Some Applications of Resolution of Singularities from the Practical Point of View

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Abstract. Recent implementations of algorithms for resolution of singularities have not only shown that it is feasible to implement these algorithms, but also that the performance allows the use of desingularisation as a tool in more complex computational tasks. In this article, we want to follow this idea and outline some rather direct applications of the algorithmic resolution of singularities such as the intersection matrix of the exceptional divisors, spectral numbers of a hypersurface singularity or the Denef-Loeser zeta function.

1 Introduction

The existence of a resolution of singularities over a field of characteristic zero has been known since the famous work of Hironaka [8] more than 40 years ago. But it was not until the late 1980s that a constructive approach to this problem had been found (see e.g. [1], [13]) which in turn allowed the development and implementation of algorithms for resolution of singularities (see e.g. [3], [6]). On this basis it is now possible to compute several invariants directly, whose definition relies on knowledge about a resolution. The aim of this article is to describe the computation of some of these invariants based on our implementation of embedded resolution of singularities in the computer algebra system SINGULAR ([10], [6]).

We start with a brief description of the notions and basic results in the field of resolution of singularities. As the final result of our algorithm is encoded in a collection of affine charts which glue together, the first problem to be solved is the identification of the exceptional divisors which is discussed in detail in section 3. In the following sections, we consider the computation of the intersection form of exceptional curves on a blown-up surface and their genus. In this context, it is necessary to pass from a given embedded resolution to a non-embedded one; the main problem then turns out to be understanding and describing the splitting of the exceptional divisors: First of all, a given divisor may be irreducible in the embedded situation, but its intersection with the surface may nevertheless consist of several components; on the other hand, all computations are done over $\mathbb{Q}$ and a $\mathbb{Q}$-irreducible components can be reducible over $\mathbb{C}$. To tackle this last problem, we use results of Gao and Ruppert [7].

In section 6, we outline how to compute the negative part of the spectrum of an isolated hypersurface singularity based on a description by Varchenko [12]. Note that, in SINGULAR, there is an implementation of M. Schulze for computing the spectrum by a completely different approach, namely using Hodge theory [9]. In section 7, we explain how to compute the local and the global Denef-Loeser zeta function for a hypersurface singularity; for functions with a non-degenerate Newton boundary there is a completely different approach to this by Denef and Hoornaert which is implemented in MAPLE [5].

We would like to thank D. van Straten for pointing out the importance of not only developing a program for resolution of singularities, but also of applications using it to allow users to benefit from the program. We are especially grateful to A. Melle and I. Luengo for turning our attention to the ζ-function which was very useful for testing our implementation and eventually led to several important improvements.

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2 Resolution of Singularities

In this section, we shall briefly recall well-known results and constructions about resolution of singularities in characteristic zero and, in particular, about algorithmic resolution of singularities. More detailed presentations of the theoretical background can, for instance, be found in [1], [2] or [4]; the practical aspects are covered in [6] and in [3].

As the problem of desingularitzation can be rephrased as the problem of resolving singularities of a given variety or scheme X by means of an appropriate finite sequence of blow-ups, the first notions we would like to recall are clearly the different notions of transforms under a blow-up:

Definition 2.1. *Let $I \subset \mathcal{O}_W$ be a sheaf of ideals on a smooth algebraic variety W and let $\phi : \tilde{W} \longrightarrow W$ be a blow-up map at a smooth center C with exceptional divisor H. Then*

$$\phi^*(I) = I\mathcal{O}_{\tilde{W}}$$

and

$$I_{\text{strict}} = (\phi^*(I) :_{\mathcal{O}_W} \mathcal{I}(H)^\infty)$$

are called the total transform respectively the strict transform of I. The weak transform of I is defined as I_{weak} such that the total transform may be factorized as

$$\phi^*(I) = \mathcal{I}(H)^k I_{\text{weak}}$$

(for a suitable k) and H is not a component of $V(I_{\text{weak}})$.

