

# ABOUT THE COMPUTATION OF THE SIGNATURE OF SURFACE SINGULARITIES $z^N + g(x, y) = 0$

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ABSTRACT. In this article we describe our experiences with a parallel SINGULAR – implementation of the signature of a surface singularity defined by  $z^N + g(x, y) = 0$ .

## 1. INTRODUCTION

Let  $g \in \mathbb{C}[x, y]$  define an isolated curve singularity at  $0 \in \mathbb{C}^2$  and  $f := z^N + g(x, y)$  for  $N \geq 2$ . The zero-set  $V := V(f) \subseteq \mathbb{C}^3$  of  $f$  has an isolated singularity at 0. For a small  $\varepsilon > 0$  let  $V_\varepsilon := V(f - \varepsilon) \subseteq \mathbb{C}^3$  be the Milnor fibre of  $(V, 0)$  and  $s : H_2(V_\varepsilon, \mathbb{R}) \times H_2(V_\varepsilon, \mathbb{R}) \rightarrow \mathbb{R}$  be the intersection form (cf. [1], [11]).  $H_2(V_\varepsilon, \mathbb{R})$  is a  $\mu$ -dimensional  $\mathbb{R}$ -vector space,  $\mu$  the Milnor number of  $(V, 0)$  (cf. [1], [7], [8], [9]), and  $s$  is a symmetric bilinear form. Let  $\sigma(f)$  be the signature of  $s$ , called the signature of the surface singularity  $(V, 0)$ . Formulas to compute the signature are given by Némethi (cf. [14], [15]) and van Doorn, Steenbrink (cf. [4]). We have implemented three approaches in SINGULAR (cf. [3]) using Puiseux expansions, the resolution of singularities and the spectral pairs of the singularity. In section 2 we will explain the different formulas to compute the signature, and finally we give examples and timings of our implementation in section 3.

## 2. THE SIGNATURE OF $(V, 0)$ IN TERMS OF $N$ AND $g$

The proofs of the following formulas (cf. Propositions 2.1, 2.2, 2.4 and 2.7) can be found in the corresponding papers of Némethi (cf. [14], [15]).

**2.1. Approach 1: Puiseux Pairs.** For the first approach assume that  $(V(g), 0) \subseteq (\mathbb{C}^2, 0)$  is the germ of an irreducible curve singularity. Let  $(m_1, n_1), \dots, (m_\ell, n_\ell)$  be the *Puiseux pairs* of  $g$  and define a sequence  $\{a_i\}_{i=1, \dots, \ell}$  by

$$a_1 = m_1 \quad \text{and} \quad a_{i+1} = m_{i+1} - n_{i+1} \cdot (m_i - n_i \cdot a_i).$$

Moreover, we set  $d_\ell = 1$  and  $d_i = \gcd(N, n_{i+1} \cdots n_\ell)$  for  $1 \leq i < \ell$ .

**Proposition 2.1.**  $\sigma(f) = \sum_{i=1}^{\ell} d_i \cdot \sigma(x^{a_i} + y^{n_i} + z^{N/d_i})$ .

The signature of a Brieskorn polynomial  $x^{c_1} + y^{c_2} + z^{c_3}$  can be computed combinatorially. Let  $S_t = \#\{(k_1, k_2, k_3) \in \mathbb{Z}^3 \mid 1 \leq k_j \leq a_j - 1, t < \sum_{j=1}^3 \frac{k_j}{a_j} < t + 1\}$  for  $t \in \mathbb{N}_0$ .

**Proposition 2.2.**  $\sigma(x^{c_1} + y^{c_2} + z^{c_3}) = S_0 - S_1 + S_2$ .

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*Remark 2.3.* The SINGULAR implementation of the first approach bases on the procedure `invariants` (cf. library `hnoether.lib`) to obtain the Puiseux pairs of  $g$ . The command `list L = invariants(g)`<sup>1</sup> provides a list  $L$  in which the third and fourth entry contain the necessary data for our application. The required combinatorics of Proposition 2.1 resp. 2.2 had to be implemented in spring 2011.

