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Reduced Hilbert schemes for irreducible curve singularities

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Abstract

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We study the Hilbert scheme of zero-dimensional subschemes of Spec(A) for a one-dimensional local noetherian k-algebra. Its connected components M_τ parametrize the ideals of colength τ in A. The M_τ are embedded in a linear subspace M of a certain Grassmanian. We study the structure of M by its intersection with the Schubert cells. The case of rings A with monomial semigroup is specially treated.

1. Introduction

Let k be a field and let A be a local integral noetherian k-algebra of dimension one, such that its normalization $\bar{A}$ is a discrete valuation ring with residue field k. Let $v : \bar{A} \to \mathbb{N} \cup \{\infty\}$ be the corresponding discrete valuation. We let $\Gamma = v(A \backslash \{0\})$ denote the semigroup of A, $\bar{I}(n) = \{f \in \bar{A} \mid v(f) \geq n\}$ and $I(n) = \bar{I}(n) \cap A$ for $n \in \mathbb{N}$. Define $c := \min\{n \mid \bar{I}(n) \subseteq A\}$ and $m =$ multiplicity of A. Then $I(c) = \mathfrak{c}$ is the conductor ideal and $c = \dim_k(\bar{A}/\mathfrak{c})$. We define $\delta = \dim_k(\bar{A}/A) = \#(\mathbb{N} \backslash \Gamma)$. Then $\delta + 1 \leq c \leq 2\delta$, and $c = 2\delta$ if and only if A is Gorenstein (cf. [6, p. 80, Proposition 7]).

In this paper we study the Hilbert scheme of zero-dimensional subschemes of Spec(A), in other words, the space Hilb(A) of nonzero ideals in A. Note that this is a *punctual Hilbert scheme* in the sense of Iarrobino [5], and not a Hilbert scheme in the usual sense [4, Exp. 221]. We will show that its connected components $\mathcal{M}_\tau$ parametrize the ideals in A of colength τ.

In Section 2 we will construct a space $\mathcal{M}$ which is a special linear section of the Grassmannian of δ-dimensional subspaces of $\bar{A}/I(2\delta)$, and for each τ a closed embedding of $\mathcal{M}_\tau$ into $\mathcal{M}$, which is an isomorphism for $\tau \geq c$. We will also study the partition of $\mathcal{M}$ by its intersection with the Schubert cells corresponding to the natural flag.

In Section 3 we will investigate the structure of $\mathcal{M}$ in the case of certain semigroups, which we baptize *monomial semigroups* (see Definition 9). In that case the strata of the partition above appear to be isomorphic to affine spaces, and we determine their dimensions.

2. Construction of $\mathcal{M}$

Let A be the local ring of a reduced and irreducible curve singularity with semigroup Γ. We fix the following notation. Let I be a nonzero ideal in A. We let $\tau(I) = l(A/I)$, $t(I) = \min\{v(f) \mid f \in I\}$, $\Gamma(I) = \{v(f) \mid f \in I\backslash\{0\}\}$, $\Gamma_0(I) = \Gamma(I) - t(I)$ and $\delta(I) = \#(\mathbb{N}\backslash\Gamma_0(I))$. For $I = A$ we get 0, 0, Γ and δ respectively. We let $[a, b] := \{x \in \mathbb{N} \mid a \leq x \leq b\}$, $[a, \infty) = \{x \in \mathbb{N} \mid x \geq a\}$ an let $c(I)$ be the conductor of $\Gamma_0(I)$, i.e. $\max\{n \in \mathbb{N} \mid n - 1 \not\in \Gamma_0(I)\} = \min\{n \in \mathbb{N} \mid [n, \infty) \subseteq \Gamma_0(I)\}$. We have the following relations:

(i) $\tau(I) = t(I) + \delta(I) - \delta$,

(ii) $c(I) \leq \delta + \delta(I)$.

Indeed, let $f_1 \in I$ with $v(f_1) = t(I)$, then

$$\begin{aligned}\tau(I) + \delta = l(\bar{A}/I) = l(f_1^{-1}\bar{A}/f_1^{-1}I) &= l(f_1^{-1}\bar{A}/\bar{A}) + l(\bar{A}/f_1^{-1}I)\\ &= t(I) + \delta(I)\,.\end{aligned}$$

Moreover, $\Gamma_0(I)$ is a Γ-module, so $\gamma \in \Gamma$ implies that $c(I) - 1 - \gamma \not\in \Gamma_0(I)$. Hence

$$\#\{\gamma \in \Gamma \mid \gamma < c(I)\} \leq \delta(I)\,.$$

Definition 1. For A, Γ as above, we let $\mathcal{M}$ be the subset of Grass(δ, $\bar{A}/I(2\delta)$) consisting of the δ-dimensional linear subspaces which are A-submodules.

Observe that the group $1 + \mathfrak{m}_A$ acts by multiplication on $\bar{A}/I(2\delta)$ and hence on Grass(δ, $\bar{A}/I(2\delta)$). The fixed points of this action are exactly the points of $\mathcal{M}$. If we embed Grass(δ, $\bar{A}/I(2\delta)$) by its Plücker embedding, we get $\mathcal{M} =$ Grass(δ, $\bar{A}/I(2\delta)) \cap \mathcal{P}(V)$, where $V \subseteq \Lambda_k^\delta(\bar{A}/I(2\delta))$ is the linear subspace of fixed points under the action of $1 + \mathfrak{m}_A$ (this works because $1 + \mathfrak{m}_A$ acts by unipotent linear transformations on $\Lambda_k^\delta(\bar{A}/I(2\delta))$). We endow $\mathcal{M}$ with the reduced scheme structure. By construction, $\mathcal{M}$ is a projective scheme.

Definition 2. For each $\tau > 0$ let $\mathcal{M}_\tau$ be the set of ideals of codimension τ in A.

Observe that each ideal of codimension τ in A satisfies $I(\tau+2\delta)\subseteq I\subseteq I(\tau)$. Choose a uniformizing parameter t in $\bar{A}$. Then $I(2\delta)\subseteq t^{-\tau}I\subseteq \bar{A}$, and $l(\bar{A}/t^{-\tau}I)=l(\bar{A}/I)-\tau=\delta$. Define the map $\phi_\tau:\mathcal{M}_\tau\to\mathcal{M}$ by $\phi_\tau(I)=t^{-\tau}I/I(2\delta)$.