Geometrically speaking, the strict transform is obtained from the total transform by dropping all components which lie inside the exceptional divisor H, whereas the weak transform originates from dropping only the component which coincides with the exceptional divisor.

There exist several variants of resolution type theorems which are closely related to each other – some requiring weak transforms, some strict transforms. For the applications which we would like to consider in this article, we can, however, restrict our considerations to the following two theorems[1]:

[1]Both of these theorems are due to Hironaka; here they are stated in a version close to [4] resp. [14]

Theorem 2.2 (Hironaka). *Let X be a subscheme of a smooth algebraic scheme W. Then there exists a sequence*

$$W = W_0 \xleftarrow{\pi_1} W_1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_r} W_r$$

of blow-ups $\pi_i : W_i \longrightarrow W_{i-1}$ at smooth centers $C_{i-1} \subset W_{i-1}$ such that

a The exceptional divisor of the induced morphism $W_i \longrightarrow W$ has only normal crossings and C_i has normal crossings with it.

b Let $X_i \subset W_i$ be the strict transform of X. All centers C_i are disjoint from $Reg(X) \subset X_i$, the set of points where X is smooth.[2]

c X_r is smooth and has normal crossings with the exceptional divisor of the morphism $W_r \longrightarrow W$.

d The morphism $(W_r, X_r) \longrightarrow (W, X)$ is equivariant under group actions.

Theorem 2.3 (Hironaka). *Let X be an algebraic variety. There exists a desingularization of X, that is there exists a variety $\tilde{X}$ and a proper birational morphism $\pi : \tilde{X} \longrightarrow X$ such that*

1. $\tilde{X}$ is non-singular.

2. π is an isomorphism over the regular points of X.

These theorems about resolution of singularities do not give any hint on how to find the appropriate centers in an algorithmic way, they merely state that they exist. The usual approach to determining these centers is to assign to each point of the given X_i an appropriate invariant and use the locus of the maximal value of the invariant as the center for the next blow-up π_{i+1}. There are several ways of defining such an invariant of which two have been implemented up to now: the invariant of Villamayor [4], implemented by Bodnar and Schicho [3], and the one introduced in [6].

Common to each of these variants of resolution of singularities is that not the value of the invariant at each point is computed, but the locus of maximal value. Moreover, calculations take place in affine charts by which the scheme X resp. the smooth ambient space W is covered. In particular, the center of a blow-up can consist of components which lie in different charts and there may not even be a chart which contains the whole center.

After these general remarks on algorithmic resolution of singularities, we can now focus on the main goal of this article, applications of resolution of singularities: intersection matrix and genus of the exceptional curves in the case of non-embedded resolution of surface singularities, negative part of the spectrum of a hypersurface singularity and the Denef-Loeser zeta-function. But before we can start considering these more complex applications, we first have to find a good way to work with the resolved object, which is described in terms of the covering by the final charts; in particular, we need to determine whether a divisor in one chart and one in another chart both belong to the same exceptional divisor and even to the same component of it or not.

[2] This is not a typographical error, it is really $Reg(X)$, not $Reg(X_i)$. This condition simply ensures that the sequence of blow-ups is an isomorphism on $Reg(X)$.

3 Identification of Exceptional Divisors

Given an embedded resolution of singularities as computed by the algorithm, we actually deal with a list of charts in which the transform and the exceptional divisors live. Therefore the first task is to find a way to identify points resp. subvarieties which appear in more than one chart; in particular we need to decide whether two given exceptional divisors living in two different charts actually belong to the same exceptional divisor of the global object. To this end, we will move through the tree of charts arising during the resolution process, first blowing-down from the first chart to the one in which the history of the two charts in question branched, and then blowing-up again to the other chart with which we want to compare (cf. figure 1).