**2.2. Approach 2: Resolution.** In the following  $(V(g), 0)$  doesn't need to be irreducible. For the second approach we use the *resolution* of the singularity  $(V(g), 0)$ . Let  $\mathcal{V} = \mathcal{W} \amalg \mathcal{A}$  be the vertices of the resolution graph,  $\mathcal{W}$  the vertices corresponding to the exceptional divisors and  $\mathcal{A}$  the vertices corresponding to the resolved branches. Let  $\mathcal{E} = \{(v_1, v_2) \mid v_1, v_2 \in \mathcal{V}\}$  be the set of edges of the resolution graph. Let  $\{m_v\}_{v \in \mathcal{W}}$  be the sequence of total multiplicities and set  $m_a = 1$  if  $a \in \mathcal{A}$ . For  $w \in \mathcal{W}$  let  $M_w = \gcd(m_v \mid v \in \mathcal{V}_w \cup \{w\})$  and for  $e = (v_1, v_2) \in \mathcal{E}$  let  $m_e = \gcd(m_{v_1}, m_{v_2})$ .

**Proposition 2.4.**  $\sigma(z^N + g) = \eta(g, N) - N \cdot \eta(g, 1)$  and <sup>2</sup>

$$\begin{aligned} \eta(g, K) = & \#(\mathcal{A}) - 1 + \sum_{e \in \mathcal{E}} (\gcd(K, m_e) - 1) - \sum_{w \in \mathcal{W}} (\gcd(K, M_w) - 1) \\ & + 4 \cdot \sum_{\substack{w \in \mathcal{W} \\ w \text{ has more} \\ \text{than 2 edges}}} \sum_{\substack{v \in \mathcal{V} \\ (v, w) \in \mathcal{E}}} \sum_{k=1}^{m_w} \left( \left( \frac{k \cdot m_v}{m_w} \right) \right) \cdot \left( \left( \frac{k \cdot K}{m_w} \right) \right). \end{aligned}$$

Moreover, it holds  $\sum_{e \in \mathcal{E}} (\gcd(K, m_e) - 1) = \sum_{w \in \mathcal{W}} (\gcd(K, M_w) - 1)$  if  $(V(g), 0)$  is irreducible.

*Remark 2.5.* The SINGULAR implementation of the second approach bases on the procedure `totalmultiplicities` (cf. library `alexpoly.lib`) to obtain necessary information about the resolution of  $(V(g), 0)$ . The command `list L = totalmultiplicities(g)` provides a list  $L$  of the following format:  $L[1]$ : incidence matrix of the resolution graph,  $L[2]$ : list of the sequences of the total multiplicities corresponding to the branches,  $L[3]$ : list of the multiplicity sequences of the branches. The required combinatorics of Proposition 2.4 had to be implemented in spring 2011.

**2.3. Approach 3: Spectral Pairs.** The third approach uses the *spectral pairs*<sup>3</sup> of the singularity  $(V(g), 0)$ . Therefore let

$$Spp(g) = \sum_{(\alpha, w)} h_{\exp(-2\pi i \alpha)}^{1+[-\alpha], w+s_\alpha-1-[-\alpha]} \cdot (\alpha, w)$$

be the set of spectral pairs where  $s_\alpha = 0$  if  $\alpha \notin \mathbb{Z}$  and  $s_\alpha = 1$  if  $\alpha \in \mathbb{Z}$ , and  $Spp(g) = \sum_{(\alpha, w) \in Spp(g)} (\alpha)$  be the spectrum of  $g$ .

<sup>1</sup>The command `list L = invariants(g)`; returns a list  $L$  of the following format:  $L[1]$ : characteristic exponents,  $L[2]$ : generators of the semigroup,  $L[3]$ : first components of Puiseux pairs,  $L[4]$ : second components of Puiseux pairs,  $L[5]$ : degree of the conductor,  $L[6]$ : sequence of multiplicities.

<sup>2</sup>For  $x \in \mathbb{Q}$  we denote by  $\{x\}$  the fractional part of  $x$  and

$$\left( \left( x \right) \right) = \begin{cases} \{x\} - \frac{1}{2}, & \text{if } x \notin \mathbb{Z} \\ 0, & \text{if } x \in \mathbb{Z} \end{cases}.$$

The definition of the *eta-invariant*  $\eta(f, K)$  can be found in [12].

