AN ALGORITHM FOR PRIMARY DECOMPOSITION IN POLYNOMIAL RINGS OVER THE INTEGERS

GERHARD PFISTER, AFSHAN SADIQ, AND STEFAN STEIDEL

ABSTRACT. We present an algorithm to compute a primary decomposition of an ideal in a polynomial ring over the integers. For this purpose we use algorithms for primary decomposition in polynomial rings over the rationals resp. over finite fields, and the idea of Shimoyama–Yokoyama resp. Eisenbud–Hunecke–Vasconcelos to extract primary ideals from pseudo–primary ideals. A parallelized version of the algorithm is implemented in SINGULAR. Examples and timings are given at the end of the article.

1. INTRODUCTION

Algorithms for primary decomposition in $\mathbb{Z}[x_1, \ldots, x_n]$ have been developed by Seidenberg (cf. [Se]) and Ayoub (cf. [A]). Within this article we present a slightly different approach which seems to be much more efficient. It uses primary decomposition in $\mathbb{Q}[x_1, \ldots, x_n]$ resp. $\mathbb{F}_p[x_1, \ldots, x_n]$ as well as the computation of the minimal associated primes of an ideal in $\mathbb{F}_p[x_1, \ldots, x_n]$[1], pseudo–primary decomposition[2], and the extraction of the primary components.

Let $x = \{x_1, \ldots, x_n\}$ always denote a set of indeterminates and let $I \subseteq \mathbb{Z}[x]$ be an ideal. We use the following known facts from commutative algebra for our algorithm:

(1) If $I \cap \mathbb{Z} = \langle 0 \rangle$, then there exists an $h \in \mathbb{Z}$ such that $I : h = I\mathbb{Q}[x] \cap \mathbb{Z}[x]$ and $I = (I : h) \cap \langle I, h \rangle$ (cf. [Se], Theorem 2).

(2) If $I \cap \mathbb{Z} = \langle 0 \rangle$ and $I\mathbb{Q}[x] = \overline{Q}_1 \cap \ldots \cap \overline{Q}_s$ is an irredundant primary decomposition with $\overline{P}_i = \sqrt{\overline{Q}_i}$, then $I\mathbb{Q}[x] \cap \mathbb{Z}[x] = (\overline{Q}_1 \cap \mathbb{Z}[x]) \cap \ldots \cap (\overline{Q}_s \cap \mathbb{Z}[x])$ is an irredundant primary decomposition and $\overline{P}_i \cap \mathbb{Z}[x] = \sqrt{\overline{Q}_i \cap \mathbb{Z}[x]}$ (cf. [Se], Theorem 3).

(3) If $I \cap \mathbb{Z} = \langle q \rangle$ such that $q \neq 0$ and $q = p_1^{\nu_1} \cdots p_r^{\nu_r}$ with $p_1, \ldots, p_r$ pairwise different primes, then $I = \bigcap_{i=1}^{r} \langle I, p_i^{\nu_i} \rangle$.

(4) If $I \cap \mathbb{Z} = \langle p^\nu \rangle$ for some prime p and $\overline{P}_1, \ldots, \overline{P}_s$ are the minimal associated primes of $I\mathbb{F}_p[x]$, then the liftings $P_1, \ldots, P_s$ to $\mathbb{Z}[x]$ are the minimal associated primes of I.

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[1]One can choose one of the modern algorithms, cf. [DGP], [EHV], [GTZ], [SY].

[2]An ideal is called *pseudo–primary* if its radical is prime, cf. [EHV], [SY].

The following result can easily be adapted to $\mathbb{Z}[x]$.

(5) If P is a minimal associated prime of I, then $I + P^m$ is a pseudo–primary component of I for a suitable $m \in \mathbb{N}$, i.e. the equidimensional part of $I + P^m$ is the primary component of I associated to P (cf. [EHV]).

Alternatively we can compute a separator[3] s of I w.r.t. P and obtain by $I : s^\infty$ a pseudo–primary component of I (cf. [SY]).