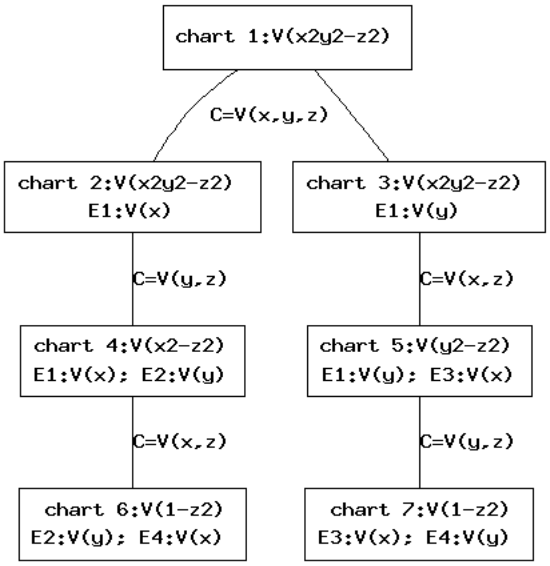

Figure 1: The tree of blow-ups in the resolution of the singularity $V(x^2y^2-z^2) \subset \mathbb{A}^3$. For simplicity of notation the variable names in all charts have been chosen to be x, y, z. To determine whether two exceptional divisors in two different charts actually belong to the same exceptional divisor, we need to move through the tree by first blowing down and then up again; for instance, the question, whether the divisors $V(x)$ in chart 6 and $V(y)$ in chart 7 belong to the same divisor, can only be answered by comparing the centers in charts 4 and 5. To this end, we have to move from chart 4 to chart one by blowing down twice and then proceed to chart 5 by blowing up twice.

As blow-ups are isomorphisms away from the center, this process of sucessively blowing-down and then blowing-up again does not cause any problems for points which do not lie on an exceptional divisor at all or only lie on exceptional divisors, which already exist in the chart at which the history of the considered charts branched. If, however, the point lies on an exceptional divisor which arises later, then blowing-down beyond the moment of birth of this divisor will inevitably lead to incorrect results, because this blow-up map is not an isomorphism. To avoid this problem, we need to represent the point on the exceptional divisor as the locus of intersection of the exceptional divisor with an auxilliary variety which is not contained in the exceptional divisor. More formally speaking, we use the following simple fact from commutative algebra:

Let $I \subset K[x_1, \ldots, x_n]$ be a prime ideal, $J \subset K[x_1, \ldots, x_n]$ another ideal such that $I + J$ is equidimensional and $ht(I) = ht(I + J) - r$ for some integer $0 < r < n$. Then there exist polynomials $p_1, \ldots, p_r \in I + J$ and a polynomial $f \in K[x_1, \ldots, x_n]$ such that

$$\sqrt{I + J} = \sqrt{(I + (p_1, \ldots, p_r)) : f}.$$

In our situation, the ideal I is, of course, the ideal of the intersection of the exceptional divisors in which the point or subvariety $V(J)$ is contained. As any sufficiently general set of polynomials $p_1, \ldots, p_r \in J \setminus (I \cap J)$ leading to the correct height of $I + (p_1, \ldots, p_r)$ will do and as the only truely restricting condition on f is that it has to exclude all extra components of $I + (p_1, \ldots, p_r)$, we also have enough freedom of choice of the $p_1, \ldots, p_r, f$ to achieve that none of them is contained in any further exceptional divisor that might be in our way when blowing-down.

Having solved the problem of identifying points which exist in more than one chart, we can now determine which exceptional divisor in one chart coincides with which one in another chart by simply comparing the centers leading to these exceptional divisors. To this end, we start at the root of the tree of charts of the resolution and work our way up to the final charts. The criteria for identifying the centers are quite simple: first of all, the centers can not be the same, if the the corresponding values of the governing function do not agree, secondly, the centers cannot be the same if the exceptional divisors in which they are contained are not the same and, in the last step, the remaining candidates are compared explicitly by mapping them through the resolution tree as described above.

