Construction of Neron Desingularization for Two Dimensional Rings

Gerhard Pfister and Dorin Popescu

Abstract Let $u : A \to A'$ be a regular morphism of Noetherian rings and B an A-algebra of finite type. Then any A-morphism $v : B \to A'$ factors through a smooth A-algebra C, that is v is a composite A-morphism $B \to C \to A'$. This theorem called General Neron Desingularization was first proved by the second author [8]. Later different proofs were given by André [1], Swan [12] and Spivakovsky [11]. All the proofs are not constructive. In [6] the authors gave a constructive proof together with an algorithm to compute the Neron Desingularization for 1-dimensional local rings. In this paper we go one step further. We give an algorithmic proof of the General Neron Desingularization theorem for 2-dimensional local rings and morphisms with small singular locus. The main idea of the proof is to reduce the problem to the one-dimensional case. Based on this proof we give an algorithm to compute the desingularization.
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1 Introduction

The General Neron Desingularization Theorem, first proved by the second author has many important applications. One application is the generalization of Artin's famous approximation theorem (Artin [2], Popescu [9], [10]).
Let us recall some definitions. A ring morphism $u : A \to A'$ has *regular fibers* if for all prime ideals $P \in \operatorname{Spec} A$ the ring A'/PA' is a regular ring, i.e. its localizations

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are regular local rings. It has *geometrically regular fibers* if for all prime ideals $P \in \operatorname{Spec} A$ and all finite field extensions K of the fraction field of A/P the ring $K \otimes_{A/P} A'/PA'$ is regular. If for all $P \in \operatorname{Spec} A$ the fraction field of A/P has characteristic 0 then the regular fibers of u are geometrically regular fibers. A flat morphism u is *regular* if its fibers are geometrically regular. If u is regular of finite type then u is called *smooth*. A localization of a smooth algebra is called *essentially smooth*.

Theorem 1. *(General Neron Desingularization, André [1], Popescu [8], [9], [10], Swan [12], Spivakovsky [11]) Let $u : A \to A'$ be a regular morphism of Noetherian rings and B an A-algebra of finite type. Then any A-morphism $v : B \to A'$ factors through a smooth A-algebra C, that is v is a composite A-morphism $B \to C \to A'$.*

The proof of this theorem is not constructive. Constructive proofs for one-dimensional rings were given in A. Popescu, D. Popescu [7], and Pfister, Popescu [6]. In this paper we will treat the 2-dimensional case. The idea is to reduce the problem to the one-dimensional case. We will choose a suitable element $a \in A$ and consider $\bar{A} = A/aA$, $\bar{B} = \bar{A} \otimes_A B$, $\bar{A}' = A'/aA'$, $\bar{v} = \bar{A} \otimes_A v : \bar{B} \to \bar{A}'$ to find a desingularization $\bar{B} \to \bar{D} \to \bar{A}'$ induced by a smooth A-algebra D. This desingularization can then be lifted to a desingularization $B \to D \to A'$.

For the computational part we have the following assumptions: Let k be a field, $x = (x_1, \ldots, x_n)$ and $J \subset k[x]$ be an ideal. We assume

$$A = (k[x]/J)_{<x>} \text{ is Cohen-Macaulay of dimension 2, } A' \text{ is its completion}$$

and u the inclusion. The images of the morphism $v : B \to A'$ need not to be in A, i.e. the input for the algorithm can only be an approximation of v by polynomials up to a given bound. The bound to obtain a desingularization of v depends also on the ring B and is usually not known in advance. If the given bound is not good enough the algorithm will fail. In this case the bound has to be enlarged and the algorith has to be restarted with new approximations of v.
The case that the image of v is already in A is trivial because in this case we can use the smooth A-algebra $C = A$ as desingularization.

2 Constructive General Neron Desingularization

Let $u : A \to A'$ be a flat morphism of Noetherian local rings of dimension 2. Suppose that the maximal ideal $\mathfrak{m}$ of A generates the maximal ideal of A' and the completions of A, A' are isomorphic. Moreover suppose that A' is Henselian, and u is a regular morphism.

