i,t-i}+\alpha_{t-i,i}\alpha_{i,t-i}\equiv 0 \mod X^i;$$

$$(3)\qquad \alpha_{ii}\alpha_{i,t-i}+\alpha_{i,t-i}\beta_{t-i,t-i}\equiv 0 \mod X^i;$$

$$(4)\qquad \alpha_{t-i,i}\alpha_{ii}+\beta_{t-i,t-i}\alpha_{t-i,i}\equiv 0 \mod X^i;$$

$$(5)\qquad X^iI_{d'}\equiv \alpha_{ii}(X^{t-2i}\alpha_{ii}^2+\alpha_{i,t-i}\alpha_{t-i,i}) \mod X^{i+1};$$

$$(6)\qquad X^iI_{d''}\equiv -(X^{t-2i}\beta_{t-i,t-i}^2+\alpha_{t-i,i}\alpha_{i,t-i})\beta_{t-i,t-i} \mod X^{i+1}$$

by [MPP, 4.2].

Lemma 1.1. *Suppose $t = 2i + 2$, $i \geq 3$. Then*

1) *there exists an infinitesimal deformation of M liftable to $R'/(Y^4)$ if and only if $d' = d''$,*

2) *every infinitesimal deformation of $P_i^{d'} \oplus P_i^{d'}$ liftable to $R'/(Y^4)$ is a direct sum of d' copies of the infinitesimal deformations defined by*

$$(w, \rho) = \left(\begin{pmatrix} X^i & Y \\ 0 & X^{t-i} \end{pmatrix}, \quad \begin{pmatrix} X^{t-i} & -Y \\ 0 & X^i \end{pmatrix} \right).$$

Proof. Using [MPP, 1.6], we may split from T more e copies of the deformation given by (w, ρ) and we may reduce to the case where $\alpha_{i,t-i} \equiv \alpha_{t-i,i} \equiv 0 \mod X$. By (1) we obtain

$$\alpha_{ii}^2 \equiv 0 \mod X^2.$$

But (1) and (5) show that α_{ii} is invertible if $d' - e > 0$ (contradiction). Thus, $d' = e$. Similarly by (2) and (6) it follows $d'' = e$. □

Let $r = \min\{i, t - 2i\} - 1$ and $u_1, \ldots, u_r \geq 0$ be some integers such that $\sum_{e=1}^{r} u_e = d'$. Put $H = \begin{pmatrix} I_{u_1} & & 0 \\ & \ddots & \\ 0 & & X^{h-1} I_{u_r} \end{pmatrix}$, $H' = \begin{pmatrix} X^{r-1} & & 0 \\ & \ddots & \\ 0 & & I_{u_r} \end{pmatrix}$ and $S := M(d' \times d',\ K[[X]])$. We have $HH' = X^{r-1} I_{d'}$.

Proposition 1.2. *Suppose that $t \neq 3i,\ 2i + 1,\ i \geq 2$. Then*

1) *there exists an infinitesimal deformation of $P_i^{d'} \oplus P_{t-i}^{d''}$, liftable to $R'/(Y^4)$ if and only if $d' = d''$,*

2) *every infinitesimal deformation of $P_i^{d'} \oplus P_{t-i}^{d'}$ liftable to $R'/(Y^4)$ is a direct sum of copies of the infinitesimal deformations of type (w, ρ) (see Lemma 1.1 and (η_{HA}, ν_{HA})*

$$\begin{aligned} \eta_{HA} &= \left(\begin{array}{c|c} X^i I_{d'} + YA & XYH \\ \hline X^{i-r} Y H' A^{-1} - X^{t-2i-r} Y H' A^2 & X^{t-i} I_{d'} - X^{t-2} Y A' \end{array} \right), \\ \nu_{HA} &= \left(\begin{array}{c|c} X^{t-i} I_{d'} - X^{t-2i} Y A & -XYH \\ \hline -X^{i-r} Y H' A^{-1} + X^{t-2i-r} Y H' A^2 & X^i I_{d'} + YA' \end{array} \right), \end{aligned}$$

where H, H' are given as above for some $u_1, \ldots, u_r$ and A, A' are invertible matrices such that $AH = HA'$.

Proof. As in the proof of Lemma 1.1 we may reduce to the case $\alpha_{i,t-i} \equiv \alpha_{t-i,i} \equiv 0 \mod X$, by splitting off some copies of deformations of type (w, ρ).