(6) If $Q_1, \ldots, Q_s$ are the primary components of I associated to the minimal associated prime ideals and $J = Q_1 \cap \ldots \cap Q_s$, then there exists a natural number m such that $I = J \cap (I + (I : J)^m)$.

Consequently, by applying (1)–(6), we can reduce the computation of the primary decomposition in $\mathbb{Z}[x]$ to the computation of the primary decomposition in $\mathbb{Q}[x]$, the computation of the minimal associated primes in $\mathbb{F}_p[x]$, and the extraction of the primary components in $\mathbb{Z}[x]$. In this connection, the extraction has to be generalized to polynomial rings over principal ideal domains (cf. Lemma 2.10). In section 2 we state the results used in the algorithm, whereupon in section 3 we explain our algorithm which has been implemented in Singular in a parallel version. Finally we give some examples and the corresponding timings in section 4.

2. Basic definitions and results

Definition 2.1. Let $I \subseteq \mathbb{Z}[x]$ be an ideal and $>$ be a monomial ordering on $\mathbb{Z}[x]$. A subset $G \subseteq I$ is called *strong Gröbner basis* of I w.r.t. $>$ if for all $f \in I$ there exists a $g \in G$ such that $\mathrm{LT}(g) | \mathrm{LT}(f)$.[4]

Lemma 2.2. *Let $G = \{g_1, \ldots, g_k\} \subseteq \mathbb{Z}[x]$ and $I = \langle G \rangle_{\mathbb{Z}[x]}$. Assume that $I \cap \mathbb{Z} = \langle 0 \rangle$ and G is a Gröbner basis of $I\mathbb{Q}[x]$ w.r.t. some ordering. Let h be the least common multiple of the leading coefficients of $g_1, \ldots, g_k$, i.e. $h = \mathrm{lcm}(\mathrm{LC}(g_1), \ldots, \mathrm{LC}(g_k))$. Then $I\mathbb{Q}[x] \cap \mathbb{Z}[x] = I : h^\infty$. Moreover, if $I : h^\infty = I : h^m$ for some natural number m, then $I = (I : h^m) \cap \langle I, h^m \rangle$.*

The proof of Lemma 2.2 is similar to the corresponding proof for polynomial rings over a field (cf. [GP], Proposition 4.3.1).

Remark 2.3. The saturation $I : h^\infty$ can be computed in $\mathbb{Z}[x]$ similarly to the case of a polynomial ring over a field by computing a Gröbner basis of $\langle I, Th - 1 \rangle_{\mathbb{Z}[x,T]}$ w.r.t. an elimination ordering for T:

$$I : h^\infty = \langle I, Th - 1 \rangle_{\mathbb{Z}[x,T]} \cap \mathbb{Z}[x].$$

A natural number m satisfying $I : h^\infty = I : h^m$ can be found by computing the normal form of $h^l g$ w.r.t. I for each generator g of $I : h^\infty$ and increasing $l \in \mathbb{N}$.

The following four Lemmata are well–known in commutative algebra.

Lemma 2.4. *Let $I \subseteq \mathbb{Z}[x]$ be an ideal with $I \cap \mathbb{Z} = \langle 0 \rangle$. Let $I\mathbb{Q}[x] = \overline{Q}_1 \cap \ldots \cap \overline{Q}_s$ be an irredundant primary decomposition with associated primes $\overline{P}_1, \ldots, \overline{P}_s$ and $Q_i = \overline{Q}_i \cap \mathbb{Z}[x]$ resp. $P_i = \overline{P}_i \cap \mathbb{Z}[x]$ for $i = 1, \ldots, s$. Then $I\mathbb{Q}[x] \cap \mathbb{Z}[x] = Q_1 \cap \ldots \cap Q_s$ is an irredundant primary decomposition with associated primes $P_1, \ldots, P_s$.*

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[3]We call s a separator of I w.r.t. P if $s \notin P$ and s is contained in any other minimal associated prime of I.