Let $B = A[Y]/I$, $Y = (Y_1, \ldots, Y_n)$. If $f = (f_1, \ldots, f_r)$, $r \leq n$ is a system of polynomials from I then we can define the ideal

$$\Delta_f \text{ generated by all } r \times r\text{-minors of the Jacobian matrix } (\partial f_i / \partial Y_j).$$

After Elkik [4] let $H_{B/A}$ be the radical of the ideal $\sum_f ((f) : I) \Delta_f B$, where the sum is taken over all systems of polynomials f from I with $r \leq n$. Then for $\mathfrak{p} \in \operatorname{Spec} B$

$$B_{\mathfrak{p}} \text{ is essentially smooth over } A \text{ if and only if } \mathfrak{p} \not\supset H_{B/A}$$

by the Jacobian criterion for smoothness. Thus $H_{B/A}$ measures the non smooth locus of B over A. B is *standard smooth* over A if there exists f in I as above such that

$$B = ((f) : I)\Delta_f B.$$

The aim of this paper is to give an algorithmic proof of the following theorem.

Theorem 2. *Any A-morphism $v : B \to A'$ such that $v(H_{B/A})A'$ is $\mathfrak{m}A'$-primary factors through a standard smooth A-algebra B'.*

Proof. The idea is to find a suitable element $a \in A$ such that we can use the one-dimensional result obtained for $\bar{A} = A/(a)$, $\bar{B} = \bar{A} \otimes_A B$, $\bar{A}' = A'/(aA')$, $\bar{v} = \bar{A} \otimes_A v$ to find a desingularization $\bar{D}$ induced by a standard smooth A-algebra D (Lemma 2). This desingularization can then be lifted using D. To simplify the proof we assume that A is Cohen-Macaulay.

We choose $\gamma, \gamma' \in v(H_{B/A})A' \cap A$ such that γ, γ' is a regular sequence in A, let us say

$$\gamma = \textstyle\sum_{i=1}^q v(b_i) z_i,\ \gamma' = \sum_{i=1}^q v(b_i) z_i' \text{ for some [1] } b_i \in H_{B/A} \text{ and } z_i, z_i' \in A'.$$

Set $B_0 = B[Z, Z']/(f, \tilde{f})$, where $f = -\gamma + \sum_{i=1}^q b_i Z_i \in B[Z]$, $Z = (Z_1, \ldots, Z_q)$, $\tilde{f} = -\gamma' + \sum_{i=1}^q b_i Z_i' \in B[Z']$, $Z' = (Z_1', \ldots, Z_q')$ and let $v_0 : B_0 \to A'$ be the map of B-algebras given by $Z \to z$, $Z' \to z'$. Replacing B by B_0 we may suppose that $\gamma, \gamma' \in H_{B/A}$.

We need the following lemmata.

Lemma 1. *1. ([8, Lemma 3.4]) Let B_1 be the symmetric algebra $S_B(I/I^2)$ of I/I^2 over[2] B. Then $H_{B/A}B_1 \subset H_{B_1/A}$ and $(\Omega_{B_1/A})_\gamma$ is free over $(B_1)_\gamma$ for any $\gamma \in H_{B/A}$.*
2. ([12, Proposition 4.6]) Suppose that $(\Omega_{B/A})_\gamma$ is free over B_γ. Let $I' = (I, Y') \subset A[Y, Y']$, $Y' = (Y_1', \ldots, Y_n')$. Then $(I'/I'^2)_\gamma$ is free over B_γ.
3. ([10, Corollary 5.10]) Suppose that $(I/I^2)_\gamma$ is free over B_γ. Then a power of γ is in $((g) : I)\Delta_g$ for some $g = (g_1, \ldots g_r)$, $r \le n$ in I.

Using (1) of Lemma 1 we can reduce the proof of Theorem 2.1 to the case when $\Omega_{B_\gamma/A}$ and $\Omega_{B_{\gamma'}/A}$ are free over B_γ respectively $B_{\gamma'}$. Let B_1 be given by (1) of Lemma 1. The inclusion $B \subset B_1$ has a retraction w which maps I/I^2 to zero. For the reduction we change B, v by B_1, vw.

Using (2) from Lemma 1 we may reduce the proof to the case when $(I/I^2)_\gamma$ (resp. $(I/I^2)_{\gamma'}$) is free over B_γ (resp. $B_{\gamma'}$). Indeed, since $\Omega_{B_\gamma/A}$ is free over B_γ we see

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[1] For the algorithm we have to choose γ, γ' more carefully: $\gamma \equiv \sum_{i=1}^q b_i(y') z_i$ modulo (γ', γ'^t), $\gamma' \equiv \sum_{i=1}^q b_i(y') z_i'$ modulo (γ', γ'^t) with $z_i, z_i' \in A$, and $y_i' \equiv v(Y_i)$ modulo $\mathfrak{m}^N$ in A , $N >> 0$.