Theorem 3. *For all τ, the map ϕ_τ is injective and its image is a Zariski-closed subset of $\mathcal{M}$. For $\tau\geq c$ the map ϕ_τ is bijective.*

Proof. Let $x\in\mathcal{M}$ correspond to the A-submodule J of $\bar{A}$ of colength δ. Then $t^\tau J$ is an A-submodule of $\bar{A}$ of colength $\tau+\delta$, and $x\in\operatorname{Im}(\phi_\tau)\Leftrightarrow t^\tau J\subseteq A$. This is clearly the case for $\tau\geq c$, hence ϕ_τ is bijective in that case. In general, the condition $t^\tau J\subset A$ defines a Zariski-closed subset of $\mathcal{M}$. The injectivity of ϕ_τ is also clear. □

Examples. (1) Let $A=k[[t^2,t^3]]$. Then $\mathcal{M}_1$ is a point, whereas $\mathcal{M}_\tau\cong\mathcal{M}\cong\mathcal{P}^1(k)$ for $\tau\geq 2$.

(2) Let $A=k[[t^2,t^5]]$. Then $\mathcal{M}$ is defined inside $\mathcal{P}^5(k)$ by the Plücker relation $\pi_{12}\pi_{34}-\pi_{13}\pi_{24}+\pi_{14}\pi_{23}=0$ (which defines the Grassmannian) and the linear equations $\pi_{12}=\pi_{14}-\pi_{23}=0$. Thus $\mathcal{M}$ is a quadratic cone. $\mathcal{M}_1$ is the vertex of this cone, and $\mathcal{M}_2$ and $\mathcal{M}_3$ are two different lines through this vertex.

(3) Let $A=k[[t^3,t^4]]$. The image of $\mathcal{M}$ under the Plücker embedding of Grass(3, 6) in $\mathcal{P}^{20}(k)$ is defined by the Plücker relations defining the Grassmannian and the linear relations

$$\begin{aligned}\pi_{123}&=\pi_{124}=\pi_{125}=\pi_{126}=\pi_{134}=\pi_{135}=\pi_{136}=\pi_{234}=\pi_{235}\\&=\pi_{145}-\pi_{245}=\pi_{156}-\pi_{246}+\pi_{345}=0\,.\end{aligned}$$

We will consider this example later in more detail.

For the study of $\mathcal{M}$, a very useful observation is that $\bar{A}/I(2\delta)$ has the canonically defined flag

$$0\subset V_1\subset\cdots\subset V_{2\delta}=\bar{A}/I(2\delta)\,,$$

where $V_i=\bar{I}(2\delta-i)/I(2\delta)$. This induces a partition of Grass(δ, $\bar{A}/I(2\delta)$) into *Schubert cells*

$$W_{a_1,a_2,\ldots,a_\delta}\quad\text{for }\delta\geq a_1\geq\cdots\geq a_\delta\geq 0\,,$$

defined by

$$\begin{aligned}&W_{a_1,\ldots,a_\delta}\\&\quad=\{\Lambda\in\operatorname{Grass}(\delta,\bar{A}/I(2\delta))\mid\dim(\Lambda\cap V_{\delta+i-a_i})=i\text{ for }i=1,\ldots,\delta\\&\qquad\qquad\text{and }\dim(\Lambda\cap V_j)<i\text{ for }j<\delta+i-a_i\}\end{aligned}$$

(see [3, p. 195]). We have

$$\dim W_{a_1,\ldots,a_\delta} = \delta^2 - \sum_{i=1}^{\delta} a_i ,$$

$$W_{b_1,\ldots,b_\delta} \subseteq \overline{W_{a_1,\ldots,a_\delta}} \quad \Leftrightarrow \quad b_i \geq a_i \text{ for } i = 1, \ldots, \delta .$$

Definition 4. Let Δ be a subset of $[0, 2\delta - 1]$ such that $\#\Delta = \delta$ and $\Delta \cup [2\delta, \infty)$ is a Γ-module. Let $\Gamma_1(I) = \Gamma(I) - \tau(I)$. Then $\mathcal{M}(\Delta) :=$ the subset of $\mathcal{M}$ parametrizing ideals I with $\Gamma_1(I) = \Delta \cup [2\delta, \infty)$.

Lemma 5. *Let $\Delta = \{b_1, \ldots, b_\delta\}$ with $0 \leq b_1 < \cdots < b_\delta < 2\delta$. Let $a_{\delta-i+1} = b_i - i + 1$ for $i = 1, \ldots, \delta$. Then $\mathcal{M}(\Delta) = \mathcal{M} \cap W_{a_1,\ldots,a_\delta}$.* □

The proof is left to the reader. In the sequel we will let $W(\Delta)$ denote the Schubert cell containing $\mathcal{M}(\Delta)$, and by abuse of language write $W(\Gamma)$, $\mathcal{M}(\Gamma)$ instead of $W(\Gamma \cap [0, 2\delta - 1])$, etc.

Theorem 6. *$\mathcal{M}$ is connected.*

Proof. Let $G = 1 + \mathfrak{m}_{\bar{A}}$. It acts on $\bar{A}/I(2\delta)$ by multiplication and leaves the standard flag invariant. Hence it also acts on Grass$(\delta, \bar{A}/I(2\delta))$ preserving the partition into Schubert cells. G obviously also acts on $\mathcal{M}$, hence it preserves each stratum $\mathcal{M}(\Delta)$. As $1 + I(2\delta)$ acts trivially on Grass$(\delta, \bar{A}/I(2\delta))$, G acts via a unipotent quotient on $\mathcal{M}$. As $\mathcal{M}$ is projective, the only closed G-orbits on $\mathcal{M}$ are points. However, G has a unique fixed point P on $\mathcal{M}$, corresponding to $\bar{I}(\delta)/I(2\delta)$. As the image of G in Aut$(\bar{A}/I(2\delta))$ is connected, each orbit is connected and has P in its closure. Hence $\mathcal{M}$ is connected. □

Remark. The stratum $\mathcal{M}(\Gamma)$ corresponds to the cyclic A-modules in $\bar{A}/I(2\delta)$, which form one G-orbit (namely $G.A/I(2\delta)$), of dimension $\dim(G/1 + \mathfrak{m}_A)) = \dim(\mathfrak{m}_{\bar{A}}/\mathfrak{m}_A) = \delta$. Observe that $\mathcal{M}(\Gamma) = \mathcal{M} \setminus \{\Lambda \mid \Lambda \subseteq V_{2\delta-1}\}$, hence $\mathcal{M}(\Gamma)$ is open in $\mathcal{M}$. Below we will see examples of strata $\mathcal{M}(\Delta)$ with $\Delta \neq \Gamma$ of dimension $\geq \delta$. Hence $\mathcal{M}$ is reducible in general.