[4]We use the notations of [GP] for the basics of Gröbner bases. Especially $\mathrm{LT}(f)$ denotes the leading term (leading monomial with leading coefficient) of f w.r.t. the ordering $>$.

Lemma 2.5. *Let $I \subseteq \mathbb{Z}[x]$ be an ideal with $I \cap \mathbb{Z} = \langle q \rangle$ and $q = p_1^{\nu_1} \cdot \ldots \cdot p_r^{\nu_r}$ be the prime factorization. Then $I = \bigcap_{i=1}^r \langle I, p_i^{\nu_i} \rangle$.*

Lemma 2.6. *Let $I \subseteq \mathbb{Z}[x]$ be an ideal such that $I \cap \mathbb{Z} = \langle p^\nu \rangle$ for some prime number p . Moreover, let $\operatorname{minAss}(I\mathbb{F}_p[x]) = \{\overline{P}_1, \ldots, \overline{P}_s\}$ be the set of minimal associated prime ideals of $I\mathbb{F}_p[x]$ and $P_1, \ldots, P_s$ be the canonical liftings to $\mathbb{Z}[x]$. Then $\operatorname{minAss}(I) = \{P_1, \ldots, P_s\}$ is the set of minimal associated primes of I.*

If $\nu = 1$ let $I\mathbb{F}_p[x] = \overline{Q}_1 \cap \ldots \cap \overline{Q}_s$ be an irredundant primary decomposition with associated primes $\overline{P}_1, \ldots, \overline{P}_s$ and $Q_1, \ldots, Q_s, P_1, \ldots, P_s$ be the canonical liftings to $\mathbb{Z}[x]$. Then $I = Q_1 \cap \ldots \cap Q_s$ is an irredundant primary decomposition with associated primes $P_1, \ldots, P_s$.

Lemma 2.7 (cf. [EHV])**.** *Let $I \subseteq \mathbb{Z}[x]$ be an ideal and P a minimal associated prime. Then there exists a natural number m such that $I + P^m$ is a pseudo–primary component of I. For any m let Q_m be the equidimensional part of $I + P^m$. Q_m is a primary component of I with associated prime P if the canonical map $((I\mathbb{Z}[x]_P \cap \mathbb{Z}[x]) : P^\infty)/(I\mathbb{Z}[x]_P \cap \mathbb{Z}[x]) \longrightarrow \mathbb{Z}[x]/Q_m$ is injective.*

Example 2.8. Consider in $\mathbb{Z}[x, y]$

$$I = \langle 9, x + 3 \rangle \cap \langle 9, y + 3 \rangle,\ P = \langle 3, x \rangle,\ m = 20.$$

Then it holds $I + P^m = \langle 9, xy + 3x + 3y, 3x^{19}, x^{20} \rangle$ which is not equidimensional. Now we have $Q_m = \langle 9, x + 3 \rangle$, and the canonical map defined above is the identity map $\mathbb{Z}[x, y]/\langle 9, x + 3 \rangle \longrightarrow \mathbb{Z}[x, y]/\langle 9, x + 3 \rangle$.

Lemma 2.9 (cf. [SY])**.** *Let $I \subseteq \mathbb{Z}[x]$ be an ideal, P a minimal associated prime and $s \notin P$ a separator, i.e. s is contained in any other associated prime. Then $I : s^\infty$ is a pseudo–primary component of I, and s can be chosen as*

$$\prod_{\substack{Q \neq P \\ Q \in \operatorname{minAss}(I)}} s_Q$$

where s_Q is an element of a Gröbner basis of Q which is not in P.