[2] Let M be a finitely represented B-module and $B^m \xrightarrow{(a_{ij})} B^n \to M \to 0$ a presentation then $S_B(M) = B[T_1, \ldots, T_n]/J$ with $J = (\{\sum_{i=1}^n a_{ij} T_i\}_{j=1,\ldots,m})$.

that changing I with $(I,Y') \subset A[Y,Y']$ we may suppose that $(I/I^2)_\gamma$ is free over B_γ. Similarly, for γ'.

Using (3) from Lemma 1 we may reduce the proof to the case when a power of γ (resp. γ') is in $((f):I)\Delta_f$ (resp. $((f'):I)\Delta_{f'}$) for some $f=(f_1,\dots f_r)$, $r\leq n$ and $f'=(f'_1,\dots f'_{r'})$, $r'\leq n$ from I.

We may now assume that a power d (resp. d') of γ (resp. γ') has the form

$$d\equiv P=\sum_{i=1}^q M_iL_i \text{ modulo } I,\ d'\equiv P'=\sum_{i=1}^{q'} M'_iL'_i \text{ modulo } I$$

for some $r\times r$ (resp. $r'\times r'$) minors M_i (resp. M'_i) of $(\partial f/\partial Y)$ (resp. $(\partial f'/\partial Y)$) and $L_i\in((f):I)$ (resp. $L'_i\in((f'):I)$).

The Jacobian matrix $(\partial f/\partial Y)$ (resp. $(\partial f/\partial Y)$) can be completed with $(n-r)$ (resp. $(n-r')$) rows from A^n obtaining a square n matrix H_i (resp. H'_i) such that $\det H_i=M_i$ (resp. $\det H'_i=M'_i$). This is easy using just the integers $0,1$.

Let $\bar{A}=A/(d^3)$, $\bar{B}=\bar{A}\otimes_A B$, $\bar{A}'=A'/(d^3A')$, $\bar{v}=\bar{A}\otimes_A v$. We will now construct a standard smooth A-algebra D and an A-morphism $\omega: D\to A'$ such that $y=v(Y)\in \Im\omega+d^3A'$.

Lemma 2. *There exists a standard smooth A-algebra D such that $\bar{v}$ factors through $\bar{D}=\bar{A}\otimes_A D$.*

Proof. Let $y'\in A^n$ be such that $y=v(Y)\equiv y'$ modulo $(d^3,d'^3)A'$, let us say $y-y'\equiv d'^2\varepsilon$ modulo d^3 for $\varepsilon\in d'A'^n$. Thus

$$I(y')\equiv 0 \text{ modulo } (d^3,d'^3)A'.$$

Recall that we have $d'\equiv P'$ modulo I and so $P'(y')\equiv d'$ modulo (d^3,d'^3) in A. Thus

$$P'(y')\equiv d's \text{ modulo } d^3 \text{ for a certain } s\in A \text{ with } s\equiv 1 \text{ modulo } d'.$$

Let G'_i be the adjoint matrix of H'_i and $G_i=L'_iG'_i$. We have $G_iH'_i=H'_iG_i=M'_iL'_i\mathrm{Id}_n$ and so

$$P'(y')\mathrm{Id}_n=\sum_{i=1}^{q'} G_i(y')H'_i(y').$$

But H'_i is the matrix [3] $(\partial f'_k/\partial Y_j)_{k\in[r'],j\in[n]}$ completed with some $(n-r')$ rows of $0,\ 1$. Especially we obtain

$$(\partial f'/\partial Y)G_i=M'_iL'_i(\mathrm{Id}_{r'}|0). \tag{1}$$

Then $t_i:=H'_i(y')\varepsilon\in d'A'^n$ satisfies

$$G_i(y')t_i=M'_i(y')L'_i(y')\varepsilon$$

and so

$$\sum_{i=1}^{q} G_i(y')t_i=P'(y')\varepsilon\equiv d's\varepsilon \text{ modulo } d^3.$$

[3] We use the notation $[n]=\{1,\dots,n\}$.

It follows that

$$s(y-y') \equiv d'\sum_{i=1}^{q'} G_i(y')t_i \text{ modulo } d^3.$$

Note that $t_{ij}=t_{i1}$ for all $i\in[r']$ and $j\in[n]$ because the first r' rows of H'_i does not depend on i (they are the rows of $(\partial f'/\partial Y)$).