Remark 3.1. *In the case of non-embedded resolutions, the situation is a little more complicated. While in the embedded case, the exceptional divisors are always ($\mathbb{Q}$-)irreducible[3], they might be reducible in the non-embedded case. To identify these components we therefore need to decompose the exceptional divisors before doing the comparison.*
Note that in this context, we are only able to deal with $\mathbb{Q}$-components in the implementation, since SINGULAR *computes over the rationals. The steps necessary to compute over the complex numbers will be mentioned in the upcoming section.*

4 Intersection Form of Exceptional Curves on a Blown-up Surface

Given an embedded resolution of a surface singularity, stored as a tree of charts, we would like to pass to a non-embedded resolution by dropping unnecessary blow-ups at the end of the branches of the tree of charts. To this end, we compute the list of exceptional divisors by identifying them in the different charts as described in the previous section 3. Starting at the final charts, we then move backwards through the resolution tree and cancel those blowing ups which are not necessary for the non-embedded resolution (see illustration 2 for an example).

Then we consider the intersection of the remaining exceptional divisors of the embedded resolution with the strict transform to obtain the exceptional locus of the non-embedded resolution. We can easily decompose these intersections into irreducible components over $\mathbb{Q}$,but

[3]This is not a consequence of the algorithm itself, but of the two facts that the components of the center are always disjoint and that one of the enhancements discussed in the previous section was to do a primary decomposition of the center and blow-up in each of the components of the center separately

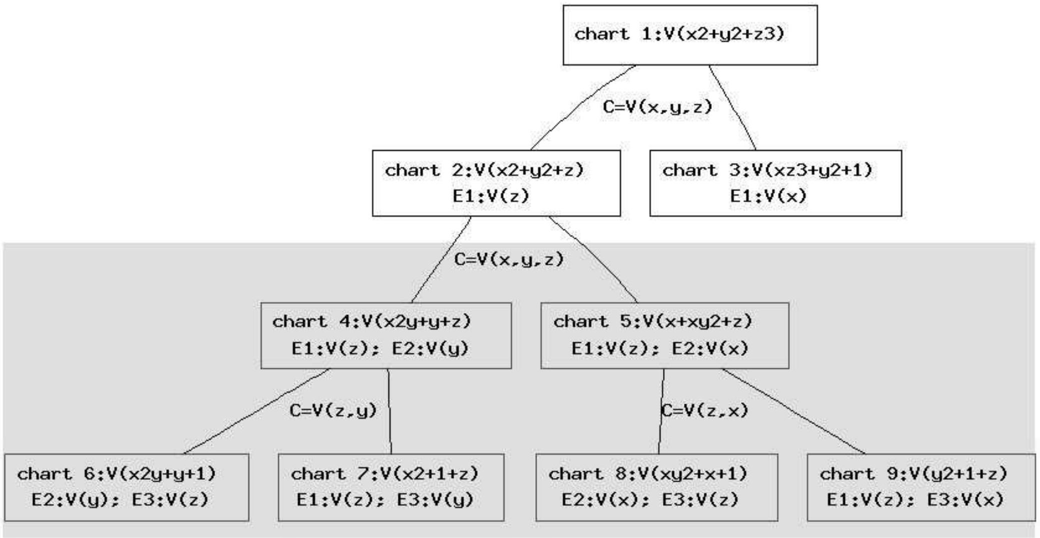

Figure 2: Tree of the embedded resolution process of an A_2 surface singularity. All charts which are marked by grey background arise from blow-ups which are only necessary in the embedded case, but not for a non-embedded resolution.

these components may still be reducible over $\mathbb{C}$ and hence we need to achieve a descomposition over $\mathbb{C}$ to compute the intersection matrix of the exceptional divisors. The following theorem (cf. [7]) is the basis for the decomposition over $\mathbb{C}$:

Theorem 4.1 (Gao/Ruppert). *Let $f \in \mathbb{Q}[x, y]$ be irreducible of bidegree (m, n). Let $G = \{g \in \mathbb{Q}[x, y] | (m - 1, n) \geq deg(g), \exists h \in \mathbb{Q}[x, y], \frac{\partial(g/f)}{\partial y} = \frac{\partial(h/f)}{\partial x}\}$. The vector space G has the following properties*

(i) f is irreducible in $\mathbb{C}[x, y]$ if and only if $dim_{\mathbb{Q}}(G) = 1$.