As a consequence of the proof of Theorem 6, one obtains a fairly good picture of the adjacencies of strata in $\mathcal{M}$ by looking at an affine neighborhood of P in Grass$(\delta, \bar{A}/I(2\delta))$. Such an affine open neighborhood U_P is given by putting $\pi_{\delta+1,\ldots,2\delta} \neq 0$, and we have an isomorphism $\mathrm{Mat}_\delta(k) \cong U_P$ given by $Z \mapsto \mathrm{rowspace}(Z \mid I)$, mapping 0 to P. In the sequel we will use the matrix coefficients Z_{ij} $(i, j = 1, \ldots, \delta)$ as affine coordinates on U_P.

The action of G on Grass$(\delta, \bar{A}/I(2\delta))$ is induced by right multiplication with matrices as follows. Let $N \in \mathrm{Mat}_\delta(k)$ be given by $N_{i,i+1} = 1$, for $i = 1, \ldots, \delta - 1$

and $N_{ij} = 0$ else. Then t^j acts by right multiplication with $\left(\begin{smallmatrix} N^j & {}^tN^{\delta-j} \\ 0 & N^j \end{smallmatrix}\right)$ if $j < \delta$ and with $\left(\begin{smallmatrix} 0 & N^{j-\delta} \\ 0 & 0 \end{smallmatrix}\right)$ if $j \geq \delta$. Hence

$$g = 1 + \sum_{i=1}^{2\delta-1} a_i t^i \text{ leaves rowspace}(Z \mid I) \text{ invariant}$$

$$\Leftrightarrow \operatorname{rank}\begin{pmatrix} Z & I \\ \sum_{j<\delta} a_j Z N^j & \sum_{j<\delta} a_j (Z\,{}^tN^{\delta-j} + N^j) + \sum_{j \geq \delta} a_j Z N^{j-\delta} \end{pmatrix} = \delta$$

$$\Leftrightarrow \Big(\sum_{j<\delta} a_j (Z\,{}^tN^{\delta-j} + N^j) + \sum_{j\geq\delta} a_j Z N^{j-\delta}\Big) Z - \sum_{j<\delta} a_j Z N^j = 0\,.$$

Hence we have the following proposition:

Proposition 7. *Equations for $\mathcal{M} \cap U_P$ can be obtained as follows. Let $\{g_i\}_{i=1}^{e}$ be a set of generators of $A/I(2\delta)$ as a k-algebra. Write $g_i = \sum_j a_{ij} t^j$. Then*

$$(Z \mid I) \in \mathcal{M} \cap U_P \quad \Leftrightarrow \quad \sum_{j<\delta} a_{ij} (Z\,{}^tN^{\delta-j} Z + [N^j, Z]) + \sum_{j \geq \delta} a_{ij} Z N^{j-\delta} Z = 0 \quad \textit{for } i = 1, \ldots, e\,. \qquad \square$$

3. Monomial semigroups

From now on we assume that k is an algebraically closed field of characteristic zero. For simplicity we will restrict to the case that A is complete.

Definition 8. A *monomial curve singularity* over k is an irreducible curve singularity with local ring isomorphic to $A = k[[t^{a_1}, \ldots, t^{a_m}]]$ for certain $a_1, \ldots, a_m \in \mathbb{N}$. Without loss of generality we may assume that $\gcd(a_1, \ldots, a_m) = 1$. In that case the semigroup Γ of A has generators $a_1, \ldots, a_m$, and again we may assume that $a_1, \ldots, a_m$ form a minimal set of generators for Γ.

Definition 9. A semigroup Γ in $\mathbb{N}$ is called *monomial* if $0 \in \Gamma$, $\#(\mathbb{N}\backslash\Gamma) < \infty$ and each reduced and irreducible curve singularity with semigroup Γ is a monomial curve singularity.

Theorem 10. *For a semigroup $\Gamma \subseteq \mathbb{N}$ the following are equivalent*:

(1) *Γ is a monomial semigroup.*

(2) *Γ is a semigroup from the following list*:

(i) $\Gamma_{m,s,b} := \{im \mid i = 0, 1, \ldots, s\} \cup [sm + b, \infty)$ *with* $1 \leq b < m$, $s \geq 1$,

(ii) $\Gamma_{m,r} := \{0\} \cup [m, m + r - 1] \cup [m + r + 1, \infty)$ *with* $2 \leq r \leq m - 1$,

(iii) $\Gamma_m := \{0, m\} \cup [m + 2, 2m] \cup [2m + 2, \infty)$ *with* $m \geq 3$.

(3) $0 \in \Gamma$, $\delta := \#(\mathbb{N}\backslash\Gamma) < \infty$ *and the following property holds*: *if* $x \in \mathbb{N}\backslash\Gamma$ *and* $c(x) := \min\{n \in \mathbb{N} \mid [n, \infty) \subseteq \Gamma \cup (x + \Gamma)\}$, *then* $\Gamma \cap (x + \Gamma) \subseteq [c(x), \infty)$.

Proof. (2)$\Rightarrow$(3) This is an easy case-by-case check.

(3)$\Rightarrow$(2) Suppose that Γ satisfies (3). Let $m := \min(\Gamma\backslash\{0\})$ and s,b be given by $1 \le b < m$ and $sm + b = \min(\Gamma\backslash m\mathbb{N})$. If $[sm + b, \infty) \subseteq \Gamma$, then Γ is of type $\Gamma_{m,s,b}$. Else there exists $r > b$ with $[sm + b, sm + r - 1] \subseteq \Gamma$ but $sm + r \not\in \Gamma$. We will prove that $s = 1$ and $b \le 2$.