Let

$$h = s(Y-y') - d'\sum_{i=1}^{q'} G_i(y')T_i, \tag{2}$$

where $T_i=(T_1,\dots,T_{r'},T_{i,r'+1}\dots,T_{i,n})$, $i\in[q']$ are new variables. We will use also $T_{ij}=T_i$ for $i\in[r']$, $j\in[n]$ because it is convenient sometimes. The kernel of the map $\bar\phi:\bar A[Y,T]\to\bar A'$ given by $Y\to y$, $T\to t$ contains h modulo d^3. Since

$$s(Y-y') \equiv d'\sum_{i=1}^{q'} G_i(y')T_i \text{ modulo } h$$

and

$$f'(Y)-f'(y') \equiv \sum_j (\partial f'/\partial Y_j)(y')(Y_j-y'_j) \text{ modulo higher order terms in } Y_j-y'_j$$

by Taylor's formula. We see that for $p'=\max_i \deg f'_i$ we have

$$s^{p'}f'(Y)-s^{p'}f'(y') \equiv \sum_j s^{p'-1}d'(\partial f'/\partial Y_j)(y')\sum_{i=1}^{q'} G_{ij}(y')T_{ij} + d'^2Q \text{ modulo } h \tag{3}$$

where $Q\in T^2A[T]^{r'}$. We have $f'(y')\equiv d'^2b'$ modulo d^3 for some $b'\in d'A^{r'}$. Then

$$g_i = s^{p'}b'_i + s^{p'}T_i + Q_i, \qquad i\in[r'] \tag{4}$$

modulo d^3 is in the kernel of $\bar\phi$. Indeed, we have $s^{p'}f'_i = d'^2 g_i$ modulo (h,d^3) because of (3). Thus $d'^2\bar\phi(g)=d'^2g(t)\in(h(y,t),f'(y))\in d^3A'$ and so $g(t)\in d^3A'$, because u is flat and d' is regular on $A/(d^3)$. Set $E=\bar A[Y,T]/(I,g,h)$ and let $\bar\psi:E\to\bar A'$ be the map induced by $\bar\phi$. Clearly, $\bar v$ factors through $\bar\psi$ because $\bar v$ is the composed map $\bar B=\bar A[Y]/I\to E\xrightarrow{\bar\psi}\bar A'$.
We will see, there are $s',s''\in E$ such that $E_{ss's''}$ is smooth over $\bar A$ and $\bar\psi$ factors through $E_{ss's''}$.

Note that the $r'\times r'$-minor s' of $(\partial g/\partial T)$ given by the first r'-variables T is from $s^{r'p'}+(T)\subset 1+(d',T)$ because $Q\in(T)^2$. Then $V=(\bar A[Y,T]/(h,g))_{ss'}$ is smooth over $\bar A$. As in [6] we claim that $I\bar A[Y,T]\subset(h,g)\bar A[Y,T]_{ss's''}$ for some $s''\in 1+(d',d^3,T)A[Y,T]$. Indeed, we have

$$P'I\bar A[Y,T]\subset(f')A[Y,T]\subset(h,g)\bar A[Y,T]_s$$

and so

$$P'(y'+s^{-1}d'G(y')T)I\subset(h,g,d^3)A[Y,T]_s.$$

Since $P'(y'+s^{-1}d'G(y')T)\in P'(y')+d'(T)V$ we get

$$P(y'+s^{-1}d'G(y')T)\equiv d's'' \text{ modulo } d^3$$

for some $s''\in 1+(T)A[Y,T]$. It follows that $s''I\subset(((h,g):d'),d^3)A[Y,T]_{ss'}$. Thus $s''I$ is contained modulo d^3 in $(0:_V d')=0$ because d' is regular on V, the map $\bar{A}\to V$ being flat. This shows our claim. It follows that $I\subset(d^3,h,g)A[Y,T]_{ss's''}$. Thus $E_{ss's''}\cong V_{s''}$ is a $\bar{B}$-algebra which is also standard smooth over $\bar{A}$.

As $u(s)\equiv 1$ modulo d' and $\bar{\psi}(s'),\bar{\psi}(s'')\equiv 1$ modulo (d',d^3,t), $d,d',t\in \mathfrak{m}A'$ we see that $u(s),\bar{\psi}(s'),\bar{\psi}(s'')$ are invertible because A' is local. Thus $\bar{\psi}$ (and so $\bar{v}$) factors through the standard smooth $\bar{A}$-algebra $E_{ss's''}$, let us say by $\bar{\omega}:E_{ss's''}\to\bar{A}'$.