(ii) $gG \subset \frac{\partial f}{\partial x} G$ mod f for all $g \in G$.

(iii) Let $g_1, \ldots, g_a \in G$ be a basis and $g \in G \setminus \mathbb{Q}\frac{\partial f}{\partial x}$, $gg_i = \sum a_{ij} g_j \frac{\partial f}{\partial x}$ mod f. Let $\chi(t) = det(tE - (a_{ij}))$ be the characteristic polynomial. Then χ is irreducible in $\mathbb{Q}[t]$.

(iv) $f = \prod_{c \in \mathbb{C}, \chi(c)=0} gcd(f, g - c\frac{\partial f}{\partial x})$ is the decomposition of f into irreducible factors in $\mathbb{C}[x, y]$.

We use this theorem for the decomposition of curves in $\mathbb{C}^n$ which are irreducible over $\mathbb{Q}$ by means of the following corollary:

Corollary 4.2. *Let $I \subset \mathbb{Q}[x_1, \ldots, x_n], ht(I) = 1$, be a prime ideal. Then there exists an irreducible polynomial $\chi(t) \in \mathbb{Q}[t]$ such that the complex zeros of $\chi(t) = 0$ correspond to the associated prime ideals of $I\mathbb{C}[x_1, \ldots, x_n]$.*

Proof. Let such a curve be defined by a prime ideal $I \subset \mathbb{Q}[x_1, \ldots, x_n]$, $ht(I) = 1$. We may assume (after a generic linear coordinate change) that

$$\begin{aligned} I = \ (\ & a_1(x_n)x_1 - b_1(x_{n-1}, x_n), \ldots, \\ & a_{n-2}(x_n)x_{n-2} - b_{n-2}(x_{n-1}, x_n), b_{n-1}(x_{n-1}, x_n) \ ) \ : \ h \end{aligned}$$

with $b_{n-1} \in \mathbb{Q}[x_{n-1}, x_n]$ irreducible and a suitable $h \in \mathbb{Q}[x_n]$. Note that the set of polynomials $a_1(x_n)x_1 - b_1(x_{n-1}, x_n), \ldots, a_{n-2}(x_n)x_{n-2} - b_{n-2}(x_{n-1}, x_n), b_{n-1}(x_{n-1}, x_n)$ is a Groebner basis of $I\mathbb{Q}(x_n)[x_1, \ldots, x_{n-1}]$ with respect to the lexicographical ordering $x_1 > \ldots > x_{n-1}$ and $h = lcm(a_1, \ldots, a_{n-2})$.
If $b_{n-1} = d_1 \cdots d_k$ is the factorization of b_{n-1} over $\mathbb{C}[x_{n-1}, x_n]$ then

$$I = \cap((I, d_i) : h)$$

is the prime decomposition of I in $\mathbb{C}[x_1 \ldots x_n]$.
We have thus reduced our problem to the problem of factorization of a bivariate $\mathbb{Q}$-irreducible polynomial over the complex numbers; but the solution to this problem is given by 4.1 which provides the desired $\chi(t)$ by (iii). The correspondence of the zeros of $\chi(t)$ and the factors of the polynomial is statement (iv). $\square$

Over the field extension L of $\mathbb{Q}$ defined by χ, we can obtain at least one $\mathbb{C}$-irreducible factor. In general, this extension $L : \mathbb{Q}$ will not be Galois and b_{n-1} will not split completely. Theoretically, one would now pass to the corresponding Galois extension, but the practical computation thereof as well as explicit calculations in it tend to be very expensive. Therefore, we only compute one irreducible factor and additionally all solutions of $\chi(t) = 0$ by means of a numerical solver.