By (1) and (5) there exists a matrix W, such that

(7) $$X^{t-2i}\alpha_{ii}^2 + \alpha_{i,t-i}\alpha_{t-i,i} = X^i W,$$

(8) $$\alpha_{ii}W \equiv I_{d'} \mod X.$$

Similarly, by (2) and (6) there exists a matrix W' such that

(9) $$X^{t-2i}\beta_{t-i,t-i}^2 + \alpha_{t-i,i}\alpha_{i,t-i} = X^i W',$$

(10) $$W'\beta_{t-i,t-i} \equiv I_{d''} \mod X.$$

By [MPP, 1.5], we may use some transformations among the matrices α defining T. For example, we may act on the last d'' columns (respectively, first d' rows) of α by a transformation τ and corresponding to the last d'' rows (respectively first d' columns) of α by τ^{-1}. After such operations, we may suppose $\alpha_{i,t-i}$ to be in the diagonal form. Since α_{ii}, W, W', $\beta_{t-i,t-i}$ are invertible, we see from (7), (9) that $\alpha_{i,t-i}$ can have no row or column divisible by X^e for $e > r$ ($t \neq 3i!$). The result is $d' = d''$ and $\alpha_{i,t-i} = XH$ for some integers $u_1, \ldots, u_r \geq 0$. Dividing by XH in (7), (9) it follows

(11) $$\alpha_{t-i,i} = X^{i-r}H'W - X^{t-2i-r}H'\alpha_{ii}^2,$$

(12) $$\alpha_{t-i,i} = X^{i-r}W'H' - X^{t-2i-r}\beta_{t-i,t-i}^2 H'.$$

By (3) we have
$$H\beta_{t-i,t-i} = -\alpha_{ii}H + X^{i-1}C$$
for a matrix C over $K[[X]]$. Then $B' := \beta_{t-i,t-i}$ and $B := -\alpha_{ii} + X^{i-r}CH'$ satisfy

(13) $$BH = HB'.$$

Using (8), (10), we see that
$$W = -B^{-1} + XD, \quad W' = -B'^{-1} \mod XS$$
for a matrix D. Since $\alpha_{ii}^2 \equiv B^2 \mod X^{i-r}H'$, we obtain from (11), (12),
$$X^{i-r}(-B'^{-1})H' - X^{t-2i-r}B'^2H' \equiv X^{i-r}H'W - X^{t-2i-r}H'B^2 \mod X^{i-r+1}SH'.$$
But (13) gives $B'^{-1}H' = H'B^{-1}$.

Then the above congruence becomes $X^{i-r+1}H'D = X^{i-r}H'(W + B^{-1}) \equiv 0 \mod X^{i-r+1}SH'$, that is, we have $H'D = D'H'$ for a certain matrix D'. Thus

(14) $$DH = HD'.$$

By (11), we have

$$\alpha_{t-i,i} \equiv X^{i-r}H'(-B^{-1}+XD) - X^{t-2i-r}H'B^2 \mod X^{i-r+1}SH'.$$

If we change D by $D + X^{i-r+1}\theta H'$ for some $\theta \in S$, we see that (14) still holds for a certain D'. After such a change we may suppose that (14) and the following identity hold:

$$\alpha_{t-i,i} = X^{i-r}H'(-B^{-1}+XD) - X^{t-2i-r}H'B^2. \tag{15}$$

Hence, we have

$$\begin{aligned}
\alpha &= \left(\begin{array}{c|c} -B + X^{i-r}CH' & XH \\ \hline -X^{i-r}H'(B^{-1}-XD) - X^{t-2i-r}H'B^2 & X^{t-2i}B' \end{array}\right) \\
\beta &= \left(\begin{array}{c|c} X^{t-2i}(-B + X^{i-r}CH') & XH \\ \hline -X^{i-r}H'(B^{-1}-XD) - X^{t-2i-r}H'B^2 & B' \end{array}\right)
\end{aligned}$$

and

$$\alpha\beta = \left(\begin{array}{c|c} X^i\left[-B^{-1} + XD - X^{t-2i-r}\big(BCH' + CH'B - X^{i-r}(CH')^2\big)\right] & X^iC \\ \hline X^{t-i}U & (B'^{-1}+XD')X^i \end{array}\right)$$