Now, let $Y'=(Y'_1,\ldots.Y'_n)$, and D be the A-algebra isomorphic with

$$(A[Y,T]/(I,h,g))_{ss's''} \text{ by } Y'\to Y,\, T\to T.$$

Since A' is Henselian we may lift $\bar{\omega}$ to a map $(A[Y,T]/(I,h,g))_{ss's''}\to A'$ which will correspond to a map $\omega:D\to A'$. Then $\bar{v}$ factors [4] through $\bar{D}$, let us say $\bar{B}\to\bar{D}\to\bar{A}'$, where the first map is given by $Y\to Y'$. This proves the lemma 2.

To continue with the proof of Theorem 2.1 let δ be the A-morphism defined by

$$\delta: B\otimes_A D\cong D[Y]/ID[Y]\to A',\, b\otimes\lambda\to v(b)\omega(\lambda).$$

Claim: δ factors through a special finite type $B\otimes_A D$-algebra $\tilde{E}$.

The proof will follow the proof of Lemma 2. Note that the map $\bar{B}\to\bar{D}$ is given by $Y\to Y'+d^3D$. Thus $I(Y')\equiv 0$ modulo d^3D. Set $\tilde{y}=\omega(Y')$. Since $\bar{v}$ factors through $\bar{\omega}$ we get

$$y-\tilde{y}=v(Y)-\tilde{y}\in d^3A'^n, \text{ let us say } y-\tilde{y}=d^2\nu \text{ for } \nu\in dA'^n.$$

Recall that $P=\sum_i L_i\det H_i$ for $L_i\in((f):I)$. We have $d\equiv P$ modulo I and so $P(Y')\equiv d$ modulo d^3 in D because $I(Y')\equiv 0$ modulo d^3D. Thus $P(Y')=d\tilde{s}$ for a certain $\tilde{s}\in D$ with $\tilde{s}\equiv 1$ modulo d. Let $\tilde{G}'_i$ be the adjoint matrix of H_i and $\tilde{G}_i=L_i\tilde{G}'_i$. We have $\sum_i\tilde{G}_iH_i=\sum_i H_i\tilde{G}_i=P\mathrm{Id}_n$ and so

$$d\tilde{s}\mathrm{Id}_n=P(Y')\mathrm{Id}_n=\sum_i\tilde{G}_i(Y')H_i(Y').$$

But H_i is the matrix $(\partial f_i/\partial Y_j)_{i\in[r],j\in[n]}$ completed with some $(n-r)$ rows from $0,1$. Especially we obtain

$$(\partial f/\partial Y)\sum_i\tilde{G}_i=(P\mathrm{Id}_r|0). \tag{5}$$

Then $\tilde{t}_i:=\omega(H_i(Y'))\nu\in dA'^n$ satisfies

$$\sum_i\tilde{G}_i(Y')\tilde{t}_i=P(Y')\nu=d\tilde{s}\nu$$

and so

[4] Note that v does not necessarily factors through D.

$$\tilde{s}(y-\tilde{y}) = d\sum_i \omega(\tilde{G}_i(Y'))\tilde{t}_i.$$

Let

$$\tilde{h} = \tilde{s}(Y-Y') - d\sum_i \tilde{G}_i(Y')\tilde{T}_i, \tag{6}$$

where $\tilde{T} = (\tilde{T}_1,\dots,\tilde{T}_n)$ are new variables. The kernel of the map $\tilde{\phi} : D[Y,\tilde{T}] \to A'$ given by $Y \to y$, $\tilde{T} \to \tilde{t}$ contains $\tilde{h}$. Since

$$\tilde{s}(Y-Y') \equiv d\sum_i \tilde{G}_i(Y')\tilde{T}_i \text{ modulo } \tilde{h}$$

and

$$f(Y) - f(Y') \equiv \sum_j (\partial f/\partial Y_j)((Y')(Y_j - Y'_j)$$

modulo higher order terms in $Y_j - Y'_j$, by Taylor's formula we see that for $p = \max_i \deg f_i$ we have

$$\tilde{s}^p f(Y) - \tilde{s}^p f(Y') \equiv \sum_j \tilde{s}^{p-1} d(\partial f/\partial Y_j)(Y')\sum_i \tilde{G}_{ij}(Y')\tilde{T}_{ij} + d^2\tilde{Q} \tag{7}$$

modulo $\tilde{h}$ where $\tilde{Q} \in \tilde{T}^2 D[\tilde{T}]^r$. We have $f(Y') = d^2\tilde{b}$ for some $\tilde{b} \in dD^r$. Then

