

A UNIFORM GENERAL NERON DESINGULARIZATION IN DIMENSION ONE

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ABSTRACT. We give a uniform General Neron Desingularization for one dimensional local rings with respect to morphisms which coincide modulo a high power of the maximal ideal. The result has interesting applications in the case of Cohen-Macaulay rings.

Key words : Smooth morphisms, regular morphisms

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INTRODUCTION

Solving equations is an old problem in mathematics. When we deal with polynomial equations over Noetherian complete local rings then a special case is the *smooth* case, that is a system where we may apply the implicit function theorem. If possible one tries to replace an arbitrary polynomial system of equations by a smooth one. This is done in the theorem below, which was used for example in Artin's Approximation Theory [2] (see also [12], [14]).

Let us recall some definitions. A ring morphism $u : A \rightarrow A'$ of Noetherian rings has *regular fibers* if for all prime ideals $\mathfrak{p} \in \text{Spec } A$ the ring $A'/\mathfrak{p}A'$ is a regular ring, i.e. its localizations are regular local rings. It has *geometrically regular fibers* if for all prime ideals $\mathfrak{p} \in \text{Spec } A$ and all finite field extensions K of the fraction field of A/\mathfrak{p} the ring $K \otimes_{A/\mathfrak{p}} A'/\mathfrak{p}A'$ is regular.

A flat morphism of Noetherian rings u is *regular* if its fibers are geometrically regular. If u is regular of finite type then u is called *smooth*. A localization of a smooth algebra is called *essentially smooth*. A Henselian Noetherian local ring A is *excellent* if the completion map $A \rightarrow \hat{A}$ is regular.

Theorem 1. (*General Neron Desingularization, André [1], Popescu [12], [13], Swan [18], Spivakovski [17]*) *Let $u : A \rightarrow A'$ be a regular morphism of Noetherian rings and B an A -algebra of finite type. Then any A -morphism $v : B \rightarrow A'$ factors through a smooth A -algebra C , that is v is a composite A -morphism $B \rightarrow C \rightarrow A'$.*

The A -algebra C is called a General Neron Desingularization (shortly GND) and in the case when (A', \mathfrak{m}') is local of dimension one it is constructive (see [10], [9]). The construction depends on formal power series defining v . We have to replace a given system of polynomials by a smooth one using only a polynomial approximation of v .

The purpose of this paper is to give such construction which depends only on a solution \bar{y} in A' of f modulo \mathfrak{m}^m for some $m \gg 0$ (see Theorem 2). Moreover, the

constructed GND works for each solutions y of f in A' lifting \bar{y} . We call such GND *uniform*.

As a consequence (see Remark 4) we get [9, Theorem 14] (see also [10, Theorem 20]), that is there exists a linear map similar as in Greenberg's case (see [5]).

An extension of [14, Theorem 10] is Theorem 10, which shows that if A, A' are Cohen-Macaulay rings and $v : B \rightarrow A'/\mathfrak{m}^{3k}$ an A -morphism, then the set of all A -morphisms $B \rightarrow A'$ equivalent with v modulo \mathfrak{m}^{3k} , is in bijection with $\mathfrak{m}^{3k}A'^s$ for some $s \in \mathbf{N}$. Theorem 10 has also a variant in the case when (A, \mathfrak{m}) is Artinian and (A', \mathfrak{m}') is a Noetherian local ring of dimension one, which says that for an A -algebra B of finite type and $v : B \rightarrow A'/\mathfrak{m}^{3k}$ an A -morphism, the set of all A -morphisms $B \rightarrow A'$ equivalent with v modulo \mathfrak{m}^{3k} , is in bijection with $\mathfrak{m}^{3k}A'^s$ for some $s \in \mathbf{N}$ (see Theorem 11).