Lemma 2.10 (Extraction Lemma)**.** *Let $I = Q \cap J$ be pseudo–primary with $\sqrt{I} = P$ and Q be P–primary with $\operatorname{ht}(Q) < \operatorname{ht}(J)$. Let $P \cap \mathbb{Z} = \langle p \rangle$ for some prime p and $u \subset x$ be a maximal independent set of variables for $\overline{P} = P\mathbb{F}_p[x]$. Let $R := \mathbb{Z}[u]_{\langle p \rangle}$, then the following holds:*

1. *$IR[x \smallsetminus u] \cap \mathbb{Z}[x] = Q$*
2. *Let G be a strong Gröbner basis of I w.r.t. a block ordering satisfying $x \smallsetminus u \gg u$. Then G is a strong Gröbner basis of $IR[x \smallsetminus u]$ w.r.t. the induced ordering for the variables $x \smallsetminus u$.*
3. *Let $G = \{g_1, \ldots, g_k\}$ be as in (2), $\operatorname{LT}_{R[x \smallsetminus u]}(g_i) = p^{\nu_i} a_i (x \smallsetminus u)^{\beta_i}$ with $a_i \in \mathbb{Z}[u] \smallsetminus \langle p \rangle$ for $i = 1, \ldots, k$, and $h = \operatorname{lcm}(a_1, \ldots, a_k)$. Then $IR[x \smallsetminus u] \cap \mathbb{Z}[x] = I : h^\infty$.*

Proof.

1. Let $K = \sqrt{J}$ and $\overline{K} = K\mathbb{F}_p[x]$ then $\overline{K} \supsetneq \overline{P} = P\mathbb{F}_p$. This implies that $\overline{K} \cap \mathbb{F}_p[u] \neq \langle 0 \rangle$ since $u \subset x$ is maximally independent for $\overline{P}$ and therefore $K \cap (\mathbb{Z}[u] \smallsetminus \langle p \rangle) \neq \emptyset$. Thus it holds $JR[x \smallsetminus u] = R[x \smallsetminus u]$. Finally, because Q is primary, we obtain $IR[x \smallsetminus u] \cap \mathbb{Z}[x] = QR[x \smallsetminus u] \cap \mathbb{Z}[x] = Q$.

(2) Let $f \in IR[x \smallsetminus u]$ and choose $s \in \mathbb{Z}[u] \smallsetminus \langle p \rangle$ such that $sf \in I$. Since G is a strong Gröbner basis of I there exists a $g \in G$ such that $\mathrm{LT}_{\mathbb{Z}[x]}(g) \mid \mathrm{LT}_{\mathbb{Z}[x]}(sf)$. As a polynomial in $x \smallsetminus u$ with coefficients in R, the element sf can be written as $sf = p^\nu a(x \smallsetminus u)^\alpha +$ (terms in $x \smallsetminus u$ of smaller order) with $a \in \mathbb{Z}[u] \smallsetminus \langle p \rangle$. If p^τ is the maximal power of p dividing the leading coefficient $\mathrm{LC}_{\mathbb{Z}[x]}(g)$ of g then $\tau \leq \nu$ since $\mathrm{LT}_{\mathbb{Z}[x]}(sf) = p^\nu \,\mathrm{LT}_{\mathbb{Z}[x]}(a)(x \smallsetminus u)^\alpha$. Now we can write g as an element of $R[x \smallsetminus u]$ w.r.t. the corresponding ordering, i.e. $g = p^\mu b(x \smallsetminus u)^\beta +$ (terms in $x \smallsetminus u$ of smaller order) with $b \in \mathbb{Z}[u] \smallsetminus \langle p \rangle$ and $\mu \leq \tau \leq \nu$. By definition we have $\mathrm{LT}_{R[x\smallsetminus u]}(g) = p^\mu b(x \smallsetminus u)^\beta$ resp. $\mathrm{LT}_{R[x\smallsetminus u]}(f) = p^\nu \frac{a}{s}(x \smallsetminus u)^\alpha$ and on the other hand it holds $\mathrm{LT}_{\mathbb{Z}[x]}(g) = p^\mu \,\mathrm{LT}_{\mathbb{Z}[x]}(b)(x \smallsetminus u)^\beta$ resp. $\mathrm{LT}_{\mathbb{Z}[x]}(sf) = p^\nu \,\mathrm{LT}_{\mathbb{Z}[x]}(a)(x \smallsetminus u)^\alpha$. Thus the assumption $\mathrm{LT}_{\mathbb{Z}[x]}(g) \mid \mathrm{LT}_{\mathbb{Z}[x]}(sf)$ implies $(x \smallsetminus u)^\beta \mid (x \smallsetminus u)^\alpha$ and consequently $\mathrm{LT}_{R[x\smallsetminus u]}(g) \mid \mathrm{LT}_{R[x\smallsetminus u]}(f)$. This proves (2).
(3) Follows from (2) similarly to the proof for fields (cf. [GP]).