$$\tilde{g}_i = \tilde{s}^p\tilde{b}_i + \tilde{s}^p\tilde{T}_i + \tilde{Q}_i, \qquad i \in [r] \tag{8}$$

is in the kernel of $\tilde{\phi}$. Indeed, we have $\tilde{s}^p f_i = d^2\tilde{g}_i$ modulo $\tilde{h}$ because of (7) and $P(Y') = d\tilde{s}$. Thus $d^2\phi(\tilde{g}) = d^2\tilde{g}(t) \in (\tilde{h}(y,\tilde{t}), f(y)) = (0)$ and so $\tilde{g}(\tilde{t}) = 0$. Set $\tilde{E} = D[Y,\tilde{T}]/(I,\tilde{g},\tilde{h})$ and let $\tilde{\psi} : \tilde{E} \to A'$ be the map induced by $\tilde{\phi}$. Clearly, v factors through $\tilde{\psi}$ because v is the composed map

$$B \to B \otimes_A D \cong D[Y]/I \to \tilde{E} \xrightarrow{\tilde{\psi}} A'.$$

Finally we will prove that there exist $\tilde{s}', \tilde{s}'' \in \tilde{E}$ such that $\tilde{E}_{\tilde{s}\tilde{s}'\tilde{s}''}$ is standard smooth over A and $\tilde{\psi}$ factors through $\tilde{E}_{\tilde{s}\tilde{s}'\tilde{s}''}$.

Note that the $r \times r$-minor $\tilde{s}'$ of $(\partial\tilde{g}/\partial\tilde{T})$ given by the first r-variables $\tilde{T}$ is from $\tilde{s}^{rp} + (\tilde{T}) \subset 1 + (d,\tilde{T})$ because $\tilde{Q} \in (\tilde{T})^2$. Then $\tilde{V} = (D[Y,\tilde{T}]/(\tilde{h},\tilde{g}))_{\tilde{s}\tilde{s}'}$ is smooth over D. We claim that $I \subset (\tilde{h},\tilde{g})D[Y,\tilde{T}]_{\tilde{s}\tilde{s}'\tilde{s}''}$ for some other $\tilde{s}'' \in 1 + (d,\tilde{T})D[Y,\tilde{T}]$. Indeed, we have

$$PID[Y] \subset (f)D[Y] \subset (\tilde{h},\tilde{g})D[Y,\tilde{T}]_{\tilde{s}}$$

and so

$$P(Y' + \tilde{s}^{-1}d\textstyle\sum_i \tilde{G}_i(Y')\tilde{T}_i)I \subset (\tilde{h},\tilde{g})D[Y,\tilde{T}]_{\tilde{s}}.$$

Since $P(Y' + \tilde{s}^{-1}d\sum_i \tilde{G}_i(Y')\tilde{T}_i) \in P(Y') + d(\tilde{T})$ we get $P(Y' + \tilde{s}^{-1}d\sum_i \tilde{G}_i(Y')\tilde{T}_i) = d\tilde{s}''$ for some $\tilde{s}'' \in 1 + (\tilde{T})D[Y,\tilde{T}]$. It follows that $\tilde{s}''I \subset ((\tilde{h},\tilde{g}) : d)D[Y,\tilde{T}]_{\tilde{s}\tilde{s}'}$. Thus $\tilde{s}''I \subset (0 :_{\tilde{V}} d) = 0$, which shows our claim. It follows that $I \subset (\tilde{h},\tilde{g})D[Y,\tilde{T}]_{\tilde{s}\tilde{s}'\tilde{s}''}$. Thus $\tilde{E}_{\tilde{s}\tilde{s}'\tilde{s}''} \cong \tilde{V}_{\tilde{s}''}$ is a B-algebra which is also standard smooth over D and A.

As $\omega(\tilde{s}) \equiv 1$ modulo d and $\tilde{\psi}(\tilde{s}'), \tilde{\psi}(\tilde{s}'') \equiv 1$ modulo $(d,\tilde{t})$, $d,\tilde{t} \in \mathfrak{m}A'$ we see that $\omega(\tilde{s}), \tilde{\psi}(\tilde{s}'), \tilde{\psi}(\tilde{s}'')$ are invertible because A' is local. Thus $\tilde{\psi}$ (and so v) factors through the standard smooth A-algebra $B' = \tilde{E}_{\tilde{s}\tilde{s}'\tilde{s}''}$. This proves Theorem 2.