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1. UNIFORM GENERAL NERON DESINGULARIZATION

We begin recalling some definitions. Let A be a Noetherian ring, $B = A[Y]/I$, $Y = (Y_1, \dots, Y_n)$. If $f = (f_1, \dots, f_r)$, $r \leq n$ is a system of polynomials in I then we can define the ideal Δ_f generated by all $r \times r$ -minors of the Jacobian matrix $(\partial f_i / \partial Y_j)$. After Elkik [4] let $H_{B/A}$ be the radical of the ideal $\sum_f ((f) : I) \Delta_f B$, where the sum is taken over all systems of polynomials f from I with $r \leq n$. Then $B_{\mathfrak{p}}$, $\mathfrak{p} \in \text{Spec } B$, is essentially smooth over A if and only if $\mathfrak{p} \not\supseteq H_{B/A}$ by the Jacobian criterion for smoothness. Thus $H_{B/A}$ measures the non smooth locus of B over A . B is *standard smooth* over A if there exists f in I as above such that $B = ((f) : I) \Delta_f B$.

Now let us fix some notations and assumptions for the next theorem. Let $u : A \rightarrow A'$ be a flat morphism of Noetherian local rings of dimension 1 and B, I be as above. Suppose that the maximal ideal \mathfrak{m} of A generates the maximal ideal of A' and the completions of A, A' are isomorphic. Moreover suppose that A' is Henselian and u is a regular morphism. Let $(0) = \bigcap_{\mathfrak{p} \in \text{Ass } A} Q_{\mathfrak{p}}$ be a reduced primary decomposition of (0) in A , where $Q_{\mathfrak{p}}$ is a primary ideal with $\sqrt{Q_{\mathfrak{p}}} = \mathfrak{p}$. Let $e \in \mathbf{N}$ be such that $\mathfrak{p}^e \subset Q_{\mathfrak{p}}$ for all $\mathfrak{p} \in \text{Ass } A$.

Theorem 2. *With the notations B, I, e fixed above, let $f = (f_1, \dots, f_r)$, $r \leq n$ be a system of polynomials in I , M an $r \times r$ -minor of the Jacobian matrix $(\partial f_i / \partial Y_j)$ and $k, c \in \mathbf{N}$. Suppose that there exist $N \in ((f) : I)$ and an A -morphism $v : B \rightarrow A'/\mathfrak{m}^{(2e+1)k+c}A' \cong A/\mathfrak{m}^{(2e+1)k+c}$, given by $Y \rightarrow y' \in A^n$, such that the ideal generated by $d = (MN)(y')$ in $A/\mathfrak{m}^{(2e+1)k}$ contains $\mathfrak{m}^k/\mathfrak{m}^{(2e+1)k}$. Then there exists a B -algebra C which is standard smooth over A with the following properties:*

- (1) *Every A -morphism $v' : B \rightarrow A'$ with $v' \equiv v$ modulo $d^{2e+1}A'$ (that is $v'(Y) \equiv v(Y)$ modulo $d^{2e+1}A'$) factors through C .*
- (2) *Every A -morphism $v' : B \rightarrow A'$ with $v' \equiv v$ modulo $\mathfrak{m}^{(2e+1)k}A'$ factors through C .*

(3) There exist an A -morphism $w : C \rightarrow A'$ which makes the following diagram commutative

$$\begin{array}{ccccc} B & \longrightarrow & C & \xrightarrow{w} & A' \\ \downarrow & & & & \downarrow \\ A/\mathfrak{m}^{(2e+1)k+c} & \longrightarrow & A/\mathfrak{m}^c & \longrightarrow & A'/\mathfrak{m}^c A' \end{array}$$

Proof. We follow the proof of [14, Theorem 10] and [9, Theorem 2]. Set $P = NM$. As A is not Artinian we see that d is not nilpotent. We have $(0 :_A d^e) = (0 :_A d^{e+1})$ because of the choice of e . Note that $\mathfrak{m}^k \subset dA + \mathfrak{m}^{(2e+1)k+c} \subset dA + \mathfrak{m}^{(2e+1)((2e+1)k+c)+c} \subset \dots$. Thus $\mathfrak{m}^k \subset dA$, in particular $\mathfrak{m}^{(2e+1)k+c} \subset dA$.

Note that $dA/\mathfrak{m}^{(2e+1)k+c}$ is the ideal corresponding to $v(NM)A'/\mathfrak{m}^{(2e+1)k+c}A'$ by the isomorphism $A/\mathfrak{m}^{(2e+1)k+c} \cong A'/\mathfrak{m}^{(2e+1)k+c}A'$.