<sup>3</sup>A definition of *spectral pairs* can be found in [9].

*Remark 2.6.* Note that  $\alpha$  is a spectral number, i.e.  $\exp(-2\pi i\alpha)$  is an eigenvalue of the monodromy.  $(V(g), 0)$  is reducible if and only if 0 is a spectral number (cf. [4]). The spectral numbers are situated in the interval  $(-1, 1)$  and the spectrum is symmetric ( $\alpha$  is a spectral number if and only if  $-\alpha$  is a spectral number, cf. [9]). If the Newton polygon of  $g$  is non-degenerate the spectral pairs can be computed combinatorially using the Newton polygon (cf. [16], [19]). There is a formula to compute the spectral pairs from the data of the resolution (cf. [17]).

**Proposition 2.7.**  $\sigma(z^N + g) = \eta(g, N) - N \cdot \eta(g, 1)$  and

$$\eta(g, K) = \sum_{\substack{\alpha \neq 0, K\alpha \in \mathbb{Z} \\ (\alpha, 2) \in Spp(g)}} h_{\exp(-2\pi i\alpha)}^{11} - 2 \sum_{\substack{\alpha \geq 0, K\alpha \notin \mathbb{Z} \\ (\alpha, w) \in Spp(g)}} h_{\exp(-2\pi i\alpha)}^{1+[-\alpha], w+s_\alpha-1-[-\alpha]} (1 - 2\{K\alpha\}).$$

*Remark 2.8.* The SINGULAR implementation of the third approach bases on the procedure `sppairs` (cf. library `gmssing.lib`) to obtain the spectral pairs of  $(V(g), 0)$ . The command `list L = sppairs(g);` provides a list `L` of the following format: `L[1]`: set of spectral numbers  $\{\alpha_1, \dots, \alpha_r\}$ , `L[2]`: set of weights  $\{w_1, \dots, w_r\}$ , `L[3]`: set of multiplicities  $\{h_1, \dots, h_r\}$  such that  $Spp(g) = \sum_{j=1}^r h_j \cdot (\alpha_j, w_j)$ . The required combinatorics of Proposition 2.7 had to be implemented in spring 2011.

**2.4. Theoretical Comparison.** The topological type of a plane curve singularity defined by  $g(x, y) = 0$  can be described by the *Puiseux pairs* (Approach 1) of the branches and their *intersection multiplicities* or equivalently by discrete *invariants of the resolution* (Approach 2) There are combinatorial formulas to get from one description to another (cf. [7]). Moreover, the *spectral pairs* (Approach 3) which are topological invariants introduced by Arnold (cf. [1]) and Steenbrink (cf. [19]) can also be computed combinatorially from the resolution data (cf. [17]). Consequently, all of the three approaches to compute the signature of the surface  $z^N + g(x, y) = 0$  as described above are based on the knowledge of three different finite sets of invariants which are related in a combinatorial way. The essential difference concerning these approaches is the method to compute the set of the corresponding invariants.

The *spectral pairs* (Approach 3) can be computed from the mixed Hodge structure. This requires several standard basis computations of certain modules over local rings which is the bottleneck of this approach. The SINGULAR library `gmssing.lib` is designed for computing the mixed Hodge structure for hypersurface singularities of any dimension. This is one reason why computing the *spectral pairs* using this method is usually comparatively slow.

The *Puiseux pairs* (Approach 1) of the branches and their *intersection multiplicities* resp. the *resolution graph* (Approach 2) and the *multiplicity sequence* can be computed via Hamburger–Noether expansion (cf. [2]) or resolution of the curve singularities. Hence, both approaches are similarly time-consuming. They only need Gröbner basis computations if field extensions of  $\mathbb{Q}$  are necessary to compute the Hamburger–Noether expansion resp. the resolution. Anyway, the field extensions are the bottlenecks of these approaches.