Assume $s > 1$. Then $sm + b \in \Gamma \cap (b + \Gamma)$ so by (3) we have $c(b) \le sm + b$, i.e. $sm + b + i \in \Gamma \cup (b + \Gamma)$ for all $i \ge 0$. But $sm + b + i \not\in b + \Gamma$ for $0 < i < b$, so $sm + b + i \in \Gamma$ for $i = 1, \dots, b - 1$. This means that $r \ge 2b$. On the other hand, $r < m + b$ and we get

$$(*) \qquad m > r - b \ge b\ .$$

If $r \equiv 1 \pmod m$, then $r = m + 1$. Now $2m - 1 \not\in \Gamma$ and $(s - 1)m + 2m - 1 = (s + 1)m - 1 \in \Gamma$. This implies by (3) that $sm + r = (s + 1)m + 1 \in \Gamma \cup (2m - 1 + \Gamma)$, i.e. $(s - 1)m + 2 \in \Gamma$. This implies $m = 2$, $b = 1$, $r = 3$ and $sm + r = 2s + 3 = (s + 1)m + b \in \Gamma$ and we get a contradiction.

If $r \not\equiv 1 \pmod m$, then $m + r - 1 \not\in \Gamma$ because of $(*)$ and $s > 1$. On the other hand, we have $(s - 1)m + m + r - 1 = sm + r - 1 \in \Gamma$. This implies using (3) that $sm + r \in \Gamma \cup (m + r - 1 + \Gamma)$, i.e. $(s - 1)m + 1 \in \Gamma$. This is a contradiction and hence $s = 1$.

Assume now that $b > 2$. First of all $r < m + b - 1$. For if $r = m + b - 1$, then $2m + 1 \in \Gamma$ and $m + 1 \not\in \Gamma$. This would imply that $m + r = 2m + b - 1 \in (m + 1 + \Gamma) \cup \Gamma$, i.e. $m + b - 2 \in \Gamma$. This is a contradiction again.

Now $r < m + b - 1$ implies $r - b + 1 \not\in \Gamma$. But $m + r > m + r - b + 1 \ge m + b + 1$ by $(*)$, i.e. $m + r - b + 1 \in \Gamma$ and $m + r \in \Gamma \cup (r - b + 1 + \Gamma)$. This implies $m + b - 1 \in \Gamma$; a contradiction. Hence we have $s = 1$ and $b \le 2$. Recall $r < m + b$.

If $b = 2$ and $r = m + 1$, then $\Gamma = \Gamma_m$. If $b = 2$ and $r < m$, then $r - 1 \not\in \Gamma$. Because $m + r - 1 \in \Gamma$ we get $m + r \in \Gamma \cup (r - 1 + \Gamma)$, i.e. $m + 1 \in \Gamma$; contradiction. Remark that $r \ne m$ as $sm + r \not\in \Gamma$.

We are left with the case $b = 1$. Now $2 \le r < m + 1$. Assume there exists $r' > r$ with $m + r' \not\in \Gamma$. Take r' minimal with this property. If $r' = r + 1$, then $m + r' - r = m + 1 \in \Gamma$. If $r' > r + 1$, then $m + r - 1 + r' - r = m + r' - 1 \in \Gamma$. This implies by (3) that $m + r' \in \Gamma \cup (r' - r + \Gamma)$, i.e. $m + r' \in r' - r + \Gamma$, so $m + r \in \Gamma$; contradiction. Hence $[m + r + 1, \infty) \subseteq \Gamma$ so $\Gamma = \Gamma_{m,r}$.

(2)$\Rightarrow$(1) Let A be a local k-subalgebra of $k[[t]]$ with semigroup Γ_m. We have $A = k[[g_m, g_{m+2}, \dots, g_{2m-1}]]$, where $g_j \equiv t^j + b_j t^{2m+1} \pmod{t^{2m+2}}$. (It is clear that a coordinate change will eliminate the coefficient of t^{m+1} in g_m.) It is sufficient to find a substitution $t = \phi(s) = s(1 + \sum_{i \ge 2} a_i s^i)$ in such a way that the coefficient of s^{2m+1} in each $g_j(\phi(s))$ is 0. This coefficient is equal to

$$b_j + \text{the coefficient of } s^{2m+1-j} \text{ in } \left(1 + \sum_{i \geq 2} a_i s^i\right)^j$$
$$= b_j + ja_{2m+1-j} + F_{2m+1-j}(a_2, \ldots, a_{2m+1-j-1}) .$$

So our equations for the a_j can be solved recursively. The cases of $\Gamma_{m,s,b}$ and $\Gamma_{m,r}$ can be treated in the same way.

(1)$\Rightarrow$(3) Suppose that Γ has not property (3). Then there exist $x \in \mathbb{N} \backslash \Gamma$, $\gamma = \gamma' + x \in \Gamma \cap (x + \Gamma)$ and $y \not\in \Gamma \cup (x + \Gamma)$ such that $y > \gamma$. Let $A_0 = k[[t^i \mid i \in \Gamma]]$ be the monomial singularity with semigroup Γ. Put

$$A = k[[t^{\gamma'} + \lambda t^{y-x}, t^{\gamma} + \mu t^y, t^i \mid i \in \Gamma \backslash \{\gamma, \gamma'\}]] .$$

Then A has semigroup Γ, but for all changes of variables $t = \phi(s)$ either s^{y-x} or s^y will have a nonzero coefficient somewhere, if λ and μ are general enough. So A is not a monomial singularity. □

Remark. For the monomial singularities we have the following invariants:

type	embedding dimension	c	δ	Gorenstein type
$\Gamma_{m,s,b}$	m	sm if $b = 1$ $sm + b$ if $b > 1$	$s(m-1) + b - 1$	$m - 1$
$\Gamma_{m,r}$	$m - 1$	$m + r + 1$	m	$m - r$
Γ_m	$m - 1$	$2m + 2$	$m + 1$	1

Here the Gorenstein type of a Cohen–Macaulay local ring is defined as the minimal number of generators of its dualizing module (cf. [6, Chapter IV, Section 3]).