where U is a matrix satisfying

$$HU = BD - DB - X^{i-r-1}(B^{-1}-XD)CH' - X^{r-2i-r-1}B^2CH' \tag{16}$$

(the existence of U is given by (4), for example!).
Now α, β satisfy (*) (T is liftable to $R'/(Y^4)$). Then there exist γ, δ such that $\alpha\beta = \gamma\psi + \varphi\delta$ and $I_{2d'} + \gamma\beta - \alpha\delta$ has the form $\left(\begin{array}{c|c} X^i\rho_{11} & X^i\rho_{12} \\ \hline X^{t-i}\rho_{21} & X^i\rho_{22} \end{array}\right)$. Note that

$$\begin{aligned}
\tilde\gamma &= \left(\begin{array}{c|c} 0 & 0 \\ \hline U & -B'^{-1} + XD' \end{array}\right), \\
\tilde\delta &= \left(\begin{array}{c|c} -B^{-1} + XD - X^{t-2i-r}\big(BCH' + CH'B - X^{i-r}(CH')^2\big) & C \\ \hline 0 & 0 \end{array}\right)
\end{aligned}$$

satisfy $\alpha\beta = \tilde\gamma\psi + \varphi\tilde\delta$. Then $\gamma' = \gamma - \tilde\gamma$, $\delta' = \delta - \tilde\delta$ satisfy $\gamma'\psi + \varphi\delta' = 0$, that is,

$$\gamma' = \left(\begin{array}{c|c} \gamma'_{11} & -\delta'_{12} \\ \hline -\delta'_{21} & -X^{t-2i}\delta'_{22} \end{array}\right), \quad \delta' = \left(\begin{array}{c|c} -X^{t-2i}\gamma'_{11} & \delta'_{12} \\ \hline \delta_{21} & \delta'_{22} \end{array}\right)$$

for some $d' \times d'$ block matrices $\gamma'_{11}, \delta'_{21}, \delta'_{12}, \delta'_{22}$. Since α, β satisfy (*) there exist $\gamma'_{11}, \delta'_{12}, \delta'_{21}, \delta'_{22}$ such that

$$\begin{aligned}
\gamma &= \left(\begin{array}{c|c} \gamma'_{11} & -\delta'_{12} \\ \hline U - \delta'_{21} & -B'^{-1} + XD' - X^{t-2i}\delta'_{22} \end{array}\right), \\
\delta &= \left(\begin{array}{c|c} -B^{-1} + XD - X^{t-2i-r}\big(BCH' + CH'B - X^{i-r}(CH')^2\big) - X^{t-2i}\gamma'_{11} & C + \delta'_{12} \\ \hline \delta'_{21} & \delta'_{22} \end{array}\right)
\end{aligned}$$

such that $I_{2d'} + \varphi\beta - \alpha\delta$ has the above form.

In particular

$$-\delta'_{12}B' + B(C + \delta'_{12}) + X(\gamma'_{11}H - H\delta'_{22}) - X^{i-r}CH'(C + \delta'_{12}) = X^i\rho_{12}. \tag{17}$$

$$\begin{aligned}(U - \delta'_{21})H + D'B' - X^{t-2i-1}(\delta'_{22}B' + B'\delta'_{22}) + X^{i-r-1}H'(B^{-1} - XD)(C + \delta'_{12})\\ + X^{t-2i-r-1}H'B^2(C + \delta'_{12}) = X^{i-1}\rho_{22}.\end{aligned} \tag{18}$$

Using (16), (13), (14), we obtain $HUH = H(B'D' - D'B') - X^{i-2}(B^{-1} - XD)C - X^{t-2i-2}B^2C = HV$, for $V = B'D' - D'B' - X^{i-r-1}H'(B^{-1} - XD)C - X^{t-2i-r-1}H'B^2C$. Hence, $UH = V$ and so (18) becomes

$$B'D' - \delta'_{21}H \equiv 0 \qquad \mod H'S. \tag{19}$$

It follows $D' \in SH + H'S$, that is, $D' = \Theta H + H'G$ for some $\Theta, G \in S$. Then $H'D = D'H' = H'GH' + X^{r-1}\Theta$. Removing multiples of X^i from $\alpha_{t-i,i}$ by [MPP, 1.2] we may suppose $\Theta = 0$. It follows