3 The Algorithm

Now we want to apply Theorem 2 to compute the Neron desingularization. We assume $A = (k[x]/J)_{<x>}$ is Cohen-Macaulay of dimension 2, A' is the completion of A and u the inclusion. The morphism $v : B \to A'$ will be given by an approximation, polynomials up to a given bound. We obtain the following algorithms (which will be implemented in SINGULAR as a library). The algorithm prepareDesingularization corresponds to Lemma 1 in the proof of Theorem 2.

Algorithm 1 `prepareDesingularization`

Input: $A := k[x]_{(x)}/J$ given by $J = (h_1,\dots,h_p) \subseteq k[x], x = (x_1,\dots,x_t), k$ a field
$B := A[Y]/I$ given by $I = (g_1,\dots,g_l) \subseteq k[x,Y], Y = (Y_1,\dots,Y_n)$
and $y' = (y'_1,\dots,y'_n) \in k[x]^n$ such that $H_{B/A}(y')$ is zero-dimensional

Output: $B := A[Y]/I$ given by $I = (g_1,\dots,g_l) \subseteq k[x,Y], Y = (Y_1,\dots,Y_n), y' = (y'_1,\dots,y'_n) \in k[x]^n$, $f, f' \in I$ and d, d' a regular sequence in A, $d \in ((f):I)\Delta_f$ resp. $d' \in ((f'):I)\Delta_{f'}$, such that $(I/I^2)_d$ resp. $(I/I^2)_{d'}$ are free B_d resp. $B_{d'}$ modules.

1: **compute** $H_{B/A} = (b_1,\dots,b_q)_B$ and $H_{B/A} \cap A$
2: **if** $\dim A/H_{B/A} \cap A = 0$ **then**
3: **choose** $\gamma, \gamma' \in H_{B/A} \cap A$, a regular sequence in A
4: **else**
5: **choose** $\gamma, \gamma' \in H_{B/A}(y')$, a regular sequence in A
write
$\gamma \equiv \sum_{i=1}^{q} b_i(y')y'_{i+n}$ modulo (γ^t, γ'^t), $\gamma' \equiv \sum_{i=1}^{q} b_i(y')y'_{i+n+q}$ modulo (γ^t, γ'^t) for some t and $y'_j \in k[x]$
6: $g_{l+1} := -\gamma + \sum_{i=1}^{q} b_i Y_{i+n}$, $g_{l+2} := -\gamma' + \sum_{i=1}^{q} b_i Y_{i+n+q}$,
$Y := (Y_1,\dots,Y_{n+2q})$; $y' := (y'_1,\dots,y'_{n+2q})$; $I := (g_1,\dots,g_{l+2})$; $l := l+2$; $n := n+2q$; $B := A[Y]/I$.
7: $B := S_B(I/I^2)$, y' trivially extended
8: **write**
$B := A[Y]/I$, $n := |Y|$, $Y := Y,Z$, $Z = (Z_1,\dots,Z_n)$, $I := (I,Z)$, $B := A[Y]/I$, y' trivially extended
9: **compute** $f = (f_1,\dots,f_r)$, and $f' = (f'_1,\dots,f'_{r'})$ **such that**
a power d of γ, resp. d' of γ' is in $((f):I)\Delta_f$, resp. in $((f'):I)\Delta'_f$
10: **return** B,y',f,f',d,d'

The next algorithm corresponds to Lemma 2 in the proof of Theorem 2.

Algorithm 2 `reductionToDimensionOne`

Input: $A := k[x]_{(x)}/J$ given by $J = (h_1, \dots, h_p) \subseteq k[x], x = (x_1, \dots, x_t), k$ a field
$B := A[Y]/I$ given by $I = (g_1, \dots, g_l) \subseteq k[x,Y], Y = (Y_1, \dots, Y_n)$, $y' = (y'_1, \dots, y'_n) \in k[x]^n$,$f' = (f'_1, \dots, f'_{r'})$, d', $d \in A$ and $\{H'_i, L'_i\}$ such that $d' \equiv P' = \sum_{i=1}^{q'} \det(H'_i) L'_i$ modulo I