Remark. Notice that the simple irreducible complete intersection curve singularities all have monomial semigroups:

type A_{2s}:	$k[[x, y]]/(x^2 - y^{2s+1})$	has semigroup $\Gamma_{2,s,1}$,
type E_6:	$k[[x, y]]/(x^3 - y^4)$	has semigroup $\Gamma_{3,2}$,
type E_8:	$k[[x, y]]/(x^3 - y^5)$	has semigroup Γ_3 ,
type W_8:	$k[[x, y, z]]/(x^2 - z^3, y^2 - xz)$	has semigroup $\Gamma_{4,3}$,
type Z_{10}:	$k[[x, y, z]]/(x^2 - yz^2, y^2 - z^3)$	has semigroup Γ_4 .

In the case of type W_8, the parametrization is given by $(x, y, z) = (t^6, t^5, t^4)$; in the case Z_{10} it is given by $(x, y, z) = (t^7, t^6, t^4)$.

Remark. Here is another view on the class of monomial semigroups. Between the invariants c, m and δ we have the following inequalities. Write $c = sm + r$ with $0 \leq r \leq m-1$. Then

$$m - 1 \leq \delta \leq s(m-1) + \begin{cases} r-1 & \text{if } r \geq 2\,, \\ 0 & \text{else}\,. \end{cases}$$

Observe that $\Gamma = \Gamma_{m,s,b} \Leftrightarrow \delta$ takes the maximal possible value. All semigroups with $\delta = m$ are of type $\Gamma_{m,r}$ and the only semigroup with $\delta = m+1$ and which is Gorenstein (i.e. with $\gamma \in \Gamma \Leftrightarrow c - 1 - \gamma \not\in \Gamma$) is Γ_m.

We fix the following notation. Suppose A is as before and that the semigroup Γ of A is monomial. Let $\Delta \subseteq [0, 2\delta - 1]$ such that $\Delta \cup [2\delta, \infty)$ is a Γ-module and $\#\Delta = \delta$. Let S be the set of minimal generators for $\Delta \cup [2\delta, \infty)$ as a Γ-module, and let $\Delta' := S \cap \Delta = \{\gamma \in \Delta \mid \forall \gamma' \in \Gamma\colon \gamma - \gamma' \not\in \Delta\}$. For each $\gamma \in \Delta$ let $J_\gamma := [\gamma + 1, 2\delta - 1] \backslash \Delta$. We fix a parameter t in $\bar{A}$ such that $A = k[[t^\gamma \mid \gamma \in \Gamma]]$.

Theorem 11. *With these notations, for each A-submodule I of length δ in $\bar{A}/I(2\delta)$ corresponding to a point in $\mathcal{M}(\Delta)$ there exist uniquely determined $u_{\gamma_j} \in k$ (for $\gamma \in \Delta'$ and $j \in J_\gamma$) such that I is generated by the elements g_γ ($\gamma \in \Delta'$), where*

$$g_\gamma := t^\gamma + \sum_{j \in J_\gamma} u_{\gamma_j} t^j\,.$$

Proof. For each $\gamma \in \Delta'$ choose $h_\gamma \in I$ such that $h_\gamma \equiv t^\gamma \bmod I(\gamma + 1)$. If $a_{\gamma_j} t^j$ is the first term in h_γ with $j \not\in J_\gamma$ and $a_{\gamma_j} \neq 0$, replace h_γ by $h_\gamma - a_{\gamma_j} t^{\gamma''} h_{\gamma'}$ where $\gamma = \gamma' + \gamma''$ with $\gamma' \in \Delta'$, $\gamma'' \in \Gamma \backslash \{0\}$. In this way we arrive at generators g_γ for I as above. Conversely, given such generators, for each $\gamma \in \Delta$ there exist uniquely determined $\gamma' \in \Delta'$ and $\gamma'' \in \Gamma$ with $\gamma = \gamma' + \gamma''$ (here we use property (3) of Theorem 10), and we get a k-basis of I by taking all $t^{\gamma''} g_{\gamma'}$ for $\gamma \in \Delta$. Because Γ is monomial, in the process of reducing this basis to reduced row echelon form, the elements $g_{\gamma'}$ with $\gamma' \in \Delta'$ are not affected. Hence these are uniquely determined by I, and I has indeed semigroup Δ. □

Corollary. *If Γ is a monomial semigroup, then each $\mathcal{M}(\Delta)$ is an affine space of dimension $\sum_{\gamma \in \Delta'} \#J_\gamma$, and the codimension $c(\Delta)$ of $\mathcal{M}(\Delta)$ inside $W(\Delta)$ is equal to $\sum_{\gamma \in \Delta \backslash \Delta'} \#J_\gamma$.* □

We now proceed to the analysis of $\mathcal{M}$ in the monomial cases. We first observe that there is a hierarchy between these, which goes as follows. Let Γ be a monomial semigroup with conductor c. Put $\Gamma^* = \Gamma \cup \{c - 1\}$. Then Γ^* appears to be a monomial semigroup again. In fact, $\Gamma^*_{m,s,b} = \Gamma_{m,s,b-1}$ if $b \geq 2$, $\Gamma^*_{m,s,1} = \Gamma_{m,s-1,m-1}$ if $s \geq 2$, $\Gamma^*_{m,1,1} = \Gamma_{m-1,1,1}$, $\Gamma^*_m = \Gamma_{m,1,2}$ and $\Gamma^*_{m,r} = \Gamma_{m,1,1}$.

Let us write $\mathcal{M}_\Gamma$ for $\mathcal{M}$, when we want to specify which semigroup is under consideration. We have a natural map $j: \mathcal{M}_{\Gamma^*} \to \mathcal{M}_\Gamma$ defined by $j(I) =$

$tI + (t^{2\delta-1}) \subseteq k[[t]]/(t^{2\delta})$ for $I \subseteq k[[t]]/(t^{2\delta-2})$. It is clear that j is injective and that $\mathrm{Im}(j)$ is the union of all strata $\mathcal{M}(\Delta)$ in $\mathcal{M}_{\Gamma}$ such that $\Delta = \{b_1, \ldots, b_\delta\}$, with $0 < b_1 < \cdots < b_\delta = 2\delta - 1$ and $b_1 + c - 1 \in \Delta \cup [2\delta, \infty)$.