$$\alpha_{t-i,i} = -X^{i-r}H'(B^{-1} - XGH') - X^{t-2i-r}H'B^2.$$

Multiplying the first d' rows of α by $E = X^{i-r-1}(B^{-1} - XGH') + X^{t-2i-r-1}B^2$ and the first d' columns of α by E^{-1} (E is invertible because $r+1$ is either i or $t-2i$ and $t \neq 3i$) we may suppose by [MPP, 1.5] that α is equivalent with $\tilde{\alpha} = \left(\begin{smallmatrix} \tilde{\alpha}_{ii} & XEH \\ -XH' & X^{t-2}B'\end{smallmatrix}\right)$ for a new $\tilde{\alpha}_{ii}$. Since $XEH = X^{i-r}B^{-1}H + X^{t-2i-r}B^2H - X^iG$, we may suppose $G = 0$ in E by removing multiples of X^i from $\alpha_{i,t-i}$ (see [MPP, 1.2]). Now multiplying the last d' columns by E'^{-1}, $E' := X^{i-r-1}B'^{-1} + X^{t-2i-r-1}B'^2$ ($EH = HE'$!) and the last d' rows of α by E' we may suppose the structure of α unchanged but with $G = 0$ (and possibly a new C). It follows $D = D' = 0$ and so (18) implies $E'H'(C + \delta'_{12}) \in SH$, thus $H'(C + \delta'_{12}) \in SH$. We need the following lemma: $\square$

Lemma 1.3. *Let $F \in S$ be a matrix such that $H'F \in SH$. then we may change C of α by $C + E^{-1}FE' - F$.*

Proof. Let $H'F = QH$ for a $Q \in S$. Adding the first d' rows of α multiplied by $X^{t-i-r}Q$ to the last d' ones and subtracting the last d' columns multiplied by $X^{i-r}Q$ from the first d' ones, we may suppose (after removing terms of (X^i) from $\alpha_{t-i,i}$)

$$\alpha \sim \left(\begin{array}{c|c} \alpha_{ii} - X^{i-r+1}HQ & XH \\ \hline -XH'E & X^{t-2i}(B' + X^{i-r+1}QH).\end{array}\right)$$

Now, adding the first d' columns of α multiplied by $X^{t-i-r}E^{-1}F$ to the last d' ones and subtracting the last d' rows multiplied by $X^{i-r-1}EF$ from the first d' ones, we may suppose (after removing terms of (X^i) from $\alpha_{i,t-i}$ by [MPP, 1.2])

$$\alpha = \left(\begin{array}{c|c} \alpha_{ii} + X^{i-r+1}(-HQ + E^{-1}FH'E) & XH \\ \hline -XH'E & X^{t-2i}B'.\end{array}\right)$$

But $HQ = FH'$ and $H'E = E'H'$ which is sufficient. $\square$

Proof. Back to our proof apply Lemma 1.3 for $F = C + \delta'_{12}$. Thus, we may change C by $\widetilde{C} := C + E^{-1}(C + \delta'_{12})E' - C - \delta'_{12} = E^{-1}B^{-1}[B(C + \delta'_{12})E' - BE\delta'_{12}]$.

By (16) we obtain $ECH' = -HU$ and so $CH' \in HS$ ($E^{-1}H = He'^{-1}$!). Thus (17) gives

$$BC \equiv \delta'_{12}B' - B\delta'_{12} \mod XSHS. \tag{20}$$

It follows

$$\begin{aligned} \widetilde{C} &\equiv E^{-1}B^{-1}(\delta'_{12}B'E' - BE\delta'_{12}) = X^{t-2i-r-1}E^{-1}B^{-1}(B^3\delta'_{12} - \delta'_{12}B'^3) \\ &\equiv -X^{t-2i-r-1}E^{-1}B^{-1}(B^3C + B^2CB' + BCB'^2) \mod XSHS. \end{aligned} \tag{21}$$

If $C \equiv 0 \mod (X^e, XSHS)$ for an $e \geq 0$, then we obtain $\widetilde{C} \equiv 0 \mod (X^{e+1}, XSHS)$ by (21) because $t \neq 3i$.

Applying this procedure by recurrence, we may arrive at a $C \equiv 0 \mod (X^r, XSHS)$. Removing from α_{ii} the terms from (X^i) as usual, we may suppose $C \in XSHS$, that is, $C = XHW + XW'H$ for some $W, W' \in S$. Thus $\alpha_{ii} \equiv -B + X^{i-r+1}HWH'$ mod X^i. Removing again from α_{ii} the terms from (X^i) we may suppose $\alpha_{ii} = -B + X^{i-r+1}HWH'$.