We may assume that $M = \det((\partial f_i/\partial Y_j)_{i,j \in [r]})$, where $[r] = \{1, \dots, r\}$. Let H be the $n \times n$ -matrix obtained by adding down to $(\partial f/\partial Y)$ as a border the block $(0|\text{Id}_{n-r})$ (we assume as above that M is given by the first r columns of the Jacobian matrix). Let G' be the adjoint matrix of H and $G = NG'$. We have

$$GH = HG = N\text{Id}_n = P\text{Id}_n$$

and so

$$d\text{Id}_n = P(y')\text{Id}_n = G(y')H(y').$$

Let $\bar{A} = A/(d^{2e+1})$, $\bar{B} = \bar{A} \otimes_A B$, $\bar{A}' = A'/(d^{2e+1}A')$. Since $\mathfrak{m}^k \subset dA$ we see that v defines modulo d^{2e+1} a map $\bar{v} : \bar{B} \rightarrow \bar{A}'$.

Let

$$(1) \quad h = Y - y' - d^e G(y')T$$

where $T = (T_1, \dots, T_n)$ are new variables. Thus

$$(2) \quad Y_l - y'_l \equiv d^e G_l(y')T \text{ modulo } h_l$$

for $l \in [n]$, where $G_l(y')$ means the row l of the matrix $G(y')$. Since

$$f(Y) - f(y') \equiv \sum_j \partial f/\partial Y_j(y')(Y_j - y'_j)$$

modulo higher order terms in $Y_j - y'_j$, by Taylor's formula we see that we have

$$f(Y) - f(y') \equiv \sum_j d^e \partial f/\partial Y_j(y') G_j(y')T + d^{2e}Q =$$

$$d^e P(y')T + d^{2e}Q = d^{e+1}(T + d^{e-1}Q) \text{ modulo } h,$$

where $Q \in T^2 A[T]^r$ and modulo h means modulo $(h)A[Y, T]^r$. This is because we replace everywhere $Y_j - y'_j$ by $d^e G_j(y')T$ modulo h_j and use that $(\partial f/\partial Y)G = (P\text{Id}_r|0)$. Since $f(y') \in \mathfrak{m}^{(2e+1)k+c} \subset d^{2e+1}\mathfrak{m}^c$ we have $f(y') = d^{e+1}a$ for some $a \in d^e \mathfrak{m}^c A^r$. Set $g_i = a_i + T_i + d^{e-1}Q_i$, $i \in [r]$ and $E = A[Y, T]/(I, g, h)$. Clearly, it holds that $d^{e+1}g \subset (f, h)$ and $(f) \subset (h, g)$.

Let s be the first $r \times r$ -minor of the Jacobian matrix of g . Thus $s = \det(\text{Id}_r + d^{e-1}(\partial Q_i/\partial T_j)_{i,j \in [r]}) \in 1 + (T)$ and $U = (A[T]/(g))_s$ is standard smooth. In

particular, U is flat over A and so $(0 :_U d^e) = (0 :_U d^{e+1})$. We claim that $E_{ss'} \cong U_{s'}$ for some $s' \in 1 + (d, T)$. It will be enough to show that $I \subset (h, g)A[Y, T]_{ss'}$ because then we get $E_{ss'} \cong A[Y, T]_{ss'}/(g, h) \cong U_{s'}$. We have $PI \subset (f) \subset (h, g)$ and so $P(y' + d^e L)I \subset (h, g)$ where $L = G(y')T$. But $P(y' + d^e L)$ has the form ds' for some $s' \in 1 + (d^{e-1}T) \subset 1 + (d^e, T)$. It follows that $s'I \subset ((h, g) : d)A[Y, T]_s$, that is $s'IA[Y, T]_s/(h, g) \subset (0 :_{A[Y, T]_s/(h, g)} d) \cong (0 :_U d)$. On the other hand, $I \equiv I(y') \equiv 0$ modulo (d^e, h) because $I(y') \subset \mathfrak{m}^{(2e+1)k} \subset (d^{2e+1})$ since $\mathfrak{m}^k \subset (d)$ and so $s'IA[Y, T]_s/(h, g) = 0$ since $(0 :_U d^e) = (0 :_U d^{e+1})$ implies $(0 :_U d) \cap d^e U = 0$. Thus $I \subset (h, g)A[Y, T]_{ss'}$ and it follows that $C = E_{ss'}$ is a B -algebra smooth over A . Note that by construction C is a smooth A -algebra of standard form.