### 3. EXAMPLES AND TIMINGS

In this section we provide examples on which we time the three approaches as described in section 2 to compute the signature of a surface singularity  $z^N + g(x, y) = 0$ . The corresponding procedures are implemented in SINGULAR in the library `surfacesignature.lib`. Timings are conducted by using SINGULAR 3-1-3 on an Intel® Xeon® X5460 with 4 CPUs, 3.16 GHz each, 64 GB RAM under the Gentoo Linux operating system.

*Example 3.1.* We consider the following polynomials:

$$\begin{aligned} g_1 &= x^{15} - 21x^{14} + 8x^{13}y - 6x^{13} - 16x^{12}y + 20x^{11}y^2 - x^{12} + 8x^{11}y - 36x^{10}y^2 \\ &\quad + 24x^9y^3 + 4x^9y^2 - 16x^8y^3 + 26x^7y^4 - 6x^6y^4 + 8x^5y^5 + 4x^3y^6 - y^8, \\ g_2 &= g_1^3 + x^{17}y^{17}, \\ g_3 &= (y^4 + 2x^3y^2 + x^6 + x^5y)^3 + x^{17}y^{17}, \\ g_4 &= g_1^5 + x^{20}y^{20}, \\ g_5 &= x^{10} + 7y^{10}, \\ g_6 &= x^{20} + 5y^{20}. \end{aligned}$$

The curve singularities in  $(\mathbb{C}^2, 0)$  defined by  $g_1$  resp.  $g_2$  resp.  $g_3$  are analytically irreducible with Puiseux pairs  $(3, 2), (7, 2), (15, 2)$  resp.  $(3, 2), (7, 2), (15, 2), (67, 3)$  resp.  $(3, 2), (7, 2), (113, 3)$ .

The curve singularities in  $(\mathbb{C}^2, 0)$  defined by  $g_4$  resp.  $g_5$  resp.  $g_6$  are analytically reducible since they are intersections of 40 resp. 10 resp. 20 lines at the origin. Consequently, the first approach is not applicable for these examples. Furthermore, the polynomials  $g_4, g_5, g_6$  are defined over  $\mathbb{Q}$ , whereas the resolution is only defined in field extensions of degree 8 resp. 10 resp. 20 over  $\mathbb{Q}$ .

Computations reveal the following results and corresponding timings which are summarized in Table 1. The symbol " $> 14$  h" indicates that the computation did not terminate after more than 14 hours. All timings are given in seconds (s).

$N$	$g_i$	$\sigma(z^N + g_i(x, y))$	Approach 1 [s]	Approach 2 [s]	Approach 3 [s]
5	$g_1$	-168	0	0	$> 14$ h
5	$g_2$	-1620	174	183	$> 14$ h
6	$g_3$	-940	2908	2912	$> 14$ h
5	$g_4$	-3192	-	19	$> 14$ h
6	$g_5$	-189	-	22	0
6	$g_6$	-779	-	14542	8

TABLE 1. Results and total running times for computing the signature of the surface singularity given by the considered examples (cf. Example 3.1) via all approaches as described in section 2.

In addition, we summarize the maximal memory allocated from system during the considered computations in Table 2. The memory consumption is given in Megabyte (MB).

$N$	$g_i$	Approach 1 [MB]	Approach 2 [MB]	Approach 3 [MB]
5	$g_1$	1	1	> 300
5	$g_2$	3442	3442	> 1100
6	$g_3$	7723	7723	> 1100
5	$g_4$	–	33	> 2300
6	$g_5$	–	8	4
6	$g_6$	–	156	72

TABLE 2. Maximal memory allocated from system while computing the signature of the surface singularity given by the considered examples (cf. Example 3.1) via all approaches as described in section 2.

*Remark 3.2* (Algorithmic Conclusion). Our experiments reveal that there exist examples where Approach 1 and Approach 2 are almost equivalent regarding time consumption resp. memory allocation, but Approach 3 is much slower (cf.  $g_1, g_2, g_3$ ). Furthermore, there exist examples where Approach 1 is not applicable, but Approach 2 consumes more time resp. allocates more memory than Approach 3 (cf.  $g_4$ ), and vice versa (cf.  $g_5, g_6$ ). Consequently, it is reasonable to summarize all approaches in one algorithm which computes the signature via every approach, if possible, in parallel such that the fastest approach wins and returns the result.

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