Now, subtracting the last d' columns of α multiplied with $X^{i-r}WH'$ from the first d' ones and correspondingly (see [MPP, 1.5]) adding the first d' rows of α multiplied by $X^{t-i-r}WH$ to the last d' ones, we may suppose $\alpha_{ii} = -B$. The entries of $\alpha_{t-i,i}, \beta_{t-i,t-i}$ are changed only with some terms of (X^i) which may be removed by [MPP, 1.2]. Consequently, taking $A := -B, A' := -B'$ we obtain

$$\alpha = \left(\begin{array}{c|c} A & XH \\ \hline X^{i-r}H'A^{-1} - X^{t-2i-r}H'A^2 & -X^{t-2i}A' \end{array} \right)$$

which finishes the proof. □

Let $T, \widetilde{T}$ be two infinitesimal deformations of $P_i^{d'} \oplus P_{t-i}^{d''}$ defined by $(\varphi + \alpha Y, \psi - \beta Y)$, respectively $(\varphi + \tilde{\alpha}Y, \psi - \tilde{\beta}Y)$, let us suppose $\alpha = \begin{pmatrix} \alpha_{ii} & \alpha_{i,t-i} \\ \alpha_{t-i,i} & X^{t-2i}\beta_{t-i,t-i} \end{pmatrix}, \tilde{\alpha} = \begin{pmatrix} \tilde{\alpha}_{ii} & \tilde{\alpha}_{i,t-i} \\ \tilde{\alpha}_{t-i,i} & X^{t-2i}\tilde{\beta}_{t-i,t-i} \end{pmatrix}$.

Suppose $T \cong \widetilde{T}$ as $\widetilde{R}$–modules. Then there exist two invertible matrices $U + YU'$, $V + YV'$ where U, V, U', V' are $2d' \times 2d'$–matrices over $K[[X]]$ such that

$$\begin{gathered} (U + YU')(\varphi + \alpha Y) = (\varphi + \tilde{\alpha}Y)(V + YV'), \text{ that is,} \\ U\varphi = \varphi V \\ U\alpha - \tilde{\alpha}V = U'\varphi + \varphi V'. \end{gathered} \tag{22}$$

The first identity implies that U, U' have the form

$$U=\left(\begin{array}{c|c} u_{ii} & u_{i,t-i} \\ \hline X^{t-2i}u_{t-i,i} & u_{t-i,t-i}\end{array}\right),\quad V=\left(\begin{array}{c|c} u_{ii} & X^{t-2i}u_{i,t-i} \\ \hline u_{t-i,i} & u_{t-i,t-i}\end{array}\right)\tag{23}$$

for some $u_{ii},u_{i,t-i},u_{t-i,i},u_{t-i,t-i}\in S$.

The second identity implies that

$$U\alpha-\tilde{\alpha}V=\left(\begin{array}{c|c} X^i w_{ii} & X^i w_{i,t-i} \\ \hline X^i w_{t-i,i} & X^{t-i}w_{t-i,t-i}\end{array}\right)\tag{24}$$

for some $w_{ii},w_{i,t-i},w_{t-i,i},w_{t-i,t-i}\in S$.

Lemma 1.4. *The $\widetilde{R}$–modules T, $\widetilde{T}$ are isomorphic if and only if there exist some matrices $u_{i,i},u_{i,t-i},u_{t-i,i},u_{t-i,t-i}\in S$ such that*

1) $u_{i,i},u_{t-i,t-i}$ *are invertible.*
2) $u_{ii}\alpha_{ii}+u_{i,t-i}\alpha_{t-i,i}\equiv\tilde{\alpha}_{ii}u_{ii}+\tilde{\alpha}_{i,t-i}u_{t-i,i}\mod X^i$,
3) $u_{ii}\alpha_{i,t-i}+X^{t-2i}u_{i,t-i}\beta_{t-i,t-i}\equiv X^{t-2i}\tilde{\alpha}_{ii}u_{i,t-i}+\tilde{\alpha}_{i,t-i}u_{t-i,t-i}\mod X^i$,
4) $X^{t-2i}u_{t-i,i}\alpha_{ii}+u_{t-i,t-i}\alpha_{t-i,i}\equiv\tilde{\alpha}_{t-i,i}u_{ii}+X^{t-2i}\tilde{\beta}_{t-i,t-i}u_{t-i,i}\mod X^i$,
5) $u_{t-i,i}\alpha_{i,t-i}+u_{t-i,t-i}\beta_{t-i,t-i}\equiv\tilde{\alpha}_{t-i,i}u_{i,t-i}+\tilde{\beta}_{t-i,t-i}u_{t-i,t-i}\mod X^i$.