It remains to see that an arbitrary A -morphism $v' : B \rightarrow A'$ with $v' \equiv v$ modulo $\mathfrak{m}^{(2e+1)k}$ factors through C . We have $v'(Y) \equiv y'$ modulo $\mathfrak{m}^{(2e+1)k}$ and so $v'(Y) \equiv y'$ modulo d^{2e+1} . Thus there exists $\varepsilon \in d^e A^n$ such that $v'(Y) - y' = d^{e+1}\varepsilon$. Then $t := H(y')\varepsilon \in d^e A^n$ satisfies

$$G(y')t = P(y')\varepsilon = d\varepsilon$$

and so

$$v'(Y) - y' = d^e G(y')t,$$

that is $h(v'(Y), t) = 0$. Note that $d^{e+1}g(t) \in (h(v'(Y), t), f(v'(Y))) = (0)$ and $g(t) \in d^e A^r$. It follows that $g(t) \in (0 :_{A^r} d^{e+1}) \cap d^e A^r = 0$ because $(0 :_{A^r} d^e) = (0 :_{A^r} d^{e+1})$. Thus v' factors through E , that is v' is a composite map $B \rightarrow E \xrightarrow{\alpha} A'$, where α is a B -morphism given by $T \rightarrow t$. As $\alpha(s) \equiv 1$ modulo d and $\alpha(s') \equiv 1$ modulo (d, t) , $t \in dA'$ we see that $\alpha(s), \alpha(s')$ are invertible because A' is local and so α (thus v') factors through the standard smooth A -algebra C . Thus (1) holds. Then (2) holds from (1), because in this case we get $v' \equiv v$ modulo d^{2e+1} since $\mathfrak{m}^k \subset dA$.

Now for (3) take the map $\hat{w} : C \cong U_{s'} \rightarrow A'/\mathfrak{m}^c A'$ given by $(Y, T) \rightarrow (y', 0)$. Then the composite map $B \rightarrow C \xrightarrow{\hat{w}} A'/\mathfrak{m}^c A'$ is given by $Y \rightarrow v(Y)$ modulo \mathfrak{m}^c . Since C is standard smooth, we may lift \hat{w} to an A -morphism $w : C \rightarrow A'$ by the Implicit Function Theorem. Clearly, w makes the above diagram commutative because $v(Y)$ modulo \mathfrak{m}^c corresponds to $w(Y)$ modulo $\mathfrak{m}^c A'$ by the isomorphism $A/\mathfrak{m}^c \cong A'/\mathfrak{m}^c A'$. \square

Remark 3. The number c in the above theorem is necessary only in (3) and for (1) (and so (2)) we could take $c = 0$ from the beginning. Since (1) and (3) have mainly the same proof, we did not want to repeat the arguments and so we unify (1), (3) in the above theorem, the disadvantage being to have c not useful in (1).

Remark 4. The third statement of the above theorem gives a variant of [9, Theorem 14] in the idea of Greenberg [5]. More precisely, let (A, \mathfrak{m}) be an excellent Henselian local ring, $A' = \hat{A}$ and B, I be as above. Suppose that \hat{y} is a solution of I in $A/\mathfrak{m}^{(2e+1)k} \cong \hat{A}/\mathfrak{m}^{(2e+1)k}\hat{A}$ such that there exist a system of polynomials $f = (f_1, \dots, f_r)$ in I , an $r \times r$ minor M of the Jacobian matrix $(\partial f/\partial Y)$ and $N \in ((f) : I)$ such that $(MN)(\hat{y})\hat{A}/\mathfrak{m}^{(2e+1)k}\hat{A} \supset \mathfrak{m}^k \hat{A}/\mathfrak{m}^{(2e+1)k}\hat{A}$. As a consequence of Theorem 2 (3), every solution \tilde{y} of I in $A/\mathfrak{m}^{(2e+1)k+c}$, $c \in \mathbf{N}$ lifting \hat{y} can be lifted to a solution y of I in A . When A is a DVR then by Greenberg [5] there exists

a linear map $\nu : \mathbf{N} \rightarrow \mathbf{N}$ such that every solution \tilde{y} of I in $A/\mathfrak{m}^{\nu(c)}$, $c \in \mathbf{N}$ can be lifted to a solution y of I in A . By [8] and [12] there exist such maps ν (not necessarily linear) for every excellent Henselian local ring A . Our above consequence gives similarly a linear map ν of a special form $c \rightarrow (2e+1)k+c$ but this is not the map given by Greenberg because here k (and so ν) depends on \tilde{y} .