□

The following Lemma is a consequence of the Lemma of Artin–Rees (cf. [GP]).

Lemma 2.11. *Let $I \subseteq \mathbb{Z}[x]$ be an ideal and J the intersection of all primary components of I associated to the minimal prime ideals of I. Then there exists a natural number m such that $I = J \cap (I + (I : J)^m)$.*

Notation 2.12. Given an ideal $I \subseteq \mathbb{Z}[x]$ we can always choose a finite set of polynomials $F_I = \{f_1, \dots, f_k\}$ such that $I = \langle F_I \rangle$ and we denote $F_I^{(m)} := \{f_1^m, \dots, f_k^m\}$ for $m \in \mathbb{N}$.

Corollary 2.13. *With the assumptions and notations of Lemma 2.11 there exists a natural number m such that $I = J \cap (I + \langle F_{I:J}^{(m)} \rangle)$.*

Proof. Due to Lemma 2.11 there exists an m such that $I = J \cap (I + (I : J)^m)$. Now we have $I \subseteq J \cap (I + \langle F_{I:J}^{(m)} \rangle) \subseteq J \cap (I + (I : J)^m) = I$ and therefore $I = J \cap (I + \langle F_{I:J}^{(m)} \rangle)$. □

Remark 2.14. The corollary is very important from a computational point of view because $\langle F_{I:J}^{(m)} \rangle$ has much less generators than $(I : J)^m$.

3. The algorithms

In this section we present the algorithm to compute a primary decomposition of an ideal in a polynomial ring over the integers by applying the results of section 2.

Algorithm 1 computes the primary decomposition of an ideal in $\mathbb{Z}[x]$[5] with the aid of algorithms 2 and 3 which we introduce subsequently in detail.

Corollary 3.1. *Algorithm 1 can easily be parallelized by computing - depending on the exponents ν_i for $i = 1, \dots, r$ - either the primary decomposition or the set of minimal associated primes in positive characteristic in parallel.*

[5]The corresponding procedures are implemented in Singular in the library `primdecZ.lib`.

Algorithm 1 PRIMDECZ

Input: $F_I = \{f_1, \ldots, f_k\}$, $I = \langle F_I \rangle_{\mathbb{Z}[x]}$, optional: a test ideal T.

Output: $L := \{(Q_1, P_1), \ldots, (Q_s, P_s)\}$, $I = Q_1 \cap \ldots \cap Q_s$ irredundant primary decomposition with $P_i = \sqrt{Q_i}$.