Proof. The necessity follows from (23), (24). The converse is also true because taking U,V as in (23), we see that (24) holds and so there exist U',V' such that (22) holds, too. $\square$

Lemma 1.5. *If $T\cong\widetilde{T}$ as $\widetilde{R}$–modules, then $\alpha_{i,t-i}$ and $\tilde{\alpha}_{i,t-i}$ have the same invariant factors modulo (X^i,X^{t-2i}). Moreover, if T is liftable to $R'/(Y^4)$ and $T\cong\widetilde{T}$, then $\alpha_{i,t-i}$ and $\tilde{\alpha}_{i,t-i}$ have the same invariant factors and their diagonal matrix associated has the form $\Delta=\begin{pmatrix} I_{u_0} & & & 0\\ & XI_{u_1} & & \\ & & \ddots & \\ 0 & & & X^rI_{u_r}\end{pmatrix}$ where $r:=\min\{i,t-2i\}-1$ and $(u_e)_{0\le e\le r}$ are some integers ≥ 0 such that $\sum_{e=0}^{r}u_e=d'$.*

Proof. If $T\cong\widetilde{T}$ then there exist two invertible matrices $u_{ii},u_{t-i,t-i}\in S$ such that

$$u_{ii}\alpha_{i,t-i}\equiv\tilde{\alpha}_{i,t-i}u_{t-i,t-i}\mod (X^i,X^{t-2i})$$

by Lemma 1.4, 1), 3). If T is liftable to $R'/(Y^4)$ and T_0 denotes the deformation given by (w,ρ) (see 1.1), then $T=T_0^{u_0}\oplus T_{HA}$ (see Proposition 1.2), where T_{HA} is given by $(\eta_{HA}, \nu_{HA}), H = \begin{pmatrix} I_{u_1} & & 0 \\ & \ddots & \\ 0 & & X^{r-1}I_{u_r} \end{pmatrix}$ and A is an invertible matrix of S, such that $AH \in HS$. Clearly, $T \sim \Delta$. Since $r < \min\{t-2i, i\}$, we see that $\widetilde{T}, T$ have the same invariant factors and so $\widetilde{T} \sim \Delta$. □

Lemma 1.6. *Let $T_{HA}, T_{H\widetilde{A}}$ be the infinitesimal deformations defined by (η_{HA}, ν_{HA}) respectively $(\eta_{H\widetilde{A}}, \nu_{H\widetilde{A}})$ for two invertible matrices $A, \widetilde{A} \in S$ with $AH, \widetilde{A}H \in HS$. Then $T_{HA} \cong T_{H\widetilde{A}}$ as $\widetilde{R}$–modules if and only if there exists an invertible matrix $C \in S$ with $CH \in HS$ such that*

$$\widetilde{A} \equiv CAC^{-1} \mod X(HS + SH').$$

Proof. By Lemma 1.4 there exist $u_{ii}, u_{i,t-i}, u_{t-i,i}, u_{t-i,t-i} \in S$ satisfying 1) – 5). In particular $C = u_{ii}$ is invertible and satisfies

$$CA - \widetilde{A}C \equiv XHu_{t-i,i} + u_{i,t-i}XH'E = X(Hu_{t-i,i} + u_{i,t-i}E'H') \mod X^i$$

and

$$XCH \equiv XHu_{t-i,t-i} \mod X^{r+1},$$

where E, E' are given in the proof of Proposition 1.2. Thus, $CA \equiv \widetilde{A}C \mod X(HS + SH')$ and $CH \in HS$. Conversely, we reduce to the case $C = I_{d'}$ by changing T_{HA} by $T_{H,CAC^{-1}} \cong T_{HA}$ the isomorphism being given by $U = \left(\begin{smallmatrix} C & 0 \\ 0 & C' \end{smallmatrix}\right) = V$, $CH = HC'$. If $\widetilde{A} - A = XGH' \mod XHS$ for a $G \in S$ then note that $(A + XGH')H = HA' + X^rG \in HS$ and $T_{HA} \cong T_{H,A+XGH'}$, the isomorphism being given by $U = \begin{pmatrix} I_{d'} & GE'^{-1} \\ 0 & I_{d'} \end{pmatrix}, V = \begin{pmatrix} I_{d'} & X^{t-2i}GE'^{-1} \\ 0 & I_{d'} \end{pmatrix}$. Thus, we reduce to the case $\widetilde{A} \equiv A \mod XHS$ which goes similarly. □