Remark 5. If A is reduced then $e = 1$ and $v : B \rightarrow A'/\mathfrak{m}^{3k+c}A'$. In fact, the proof goes similarly with $v : B \rightarrow A'/\mathfrak{m}^{3k+c}A'$ (in (1) with $v : B \rightarrow A'/d^3A'$) if A is Cohen-Macaulay, because for the proof in this case it is enough to see that d is regular which follows from $\mathfrak{m}^k \subset (d)$.

The following proposition follows in particular from [4, Theorem 2] but it is also a consequence of Theorem 2.

Proposition 6. (*Elkik*) *Let (A, \mathfrak{m}) be a Noetherian Henselian local ring of dimension one and $B = A[Y]/I$, $Y = (Y_1, \dots, Y_n)$ an A -algebra of finite type. Then for every $k \in \mathbf{N}$ there exist two integers $m_0, p \in \mathbf{N}$ such that if $y' \in A^n$ satisfies $\mathfrak{m}^k \subset H_{B/A}(y')$ and $I(y') \equiv 0$ modulo \mathfrak{m}^m for some $m > m_0$ then there exists $y \in A^n$ such that $I(y) = 0$ and $y \equiv y'$ modulo \mathfrak{m}^{m-p} .*

Proof. Suppose that $A' = A$. In the notations of Theorem 2 given k set $m_0 = p = (2e+1)k$ and suppose that $y' \in A^n$ satisfies $\mathfrak{m}^k \subset H_{B/A}(y')$ and $I(y') \equiv 0$ modulo \mathfrak{m}^m for some $m > m_0$. Let $v : B \rightarrow A/\mathfrak{m}^m$ be given by $Y \rightarrow y'$. Set $c = m - p$. By Theorem 2 there exists a standard smooth A -algebra C and a map $w : C \rightarrow A$ which makes the diagram commutative from (3) of Theorem 2. Let y be the image of Y by the composite map $B \rightarrow C \xrightarrow{w} A$. Then we have $I(y) = 0$ and $y \equiv y'$ modulo $\mathfrak{m}^c = \mathfrak{m}^{m-p}$. \square

Remark 7. Proposition 6 still holds when A is a Noetherian Henselian local ring of arbitrary dimension (see [4, Theorem 2]). Thus for fixed $k \in \mathbf{N}$ and $B = A[Y]/I$ as above there exist $m_0, p \in \mathbf{N}$ such that the linear map $c \rightarrow p+c$ behaves almost like the Artin function of B when $y' \in A^n$ satisfies $H_{B/A}(y') \supset \mathfrak{m}^k$ and $c \geq m_0 - p$. As in Remark 4, k depends on y' and we cannot say that the Artin function of B is linear.

Let f be a system of r polynomials from I as above, M an $r \times r$ -minor of $(\partial f / \partial Y)$ and $N \in ((f) : I)$. The above sentences give the idea that there exists a $\rho \in \mathbf{N}$ such that if there exists an A -morphism $v : B \rightarrow A/\mathfrak{m}^\rho$ with $v(MN) \supset \mathfrak{m}^k/\mathfrak{m}^\rho$ for some $k \in \mathbf{N}$ then there exists a B -algebra C , which is smooth over A , such that any A -morphism $v' : B \rightarrow \hat{A}$ with $v' \equiv v$ modulo $\mathfrak{m}^\rho \hat{A}$ factors through C , \hat{A} being the completion of A . In other words, we believe that Theorem 2 holds in arbitrary dimension if the smooth locus of B is big with respect to v , that is if $v(H_{B/A})\hat{A}$ is $\mathfrak{m}\hat{A}$ primary. Note that [16, Remark 4.7] (see also [9, Remark 16]) almost disagrees with our belief.

2. THE COHEN-MACAULAY CASE.

Here we present some consequences of Theorem 2 when A is Cohen-Macaulay and so $e = 1$ by Remark 5.