if T is not given in the input **then**
  $T := \langle 1 \rangle$;
$G :=$ strong Gröbner basis of I;
$q :=$ generator of $I \cap \mathbb{Z}$;[6]
if $q = 0$ **then**
  compute $h \in \mathbb{Z}$ such that $I : h = I\mathbb{Q}[x] \cap \mathbb{Z}[x]$;[7]
  compute $\overline{Q}_1, \ldots, \overline{Q}_s$, an irredundant primary decomposition of $I\mathbb{Q}[x]$ and $\overline{P}_i = \sqrt{\overline{Q}_i}$ the associated primes;
  compute $Q_i = \overline{Q}_i \cap \mathbb{Z}[x]$, $P_i = \overline{P}_i \cap \mathbb{Z}[x]$;[8]
  $L := \{(Q_1, P_1), \ldots, (Q_s, P_s)\}$;
  $M :=$ PRIMDECZ$(\langle I, h \rangle)$;
  remove redundant primary ideals from M;
  return $L \cup M$;
else
  compute $q = p_1^{\nu_1} \ldots p_r^{\nu_r}$, the prime factorization of q;
  for $i = 1, \ldots, r$ **do**
    if $\nu_i = 1$ **then**
      compute $\overline{L}_i = \{(\overline{Q}_1^{(i)}, \overline{P}_1^{(i)}), \ldots, (\overline{Q}_{s_i}^{(i)}, \overline{P}_{s_i}^{(i)})\}$, the primary decomposition of $I\mathbb{F}_{p_i}[x]$;
      $L_i := \{(Q_1^{(i)}, P_1^{(i)}), \ldots, (Q_{s_i}^{(i)}, P_{s_i}^{(i)})\}$, the lifting of $\overline{L}_i$ to $\mathbb{Z}[x]$;[9]
    else
      compute $\overline{A}_i = \{\overline{P}_1^{(i)}, \ldots, \overline{P}_{s_i}^{(i)}\}$, the set of minimal associated primes of $I\mathbb{F}_{p_i}[x]$ and independent sets of variables $\overline{u}_1^{(i)}, \ldots, \overline{u}_{s_i}^{(i)}$ for $\overline{P}_1^{(i)}, \ldots, \overline{P}_{s_i}^{(i)}$;
      $A_i := \{P_1^{(i)}, \ldots, P_{s_i}^{(i)}\}$, the lifting of $\overline{A}_i$ to $\mathbb{Z}[x]$;
      for $j = 1, \ldots, s_i$ **do**
        $Q_j^{(i)} :=$ EXTRACTZ$(I, A_i, P_j^{(i)}, \overline{u}_j^{(i)})$;
        $L_i := \{(Q_1^{(i)}, P_1^{(i)}), \ldots, (Q_{s_i}^{(i)}, P_{s_i}^{(i)})\}$;
  $L := L_1 \cup \ldots \cup L_r$;
  compute J, the intersection of all primary ideals in L and T;
  if $J = I$ **then**
    return L;
  compute $F_{I:J}$ such that $\langle F_{I:J} \rangle = I : J$;
  compute m such that $J \cap (I + \langle F_{I:J}^{(m)} \rangle) = I$;
  $M :=$ PRIMDECZ$(I + \langle F_{I:J}^{(m)} \rangle, J)$;
  return $L \cup M$;

[6] q is either 0 or the unique element in G of degree 0.

[7] h is a suitable power of the least common multiple of all leading coefficients of elements in G, cf. Lemma 2.2.

[8] The computation of Q_i resp. P_i is based on Lemma 2.4.

[9] If $I = \langle F_I \rangle \subseteq \mathbb{F}_p[x]$ then its lifting is obtained by $\langle p, F_I \rangle$ with the canonical lifting of F_I.

The algorithm to compute the separators is based on Lemma 2.9.

Algorithm 2 SEPARATORSZ

Input: B a list of prime ideals generated by a Gröbner basis w.r.t. some ordering, not contained in each other, $P \in B$.
Output: Polynomial s such that $s \notin P$, $s \in Q$ for all $Q \in B \backslash \{P\}$.

for $Q \in B \backslash \{P\}$ **do**
  choose s_Q in the Gröbner basis of Q such that $s_Q \notin P$;
return $\prod_{Q \in B \backslash \{P\}} s_Q$;

The algorithm to extract the primary component from the pseudo–primary component is based on the Extraction Lemma 2.10.

Algorithm 3 EXTRACTZ

Input: $I \subseteq \mathbb{Z}[x]$ an ideal, B the list of minimal associated primes of I, $P \in B$ with $P \cap \mathbb{Z} = \langle p \rangle$ for some prime p, $u \subset x$ an independent set of variables for $P\mathbb{F}_p[x]$.
Output: The primary component Q of I associated to P.