Remark 1.7. If $\widetilde{A} \equiv A \mod X^r$ then $T_{HA} \cong T_{H\widetilde{A}}$. Moreover, A has the form

$$A = \begin{pmatrix} a_{11} & \cdots & & a_{1r} \\ Xa_{21} & \ddots & & \\ \vdots & & \ddots & \vdots \\ X^{r-1}a_{r1} & \cdots & Xa_{r,r-1} & a_{rr} \end{pmatrix},$$

where a_{pq} is a $u_p \times u_q$–block matrix because $AH = HA'$. If $\tilde{a}_{pq} \equiv a_{pq} \mod X^{s_{pq}}, s_{pq} := \min\{p, q, r-p, r-q\}$, then $T_{HA} \cong T_{H\widetilde{A}}$ for $\widetilde{A}$ defined by $(\tilde{a}_{pq})$ similarly to A.

2 Maximal Cohen–Macaulay Modules

Let i, $2 \le i < t/2$, $i \ne t/3$ be an integer and $r = \min\{t-2i, i\} - 1$.

Theorem 2.1. *Then*

1) *there exists a MCM R'–module N such that $N/YN \cong P_i^{d'} \oplus P_{t-i}^{d''}$ if and only if $d' = d''$,*

2) *every MCM R'–module N such that $N/YN \cong P_i^{d'} \oplus P_{t-i}^{d'}$ is a direct sum of copies of the ideals $(X^i, Y), (X^{t-i}, Y)$ and of a MCM R'–module N_{HA} given by $(\hat{\eta}_{HA}, \hat{\nu}_{HA})$*

$$\hat{\eta}_{HA} = \left(\begin{array}{c|c} X^i I_{d'} + YA & XYH \\ \hline X^{i-r}YH'A^{-1} - X^{t-2i-r}YH'A^2 & X^{t-i}I_{d'} - X^{t-2i}YA' + Y^2A'^{-1} \end{array}\right)$$

$$\hat{\nu}_{HA} = \left(\begin{array}{c|c} X^{t-i}I_{d'} - X^{t-2i}YA + Y^2A^{-1} & -XYH \\ \hline -X^{i-r}YH'A^{-1} + X^{t-2i-r}YH'A^2 & X^i I_{d'} + YA' \end{array}\right),$$

where $H = \begin{pmatrix} I_{u_1} & & 0 \\ & \ddots & \\ 0 & & X^{r-2}I_{u_r} \end{pmatrix}$ for some integers $(u_e)_{1 \le e \le r}$, $u_e \ge 0$, $H' := \begin{pmatrix} X^{r-1}I_{u_1} & & 0 \\ & \ddots & \\ 0 & & I_{u_r} \end{pmatrix}$ and A, A' are invertible $q \times q$–matrices over $K[X]$ with $AH = HA'$, whose entries have degree $< r$,

3) *if $N_{\widetilde{H}\widetilde{A}}$ is given by $(\hat{\eta}_{\widetilde{H}\widetilde{A}}, \hat{\nu}_{\widetilde{H}\widetilde{A}})$ for $\widetilde{H} = \begin{pmatrix} I_{\tilde{u}_1} & & \\ & \ddots & \\ & & X^{r-1}I_{\tilde{u}_r} \end{pmatrix}$ and $\widetilde{A}, \widetilde{A}'$ are invertible matrices such that $\widetilde{A}\widetilde{H} = \widetilde{H}\widetilde{A}'$ then $N_{\widetilde{H}\widetilde{A}} \cong N_{HA}$ if and only if $u_e = \tilde{u}_e$ for $1 \le e \le r$ (that is, $H = \widetilde{H}$) and there exists an invertible matrix C over $K[[X]]$ such that $CH \in HS_q$, $S_q := M(q \times q, K[[X]])$ and*