Corollary 8. *With the assumptions and notations of Theorem 2, suppose that A is Cohen-Macaulay, and let $\rho : B \rightarrow C$ be the structural algebra map. Then ρ induces bijections ρ^* given by $\rho^*(w) = w \circ \rho$, between*

- (1) $\{w \in \text{Hom}_A(C, A') : w \circ \rho \equiv v \text{ modulo } d^3 A'\}$ and $\{v' \in \text{Hom}_A(B, A') : v' \equiv v \text{ modulo } d^3 A'\}$,
- (2) $\{w \in \text{Hom}_A(C, A') : w \circ \rho \equiv v \text{ modulo } \mathfrak{m}^{3k} A'\}$ and $\{v' \in \text{Hom}_A(B, A') : v' \equiv v \text{ modulo } \mathfrak{m}^{3k} A'\}$.

Proof. (1) By Theorem 2, (1), also by Remark 5, ρ^* is surjective. Let $w, w' \in \text{Hom}_A(C, A')$ be such that $w \circ \rho = w' \circ \rho \equiv v$ modulo $d^3 A'$. In particular $w|_B = w'|_B$. In the notations which are introduced in the proof of Theorem 2, and from equation 2 we have $H(y')(Y - y') \equiv d^2 T \text{ mod } h$ by using $G(y')H(y') = d \text{Id}_n$. Applying w and w' on the above congruence and subtracting we get $H(y')(w(Y) - w'(Y)) = d^2(w(T) - w'(T))$, that is $d^2(w(T) - w'(T)) = 0$ because $w(Y) = w'(Y)$ and so $w(T) = w'(T)$ because d is regular on A' since d is regular in A (see Remark 5) and u is flat. It follows that $w = w'$.

- (2) Apply Theorem 2, (2) and Remark 5 for surjectivity, and injectivity goes as above. □

By construction, C has the form $(A[T]/(g))_{ss'}$ in the notations of Theorem 2, where $s = \det(\partial g_i / \partial T_j)_{i,j \in [r]}$. Let $\tilde{w} : C \rightarrow A'$ be the unique A -morphism such that $\tilde{w} \circ \rho \equiv v$ modulo $d^3 A'$ in the notations of Corollary 8. We have $\tilde{w}(ss') \notin \mathfrak{m} A'$.

Lemma 9. *There exist canonical bijections*

- (1) $A'^{m-r} \rightarrow \{w' \in \text{Hom}_A(C, A') : w' \equiv \tilde{w} \text{ modulo } d^3 A'\}$.
- (2) $\mathfrak{m}^{3k} A'^{m-r} \rightarrow \{w' \in \text{Hom}_A(C, A') : w' \equiv \tilde{w} \text{ modulo } \mathfrak{m}^{3k} A'\}$.

Proof. Let $w' \in \text{Hom}_A(C, A')$ be with $w' \equiv \tilde{w}$ modulo $d^3 A'$. We have

$$g(w'(T)) \equiv g(\tilde{w}(T)) = 0 \text{ modulo } d^3 A',$$

$$(ss')(w'(T)) \equiv (ss')(\tilde{w}(T)) \not\equiv 0 \text{ modulo } \mathfrak{m}.$$

Set $V = (T_1, \dots, T_r)$, $Z = (T_{r+1}, \dots, T_n)$. Thus $g(V, w'(Z)) = 0$ has a unique solution (namely $w'(V)$) in $\tilde{w}(Z) + d^3 A'$ by the Implicit Function Theorem. It follows that the restriction

$$\begin{aligned} & \{w' \in \text{Hom}_A(C, A') : w' \equiv \tilde{w} \text{ modulo } d^3 A'\} \rightarrow \\ & \{w'' \in \text{Hom}_A(A[Z], A') : w'' \equiv \tilde{w}|_{A[Z]} \text{ modulo } d^3 A'\} \end{aligned}$$

is bijective.

On the other hand, the map

$$\{w'' \in \text{Hom}_A(A[Z], A') : w'' \equiv \tilde{w}|_{A[Z]} \text{ modulo } d^3 A'\} \rightarrow \tilde{w}(Z) + d^3 A'^{m-r}$$

given by $w'' \rightarrow w''(Z)$ is a bijection too. It follows that there exists a canonical bijection between $\{w' \in \text{Hom}_A(C, A') : w' \equiv \tilde{w} \text{ modulo } d^3 A'\}$ and $\tilde{w}(Z) + d^3 A'^{m-r}$,

the last one being in bijection with $d^3 A^{m-r}$, that is with A^{m-r} , d being regular in A' . The proof of (2) goes similarly. \square

The following theorem extends [14, Theorem 15].