Remarks. (1) If Γ is Gorenstein, then $\mathcal{M}(\Delta) \subseteq \mathrm{Im}(j) \Leftrightarrow b_1 > 0 \Leftrightarrow \Delta \neq \Gamma$. So in this case we have $\mathcal{M} = \mathcal{M}(\Gamma) \cup \mathrm{Im}(j)$.

(2) If $b_1 + c - 1 \in \Delta$, then $b_\delta = 2\delta - 1$.

(3) In general, $\mathrm{Im}(j)$ is closed in $\mathcal{M}$ and $\mathcal{M} = \mathrm{Im}(j) \cup \bigcup \mathcal{M}(\Delta)$, where we take the union over all Δ such that $b_1 = 0$, $b_1 + c - 1 \not\in \Delta \cup [2\delta, \infty)$ or $b_\delta < 2\delta - 1$.

We start with the study of $\mathcal{M}$ for the semigroups which are lowest in this hierarchy.

(I) $\Gamma = \Gamma_{m,1,1}$. We have $\delta = m - 1$. From the Corollary above we see that $\mathcal{M}(\Delta) = W(\Delta)$ for each Δ. Hence the adjacencies are determined by the remark before Definition 4. We have $m - 1$ irreducible components $\mathcal{M}_1, \ldots, \mathcal{M}_{m-1}$. The intersection of $\mathcal{M}_r$ with the open patch U_P consists of row spaces of matrices $(Z \mid I)$ such that $Z_{ij} = 0$ if $i > r$ or $j < r$. In particular, $\dim(\mathcal{M}_r) = r(m - r)$ and $\dim(\mathcal{M}) = \max\{r(m - r) \mid r = 1, \ldots, m - 1\} = [m^2/4]$.

(II) $\Gamma = \Gamma_{m,1,2} = \{0, m\} \cup [m + 2, \infty)$. We have $\delta = m$. Claim: if Δ has the minimal element b_1, then $c(\Delta) = 1$ if $b_1 + m + 1 \not\in \Delta$ and else $c(\Delta) = 0$. Moreover, in the former case, $\mathcal{M}(\Delta) \cap U_P$ is defined inside $W(\Delta) \cap U_P$ by the single equation $\mathrm{tr}(Z) = 0$. Again the adjacencies are as for the corresponding Schubert cells. This follows from Proposition 7; the equations for $\mathcal{M} \cap U_P$ are: $ZN^jZ = 0$ for $j = 0, 2, \ldots, m - 1$.

Let

$$\Delta_b = \{b, b + 2, \ldots, 2b + 1, b + m, b + m + 2, \ldots, 2m - 1\}\ ,$$

$$\Delta'_b = \{b, b + 2, \ldots, 2b, b + m, \ldots, 2m - 1\}\ ,$$

for $b = 0, \ldots, m - 2$ (notice that $\Delta_0 = \Gamma$). Remark that $\Gamma^* = \Gamma_{m,1,1}$. Then

$$\mathcal{M} = \mathrm{Im}(j) \cup \bigcup_{b=0}^{m-2} \overline{\mathcal{M}(\Delta_b)}\ , \quad \text{with } \mathrm{Im}(j) \cap \overline{\mathcal{M}(\Delta_b)} = \overline{\mathcal{M}(\Delta'_b)}\ ,$$

$$\dim \mathcal{M}(\Delta_b) = (b + 1)(m - b) - b + 1\ .$$

Especially

$$\dim \mathcal{M}(\Delta_{[m/2]-1}) = [m^2/4] + 1$$

implies

$$\dim \mathcal{M} = \dim \mathrm{Im}(j) + 1 = [m^2/4] + 1\ .$$

Using Remark 3 above, we have to prove that if $W(\Delta) \subseteq \overline{W(\Delta_b)}$ for some Δ with $b = \min(\Delta)$, then $\mathcal{M}(\Delta) \subseteq \overline{\mathcal{M}(\Delta_b)}$. But this is clear, because in U_P, $\mathcal{M}(\Delta)$ and $\mathcal{M}(\Delta_b)$ are defined by the single equation $\mathrm{tr}(Z) = 0$ inside $W(\Delta)$ and $W(\Delta_b)$.

(III) $\Gamma = \Gamma_{m,r}$. Recall that $\Gamma^* = \Gamma_{m,1,1}$. Let

$$\begin{aligned} \Delta_b &= \{b\} \cup [b+r+1, 2b+r] \cup [b+m, b+m+r-1] \\ &\qquad \cup [b+m+r+1, 2m-1]\,, \\ \Delta'_b &= \{b\} \cup [b+r+1, 2b+r-1] \cup [b+m, 2m-1]\,, \end{aligned}$$

for $b = 0, \dots, m-r-1$. Then

$$\begin{aligned} &\mathcal{M} = \mathrm{Im}(j) \cup \bigcup_{b=0}^{m-r-1} \overline{\mathcal{M}(\Delta_b)} \quad \text{with } \mathrm{Im}(j) \cap \overline{\mathcal{M}(\Delta_b)} = \overline{\mathcal{M}(\Delta'_b)}\,, \\ &\dim \mathcal{M}(\Delta_b) = (b+1)(m-b-r) + r\,, \end{aligned}$$

i.e.

$$\dim \mathcal{M} = \begin{cases} 3 & \text{if } m = 3\,, \\ [m^2/4] & \text{if } m \geq 4\,. \end{cases}$$

Again it is enough to prove that if $W(\Delta) \subseteq \overline{W(\Delta_b)}$ for some Δ with $b = \min(\Delta)$, then $\mathcal{M}(\Delta) \subseteq \overline{\mathcal{M}(\Delta_b)}$. We may assume that $b+m+r \not\in \Delta$ (otherwise $\mathcal{M}(\Delta) = W(\Delta)$). In this case $\mathcal{M}(\Delta) \cap U_P$ is defined in $W(\Delta) \cap U_P$ by the equations $\mathrm{tr}(ZN^j) = 0$, $j = 0, \dots, r-1$. As in the case $\Gamma_{m,1,2}$ we conclude that $\mathcal{M}(\Delta) \subseteq \overline{\mathcal{M}(\Delta_b)}$.