$$\widetilde{A} \equiv CAC^{-1} \mod X(HS_q + S_qH').$$

Proof. If N is a MCM R'–module such that $N/YN \cong P_i^{d'} \oplus P_{t-i}^{d''}$, then N/Y^2N is an infinitesimal deformation of N/YN and so by Proposition 1.2 $d' = d''$ and $N/Y^2N = T_0^{u_0} \oplus T_{HA}$ as in the proof of Lemma 1.5. Clearly the ideals $(X^i, Y), (X^{t-i}, Y)$ lift both T_0 and there exist none other with this property by [EPP, 1.6]. Splitting off all direct summands of N which are isomorphic with the above ideals, we may reduce to the case $N/Y^2N \cong T_{HA}$. Indeed, if T_0 appears still as a direct summand of N/Y^2N, then $(X^i, Y) \oplus (X^{t-i}, Y)$ is a direct summand of $\Omega^1_{R'}(N/Y^2N) \cong N \oplus \Omega^1_{R'}(N)$ by [Po], or [HP, 2.8], which is not possible.

For the last part of 2), we may suppose N to be indecomposable. Since $N_{HA}/Y^2N_{HA} \cong T_{HA}$ and $N_{HA} \cong \Omega^1_{R'}(N_{HA})$ (see the matrices $(\hat{\eta}_{HA}, \hat{\nu}_{HA})$), we see that N is a direct summand of $\Omega^1_{R'}(T_{HA}) = N^2_{HA}$ and so of N_{HA}. Hence, $N \cong N_{HA}$ because they have the same number of a minimal system of generators ($N/Y^2N \cong N_{HA}/Y^2N_{HA}$!). By Remark 1.7 we may suppose the entries of A of degree $< r$.

For 3), if $N_{\widetilde{H}\widetilde{A}}$, N_{HA} are isomorphic, then they induce isomorphic infinitesimal deformations $\mod Y^2$ and so the necessity follows by Lemma 1.6.

Conversely, we see by Lemma 1.6 that $N_{H\widetilde{A}}$, N_{HA} induce isomorphic infinitesimal deformations $\mod Y^2$ and so $N_{H\widetilde{A}}$ is a direct summand of $\Omega^1_{R'}(T_{HA}) \cong N^2_{HA}$. Hence, $N_{H\widetilde{A}} \cong N_{HA}$. (Note that we proved here, in fact, that there exists just one MCM R'–module lifting T_{HA}). □

Remark 2.2.

1) If $t = 3i$, then for any d', d'' there exists a MCM R'–module N such that $N/YN \cong P_i^{d'} \oplus P_{t-i}^{d''}$. For example, $R'/(X^i+Y) \cong K[[X,Y]]/(X^i+Y)$ lifts P_i and $R'/(X^{2i} - X^iY + Y^2)$ lifts P_{t-i}. Moreover, the MCM modules given by Theorem 2.1 have sense also if $t = 3i$.

2) The minimal number of generators of N_{HA} is $2q$. Since $N_{HA} \cong \Omega^1_{R'}(N_{HA})$ it follows $\operatorname{rank}_{R'} N_{HA} = q$.

3) If $i = 1$, or $t = 2i+1$, then $r = 0$ and Theorem 2.1 states that each N above is a direct summand of copies of the ideals $(X^i, Y), (X^{t-i}, Y)$.

4) If $i = 2$ then $r = 1$ and Theorem 2.1 states that each N above is a direct summand of copies of the ideals $(X^i, Y), (X^{t-i}, Y)$ and a MCM R'–module N_{I_qA} obtained for $H = I_r$ and an invertible $q \times q$–matrix A over K. Applying the Jordan decomposition in cells for A, we obtain exactly Theorem 0.2.

5) If $t = 2i+2$ then $r = s$ and, as above, we obtain a variant of Theorem 0.2 in this case.

6) If $t = 16$, $i = 5$ then $r = 4$,

$$H = \begin{pmatrix} I_{u_1} & & & \\ & XI_{u_2} & & 0 \\ & 0 & X^2I_{u_3} & \\ & & & X^3I_4 \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & \cdots\cdots & & a_{41} \\ Xa_{21} & \ddots & & a_{42} \\ X^2a_{31} & \ddots & \ddots & a_{43} \\ X^3a_{41} & & Xa_{43} & a_{44} \end{pmatrix}$$

and by Remark 1.7 we may take the entries of the blocks a_{jp} to be in K, if $j \neq 2$, $p \neq 2$. The entries of a_{22} can be chosen to be linear polynomial in X over K. Thus, (N_{HA}) form a family with $(u_{22}^2 + 1)$–parameters. As u_{22} can be chosen arbitrarily high, we see that MCM (R') is a wild category in this case (see also [DG].

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