Theorem 10. *With the assumptions and notations of Corollary 8 there exist canonical bijections*

$$(1) \quad A^{m-r} \rightarrow \{v' \in \text{Hom}_A(B, A') : v' \equiv v \text{ modulo } d^3 A'\}.$$

$$(2) \quad \mathfrak{m}^{3k} A^{m-r} \rightarrow \{v' \in \text{Hom}_A(B, A') : v' \equiv v \text{ modulo } \mathfrak{m}^{3k} A'\}.$$

For the proof apply Corollary 8 and the above lemma.

Let (Λ, \mathfrak{p}) be a local Artinian ring, (A', \mathfrak{m}') be a local Noetherian ring of dimension 1 and $u : \Lambda \rightarrow A'$ a regular morphism. Suppose that A' is Henselian and has the same residue field with Λ . Let F be an Λ -algebra of finite type, let us say $F = \Lambda[U]/J$, $U = (U_1, \dots, U_n)$. A constructive GND of F with respect to an A -morphism $F \rightarrow A'$ follows easily from the classical Néron Desingularization [7] as it is shown in [6]. Note that $A'/\mathfrak{p}A'$ is a DVR because it is a one dimensional regular local ring, u being regular. Let $x \in A$ be a lifting of a local parameter of A/\mathfrak{p} . Clearly, x is a regular element in A and Λ, A' are Cohen-Macaulay rings. Set $A = \Lambda[X]_{(\mathfrak{p}, X)}$, $B = A \otimes_{\Lambda} F$, and let $u' : A \rightarrow A'$ be the regular morphism extending u by $X \rightarrow x$. Let $f = (f_1, \dots, f_r)$, $r \leq n$ be a system of polynomials from J and M an $r \times r$ -minor of the Jacobian matrix $(\partial f_i / \partial U_j)$, $g : F \rightarrow A'/\mathfrak{m}'^{3k}$ a Λ -morphism, and $N \in ((f) : J)$. Suppose that $g(NM)A'/\mathfrak{m}'^{3k} \supset \mathfrak{m}'^k / \mathfrak{m}'^{3k}$.

The following theorem extends [14, Corollary 16].

Theorem 11. *In the assumptions and notations from above, the set $\mathcal{G} = \{g' \in \text{Hom}_{\Lambda}(F, A') : g' \equiv g \text{ modulo } \mathfrak{m}'^{3k}\}$ is in bijection with $\mathfrak{m}'^{3k} A^{m-r}$.*

For the proof apply Theorem 10 to the morphism $A \otimes_{\Lambda} F \rightarrow A'$ induced by g and u' and note that $A \otimes_{\Lambda}$ induces a bijection $\text{Hom}_{\Lambda}(F, A') \rightarrow \text{Hom}_A(B, A')$ by adjunction.

3. ALGORITHM NÉRON DESINGULARIZATION

Input: $A = k[x]_{(x)}/J$, $J = (h_1, \dots, h_q)$, $h_i \in k[x]$, $x = (x_1, \dots, x_t)$, $\dim(A) = 1$, k is a field. $B = A[Y]/I$, $I = (f_1, \dots, f_r)$, $f_i \in k[x, Y]$, $Y = (Y_1, \dots, Y_n)$, integers k, r , $y' \in k[x]^n$, $N \in (f_1, \dots, f_r) : I$.

Output: A Neron Desingularization induced by y' or the message “ $y', N, (f_1, \dots, f_r)$ are not well chosen”.