Using Proposition 7 we see that $\mathcal{M} \cap U_P$ is defined by the equations $ZN^jZ = 0$, $j = 0, \dots, m-1$, $j \neq r$. We conclude the following facts:

(1) $$\{(Z \mid I) \in \mathcal{M} \cap U_P \mid \mathrm{rk}(Z) \leq 1\} = (\overline{\mathcal{M}(\Gamma)} \cup \overline{\mathcal{M}(\Delta_{m-r-1})}) \cap U_P$$

(notice that for $r = m-1$ we have just one component);

(2) $\overline{\mathcal{M}(\Gamma)} \cap U_P$ is defined by the equations

$$\begin{aligned} &\mathrm{rk}(Z) \leq 1\,, \\ &\mathrm{tr}(ZN^j) = 0 \quad j = 0, \dots, m-1\,,\ j \neq r\,, \\ &Z_{ij} = 0 \qquad \text{for } i = r+2, \dots, m\,,\ \text{all } j\,, \end{aligned}$$

whereas $\overline{\mathcal{M}(\Delta_{m-r-1})} \cap U_P$ is defined by

$$\begin{aligned} &\mathrm{rk}(Z) \leq 1\,, \\ &\mathrm{tr}(ZN^j) = 0 \quad j = 0, \dots, m-1\,,\ j \neq r\,, \\ &Z_{ij} = 0 \qquad \text{for } j = 1, \dots, m-r-1\,,\ \text{all } i\,; \end{aligned}$$

(3) if $m=3$, then $\mathcal{M}=\overline{\mathcal{M}(\Gamma)}\cup\overline{\mathcal{M}(\Delta_{2-r})}$.

It is enough to prove (1) and (2), because of the fact that in the case $m=3$, $\mathrm{rk}(Z)\le 1$ follows from $Z^2=0$, hence (3) holds. It suffices to prove that

$$\{Z\in \mathrm{Mat}_m(k)\mid \mathrm{rk}(Z)\le 1,\ ZN^jZ=0 \text{ for } j=0,\dots,m-1,\ j\neq r\}$$

is the union of the two irreducible components

$$\{Z\in \mathrm{Mat}_m(k)\mid \mathrm{rk}(Z)\le 1,\ ZN^jZ=0 \text{ for } j=0,\dots,m-1,\ j\neq r,\ Z_{ij}=0 \text{ for } i=r+2,\dots,m, \text{ all j}\}$$

and

$$\{Z\in \mathrm{Mat}_m(k)\mid \mathrm{rk}(Z)\le 1,\ ZN^jZ=0 \text{ for } j=0,\dots,m-1,\ j\neq r,\ Z_{ij}=0 \text{ for } j=1,\dots,m-r-1, \text{ all } i\}$$

because obviously $\mathcal{M}(\Delta_{m-r-1})$ is contained in the first of these components and $\mathcal{M}(\Gamma)$ in the other one, and moreover $ZN^jZ=0\Leftrightarrow \mathrm{tr}(ZN^j)=0$ (as $\mathrm{rk}(Z)\le 1$).

Now consider the Segre embedding $\mathcal{P}^{m-1}\times\mathcal{P}^{m-1}\to\mathcal{P}^{m^2-1}$, $(x,y)\mapsto x.{}^{t}y$. The image corresponds to all $m\times m$ matrices Z of rank 1. Writing $Z=x.{}^{t}y$ we find that $ZN^jZ=0\Leftrightarrow {}^{t}yN^jx=0$. We will prove that

$$\{(x,y)\in\mathcal{P}^{m-1}\times\mathcal{P}^{m-1}\mid {}^{t}yN^jx=0 \text{ for } j=0,\dots,m-1,\ j\neq r\}$$

is irreducible if $r=m-1$ and the union of two components defined as the closure of $y_1\neq 0$ resp. $x_m\neq 0$ (where $x=(x_1,\dots,x_m)$, $y=(y_1,\dots,y_m)$).

If $r=m-1$, we have the following system of equations:

$$\begin{cases} x_1y_1+\cdots+x_my_m=0\\ x_2y_1+\cdots+x_my_{m-1}=0\\ \dots\\ x_{m-1}y_1+x_my_2=0\,. \end{cases}$$

If $r<m-1$, we have

$${}^{t}yN^{m-1}x=\cdots={}^{t}yN^{r+1}x=0\,,$$

so

$$\begin{cases} y_1x_m=0\\ y_1x_{m-1}+y_2x_m=0\\ \dots\\ y_1x_{r+2}+\cdots+y_{m-r-1}x_m=0\,. \end{cases}$$

If $y_1 \neq 0$, then $x_{r+2} = \cdots = x_m = 0$, i.e. $Z_{ij} = 0$ for $i \geq r+2$.

If $x_m \neq 0$, then $y_1 = \cdots = y_{m-r-1} = 0$, i.e. $Z_{ij} = 0$ for $j \leq m-r-1$.

We finish with the remark that

$$S = \left\{ (x, y) \in \mathcal{P}^r \times \mathcal{P}^{m-1} \mid \sum_{i-j=\nu} x_i y_i = 0 \text{ for } \nu = 0, \ldots, r-1 \right\}$$

is irreducible of dimension $m-1$. Indeed, it is clear that each irreducible component of S has dimension $\geq m-1$. Looking at these equations for fixed y, we see that there is a unique component of S of dimension $m-1$ projecting birationally onto $\mathcal{P}^{m-1}$ and that the set of points in $\mathcal{P}^{m-1}$ for which the fibre has dimension at least i has codimension $i+1$. This implies that S has no other component of dimension $\geq m-1$.

If one takes $r = 1$ in the above description, the results remain valid, and apply to the case of $\Gamma_{m,1,2}$.