Output: $D := (A[Y', T]/(I, g, h))_{ss's''}$ given by $I, g, h, s, s', s" \in k[x, Y', T], Y' := (Y'_1, \dots, Y'_n)$;

write
  $P'(y') = d's$ modulo d^3 for $s \in A$, $s \equiv 1$ modulo d'
for $i = 1$ to q' **do**
  compute G'_i the adjoint matrix of H'_i and $G_i = L'_i G'_i$
$h := s(Y - y') - d' \sum_i G_i(y') T_i$, $T_i = (T_1, \dots, T_{r'}, T_{i,r'+1}, \dots, T_{i,n})$
$p' := \max_i \{\deg f'_i\}$
write
  $s^{p'} f'(Y) - s^{p'} f'(y') = \sum_j s^{p'-1} d' \partial f'/\partial Y(y') \sum_i G_{ij}(y') T_{ij} + d'^2 Q$ modulo h
write $f'(y') = d'^2 b'$ modulo d^3
for $i = 1$ to r' **do**
  $g_i := s^{p'} b'_i + s^{p'} T_i + Q_i$
compute s' the r'-minor of $(\partial g/\partial T)$ given by the first r' variables and s'' **such that**
  $P(y' + s^{-1} d' \sum_i G_i(y') T) = d' s''$ modulo d^3
$D := (A[Y', T]/(I, g, h))_{ss's''}$; $Y' := (Y'_1, \dots, Y'_n)$; $g := g(Y')$; $I := I(Y')$; $h := h(Y')$
return D

Algorithm 3 `NeronDesingularization`

Input: $N \in \mathbb{Z}_{>0}$ a bound
$A := k[x]_{(x)}/J$ given by $J = (h_1, \dots, h_p) \subseteq k[x], x = (x_1, \dots, x_t), k$ a field
$B := A[Y]/I$ given by $I = (g_1, \dots, g_l) \subseteq k[x, Y], Y = (Y_1, \dots, Y_n)$ $v : B \to A' \subseteq K[[x]]/JK[[x]]$ an A-morphism given by $y' = (y'_1, \dots, y'_n) \in k[x]^n$, approximations modulo $(x)^N$ of $v(Y)$.

Output: A Neron desingularization of $v : B \to A'$ or the message "the bound is too small"

(B, y', f, f', d, d'):=prepareDesingularization(A, B, y')
if $(d^3, d'^3) \not\supseteq (x)^N$ **then**
  return "the bound is too small"
choose r-minors M_i (resp. r'-minors M'_i) of $(\partial f/\partial Y)$, (resp. $(\partial f'/\partial Y)$) and $L_i \in ((f) : I)$, (resp. $L'_i \in ((f') : I)$) **such that**
  for $P = \sum_i M_i L_i$ (resp. $P' = \sum_i M'_i L'_i$), $d \equiv P$ modulo I (resp. $d' \equiv P'$ modulo I)
complete the Jacobian matrix $(\partial f/\partial Y)$ (resp. $(\partial f'/\partial Y)$) by $(n-r)$ (resp. $(n-r')$ rows of $0, 1$ to obtain square matrices H_i (resp. H'_i) **such that**
  $\det H_i = M_i$ (resp. $\det H'_i = M'_i$)
D:=reductionToDimensionOne$(A, B, y', f', d', d, \{H'_i, L'_i\})$
write $P(Y') = d\tilde{s}$; $\tilde{s} \equiv 1$ modulo d
compute $\tilde{G}'_i$ the adjoint matrix of H_i and $\tilde{G}_i = L_i \tilde{G}'_i$
$\tilde{h} := \tilde{s}(Y - Y') - d \sum_{i=1}^{q} \tilde{G}_i \tilde{T}_i$, $\tilde{T}_i = (\tilde{T}_1, \dots, \tilde{T}_r, \tilde{T}_{i,r+1}, \dots, \tilde{T}_{i,n})$
$p := \max_i \{\deg f_i\}$
write $\tilde{s}^p f(Y) - \tilde{s}^p f(Y') = \sum_j \tilde{s}^{p-1} d \partial f/\partial Y(Y') \sum_i \tilde{G}_{ij}(Y') \tilde{T}_{ij} + d'^2 \tilde{Q}$ modulo $\tilde{h}$ and
write $f(Y') = d^2 \tilde{b}$, $\tilde{b} \in dD^r$
for $i = 1$ to r **do**
  $\tilde{g}_i := \tilde{s}^p \tilde{b}_i + \tilde{s}^p \tilde{T}_i + \tilde{Q}_i$
compute $\tilde{s}'$ the $r \times r$-minors of $(\partial \tilde{g}/\partial \tilde{T})$ given by the first r variables of $\tilde{T}$
compute $\tilde{s}''$ **such that**
  $P(Y' + \tilde{s}^{-1} d \sum_i \tilde{G}_i(Y') \tilde{T}) = d\tilde{s}''$
return $D[Y, \tilde{T}]/(I, \tilde{g}, \tilde{h})_{\tilde{s}\tilde{s}'\tilde{s}''}$

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