$s :=$ SEPERATORSZ(P, B);
$I = I : s^\infty$;
compute $G = \{g_1, \ldots, g_k\}$, a strong Gröbner basis of I w.r.t. a block ordering satisfying $x \smallsetminus u \gg u$;
compute $\{a_1, \ldots, a_k\}$ such that $\mathrm{LC}_{\mathbb{Z}[u]_{\langle p \rangle}[x \smallsetminus u]}(g_i) = p^{\nu_i} \cdot a_i$ with $a_i \in \mathbb{Z}[u] \smallsetminus \langle p \rangle$;
compute $h = \mathrm{lcm}(a_1, \ldots, a_k)$, the least common multiple of $a_1, \ldots, a_k$;
return $I : h^\infty$;

Example 3.2. Consider $I = \langle 9, 3x, 3y \rangle$, $P = \langle 3 \rangle$, $u = \{x, y\}$ and $B = \{P\}$ in $\mathbb{Z}[x, y]$. Then we obtain $s = 1$, $h = xy$ and thus $I : h^\infty = \langle 3 \rangle$.

4. Examples and timings

In this section we provide examples on which we time the algorithm `primdecZ` (cf. section 3) and its parallelization implemented in Singular. Timings are conducted by using the 32-bit version of Singular 3-1-1 on an Intel® Xeon® X5460 with 4 CPUs, 3.16 GHz each, 64 GB RAM under the Gentoo Linux operating system. All examples are chosen from The SymbolicData Project (cf. [G]).

We choose the following examples:

Example 4.1. Coefficients: `integer`, ordering: `dp`[10], `Gerdt-93a.xml` (cf. [G]) considered with another integer generator $2 \cdot 3 \cdot 5 \cdot 13 \cdot 17 \cdot 181$.

Example 4.2. Coefficients: `integer`, ordering: `dp`, `Gerdt-93a.xml` (cf. [G]) considered with another integer generator $2 \cdot 3 \cdot 5 \cdot 13 \cdot 17 \cdot 31 \cdot 181$.

Example 4.3. Coefficients: `integer`, ordering: `dp`, `Gerdt-93a.xml` (cf. [G]) considered with another integer generator $2 \cdot 3 \cdot 5 \cdot 13 \cdot 17 \cdot 31 \cdot 37 \cdot 181$.

[10]For definitions of the orderings cf. [GP].

Example 4.4. Coefficients: `integer`, ordering: `dp`, `Steidel_6.xml` (cf. [ES]) considered with another integer generator $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$.

Example 4.5. Coefficients: `integer`, ordering: `dp`, `Gonnet-83.xml` (cf. [BGK]) considered with another integer generator $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$.

Table 1 summarizes the results where `primdecZ`$^*(n)$ denotes the parallelized version of the algorithm using n processes. All timings are given in seconds.

Example	`primdecZ`	`primdecZ`$^*(2)$	`primdecZ`$^*(3)$	`primdecZ`$^*(4)$
4.1	344	229	208	157
4.2	426	309	236	215
4.3	508	331	258	248
4.4	11	7	6	5
4.5	13	9	8	6

TABLE 1. Total running times for computing a primary decomposition of the considered examples via `primdecZ` and its parallelized variant `primdecZ`$^*(n)$ for $n = 2, 3, 4$.

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Gerhard Pfister, Department of Mathematics, University of Kaiserslautern, Erwin-Schrödinger-Str., 67663 Kaiserslautern, Germany
E-mail address: pfister@mathematik.uni-kl.de
URL: http://www.mathematik.uni-kl.de/∼pfister

Afshan Sadiq, Abdus Salam School of Mathematical Sciences, GC University, Lahore, 68-B, New Muslim Town, Lahore 54600, Pakistan
E-mail address: afshanatiq@gmail.com

Stefan Steidel, Department of Mathematics, University of Kaiserslautern, Erwin-Schrödinger-Str., 67663 Kaiserslautern, Germany
E-mail address: steidel@mathematik.uni-kl.de
URL: http://www.mathematik.uni-kl.de/∼steidel