Let us consider the special case of $\Gamma_{3,2}$ (the E_6 singularity). Then for Δ we have the possibilities

$$\{0,3,4\} \quad \{2,3,5\} \quad \{3,4,5\} \quad \{1,4,5\} \quad \{2,4,5\}.$$

$\mathcal{M} \cap U_P$ is defined by $\mathrm{rk}(Z) \leq 1$, $\mathrm{tr}(Z) = 0 = \mathrm{tr}(ZN)$, i.e. by

$$\operatorname{rank} \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & -Z_{21} & -Z_{11} - Z_{22} \end{pmatrix} \leq 1.$$

One can prove that this defines a threefold with a singular line with transverse singularity of type A_2. For the different Δ we get for $\mathcal{M}(\Delta) \cap U_P$ the following:

Δ	$\mathcal{M}(\Delta) \cap U_P$,
$\{3,4,5\}$	$\{(0 \mid I)\}$,
$\{2,3,5\}$	$\left\{ \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix} : v \neq 0 \right\}$,
$\{1,4,5\}$	$\left\{ \begin{pmatrix} 0 & v & u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : v \neq 0 \right\}$,
$\{2,4,5\}$	$\left\{ \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : u \neq 0 \right\}$.

(IV) Let $\Gamma = \Gamma_m$. Here $\Gamma^* = \Gamma_{m,1,2}$. As Γ is Gorenstein, $\mathcal{M} = \mathrm{Im}(j) \cup \mathcal{M}(\Gamma)$ and

$$\dim \mathcal{M} = \begin{cases} 4 & \text{if } m = 3\,, \\ [m^2/4] + 1 & \text{if } m \geq 4\,. \end{cases}$$

Using Proposition 7 we see that $\mathcal{M} \cap U_P$ is defined by the equations

$$Z^{t}NZ = [Z, N^{m}] \quad \text{and} \quad ZN^{j}Z = 0 \text{ for } j = 1, \ldots, m-1\,.$$

As before, we are interested in characterizing $\overline{\mathcal{M}(\Gamma)}$. The situation is not as easy as before. It is no longer true that $\overline{\mathcal{M}(\Gamma)}$ is a union of $\mathcal{M}(\Delta)$'s (as is the case for the corresponding Schubert cells): if $m = 4$, then $(t^2, t^4) \in \overline{\mathcal{M}(\Gamma)} \cap \mathcal{M}(\Delta)$ with $\Delta = \{2, 4, 6, 8, 9\}$. Obviously $W(\Delta) \subseteq \overline{W(\Gamma)}$. On the other hand, $\dim \mathcal{M}(\Delta) = \dim W(\Delta) - 1 = 5 = \dim \mathcal{M}(\Gamma)$, hence $\mathcal{M}(\Delta)$ is not contained in $\overline{\mathcal{M}(\Gamma)}$.

For $m = 3$ (the singularity E_8) one can check by explicit computation that $\mathcal{M} = \overline{\mathcal{M}(\Gamma)}$. We omit the details. We just mention that $\mathrm{Sing}(\mathcal{M})$ is the union of the closures of $\mathcal{M}(\{2, 5, 6, 7\})$ and $\mathcal{M}(\{3, 4, 6, 7\})$ with transverse types A_3 resp. D_4.

(V) $\Gamma = \Gamma_{2,s,1}$ (the singularity of type A_{2s}). Then Γ is Gorenstein and $\Gamma^* = \Gamma_{2,s-1,1}$ (if $s \geq 2$). Hence again $\mathcal{M} = \mathrm{Im}(j) \cup \mathcal{M}(\Gamma)$ and $\dim \mathcal{M} = s$ by induction on s. Furthermore, $\mathcal{M}(\Gamma)$ is dense in $\mathcal{M}$ and $\mathrm{Im}(j) = \bigcup_{x=1}^{s} \mathcal{M}(\langle x, 2s+1-x \rangle)$ (here $\langle x, y \rangle$ denotes the semigroup generated by x and y). With Proposition 7 we see that $\mathcal{M} \cap U_P$ has equations $Z({}^{t}N)^{s-2}Z = [Z, N^2]$. For small s we get:

$s = 1$: $\mathcal{M} \cong \mathcal{P}^1(k)$,

$s = 2$: $\mathcal{M}$ is a quadratic cone in $\mathcal{P}^3(k)$,

$s = 3$: $\mathcal{M}$ is a threefold with a singular line with transverse singularity of type A_2. At the point P, $\mathcal{M}$ has embedding dimension 5.

Remark. Consider a ring A and A-modules E with a resolution of the form

$$0 \to A \to A^n \to E \to 0\,.$$

Let us call such modules 1–n A-modules. The isomorphism classes of 1–n A-modules are in 1–1 correspondence with their Fitting ideals, i.e. with ideals in A which are generated by at most n elements (see [1, p. 146]). Hence for A the local ring of a reduced and irreducible curve singularity, the isomorphism classes of such modules are parametrized by open subsets of $\mathcal{M}$ (and by the whole of $\mathcal{M}$ if n is large enough).

Remark. There is some more structure on $\mathcal{M}$ which we have not exploited yet. First, we have the residue pairing R on $k[[t]]/(t^{2\delta})$. If I is an A-submodule of $k[[t]]/(t^{2\delta})$ of length δ, then also $I^{\perp} = \{x \in k[[t]]/(t^{2\delta}) \mid R(x, I) = 0\}$ is such an A-submodule. This defines an involution on $\mathcal{M}$. In the neighborhood U_p it is given by $Z \mapsto -J^{t}ZJ$, where $J_{ij} = 1$ if $i + j = \delta - 1$ and 0 else. The stratum $\mathcal{M}(\Delta)$ is mapped to $\mathcal{M}(\Delta')$, where $\Delta' = [0, 2\delta - 1] \setminus (2\delta - 1 - \Delta)$. In particular, $\mathcal{M}(\Gamma)$ is stable under this involution $\Leftrightarrow$ Γ is Gorenstein. Hence, in the non-Gorenstein case, $\mathcal{M}(\Gamma)$ cannot be dense in $\mathcal{M}$, as there exists another stratum with the same dimension.

In the monomial case, we have an action of k^* on $\mathcal{M}$, induced by the k^*-action on A, $(\lambda f)(t) = f(\lambda t)$ for $\lambda \in k^*$. There is a unique fixed point in each stratum, the zero point in the corresponding Schubert cell. In particular, the equations for $\mathcal{M} \cap U_P$ are quasi-homogeneous with weight $(Z_{ij}) = \delta + i - j$.

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