- (1) Compute $M = \det((\partial f_i / \partial Y_j)_{i,j \in [r]})$, $P := NM$ and $d := P(y')$
- (2) $f := (f_1, \dots, f_r)$
- (3) Compute e such that $(0 :_A d^e) = (0 :_A d^{e+1})$
- (4) If $I(y') \not\subseteq (x)^{(2e+1)k} + J$ or $(x)^k \not\subseteq (d) + J$, return “ $y', N, (f_1, \dots, f_r)$ are not well chosen”
- (5) Complete $(\partial f_i / \partial Y_j)_{i \leq r}$ by $(0 | (\text{Id}_{n-r}))$ to obtain a square matrix H

- (6) Compute G' the adjoint matrix of H and $G := NG'$
- (7) $h = Y - y' - d^e G(y')T$, $T = (T_1, \dots, T_n)$
- (8) Write $f(Y) - f(y') = \sum_j d^e \partial f / \partial Y_j(y') G_j(y')T + d^{2e} Q$
- (9) Write $f(y') = d^{e+1} a$
- (10) For $i = 1$ to r , $g_i = a_i + T_i + d^{e-1} Q_i$
- (11) $E := A[Y, T] / (I, g, h)$
- (12) Compute s the $r \times r$ minor defined by the first r columns of $(\partial g / \partial T)$
- (13) Write $P(y' + d^e G(y')T) = ds'$
- (14) return $E_{ss'}$.

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REFERENCES

- [1] M. André, *Cinq exposés sur la désingularisation*, Handwritten manuscript Ecole Polytechnique Fédérale de Lausanne, (1991).
- [2] M. Artin, *Algebraic approximation of structures over complete local rings*, Publ. Math. IHES, **36**, (1969), 23-58.
- [3] W. Decker, G.-M. Greuel, G. Pfister, H. Schönemann: SINGULAR 4-1-0 — A computer algebra system for polynomial computations. <http://www.singular.uni-kl.de> (2016).
- [4] R. Elkik, *Solutions d'équations à coefficients dans un anneaux hensélien*, Ann. Sci. Ecole Normale Sup., **6** (1973), 553-604.
- [5] M. Greenberg, *Rational points in henselian discrete valuation rings*, Publ. Math. IHES, **31**, (1966), 59-64.
- [6] A. Khalid, Z. Kosar, *An Easy proof of the General Neron Desingularization in dimension 1*, Bull. Math. Soc. Sci. Math. Roumanie, **59 (107)**, (2016), 349-353.
- [7] A. Néron, *Modèles minimaux des variétés abéliennes sur les corps locaux et globaux*, Publ. Math. IHES, **21**, (1964), 5-128.
- [8] G. Pfister, D. Popescu, *Die strenge Approximationseigenschaft lokaler Ringe*, Inventiones Math. **30**, (1975), 145-174.
- [9] G. Pfister, D. Popescu, *Constructive General Neron Desingularization for one dimensional local rings*, Journal of Symbolic Computation, **80**, (2017), 570-580.
- [10] A. Popescu, D. Popescu, *A method to compute the General Neron Desingularization in the frame of one dimensional local domains*, in "Singularities and Computer Algebra - Festschrift for Gert-Martin Greuel, On the Occasion of his 70th Birthday", Editors Wolfram Decker, Gerhard Pfister, Mathias Schulze, Springer Monograph., DOI 10.1007/978-3-319-28829-1, 2017, 199-222, arXiv:AC/1508.05511.
- [11] D. Popescu, *General Neron Desingularization*, Nagoya Math. J., **100** (1985), 97-126.
- [12] D. Popescu, *General Neron Desingularization and approximation*, Nagoya Math. J., **104**, (1986), 85-115.
- [13] D. Popescu, *Artin Approximation*, in "Handbook of Algebra", vol. 2, Ed. M. Hazewinkel, Elsevier, 2000, 321-355.
- [14] D. Popescu, *Around General Neron Desingularization*, Journal of Algebra and Its Applications, **16**, No. 4 (2017), doi: 10.1142/S0219498817500724, arXiv:AC/1504.06938.
- [15] D. Popescu, *Artin approximation property and the General Neron Desingularization*, Revue Roum. Math. Pures et Appl., **62**, (1), (2017), 171-189, arXiv:AC/1511.06967.
- [16] G. Rond, *Lojasiewicz inequality over the ring of power series in two variables*, Math. Res. Lett., **20**, (2013), 1-13.

- [17] M. Spivakovski, *A new proof of D. Popescu's theorem on smoothing of ring homomorphisms*, J. Amer. Math. Soc., **294**, (1999), 381-444.
- [18] R. Swan, *Neron-Popescu desingularization*, in "Algebra and Geometry", Ed. M. Kang, International Press, Cambridge, (1998), 135-192.

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