To illustrate this aspect, let us consider the factorization of the polynomial $x^3 + 2y^3$. In our implementation, the procedure `getMinpoly` computes the minimal polynomial described above and its numerical roots.

```
ring R=0,(x,y),dp;
poly p=x3-2y3;
getMinpoly(p);

[1]:
   poly p=t^3-2;
[2]:
   [1]:
      (-0.6299605249474365823836+i*1.0911236359717214035601)
   [2]:
      (-0.6299605249474365823836-i*1.0911236359717214035601)
   [3]:
      1.2599210498948731647672
[3]:
   3
```

If we factorize $x^3 + 2y^3$ over the field extension $\mathbb{Q}[t]/t^3 - 2$ we obtain two factors.

```
 ring T=(0,t),(x,y),dp;
 minpoly=t3-2;
 factorize(x3-2y3);

[1]:
   _[1]=1
```

```
   _[2]=x2+(t)*x*y+(t2)*y^2
   _[3]=x+(-t)*y
[2]:
   1,1,1
```

To obtain a complete factorization we need a Galois extension which is of higher degree. Therefore the factorization takes more time and so do all further calculations in this field (Just consider the large coefficients of y in our very simple example!).

```
 ring T=(0,t),(x,y),dp;
 minpoly=t6+3t5+6t4+11t3+12t2-3t+1;
 factorize(x3-2y3);

[1]:
   _[1]=1
   _[2]=x+(2/9t5+7/9t4+14/9t3+26/9t2+37/9t+2/9)*y
   _[3]=x+(1/9t5+2/9t4+4/9t3+4/9t2-1/9t-11/9)*y
   _[4]=x+(-1/3t5-t4-2t3-10/3t2-4t+1)*y
[2]:
   1,1,1,1
```

To identify the $\mathbb{C}$-components of an exceptional divisor E (irreducible over $\mathbb{Q}$) in a chart, we, therefore, store E, $\chi(t)$ and the respective numerical root of $\chi(t)$. Given these data, we can then proceed in the same way as for the identification of the $\mathbb{Q}$-components in section 3. As soon as the exceptional divisors in the different charts are identified, we can directly compute the intersection numbers $E_i.E_j$ for all $i \neq j$.

The computation of the self-intersection numbers E_i^2 is done by the following well-known method:
Let $\pi : X \longrightarrow Y$ be a resolution of the surface Y and $E_1, \ldots, E_s$ the exeptional divisors and let $h : Y \longrightarrow \mathbb{C}$ be a non-trivial linear form. Then $\pi^*(h).E_i = 0$.
Now we can write

$$\pi^*(h) = \sum_{i=1}^{s} c_i E_i + H,$$

where H denotes the strict transform, and obtain the equations

$$0 = \pi^*(h)E_i = \sum_{i=1}^{s} c_j E_j.E_i + H.E_i \quad \forall 1 \leq i \leq s$$

which provide us with the desired self-intersection numbers.

Example 4.3. *In our example at the beginning of this section (figure 2), the non-embedded resolution finishes after the first blow-up leading to the following situation in the charts:*

chart 2: *Forming the intersection of $X = V(x^2 + y^2 + z)$ and $E = V(z)$ leads to the $\mathbb{Q}$-irreducible curve $E_1 = V(x^2 + y^2, z)$, which obviously decomposes over $\mathbb{C}$ into the two curves $E_{1,1} = V(x + iy, z)$ and $E_{1,2} = V(x - iy, z)$. These meet in one point.*

chart 3: *In this chart, we see the same two curves $E_{1,1}$ and $E_{1,2}$, but we do not see the point in which they intersect.*

Therefore we are dealing with two exceptional divisors for which the intersection matrix is

$$\begin{pmatrix} * & 1 \\ 1 & * \end{pmatrix},$$

where the places for the (not yet computed) self-intersection numbers are filled by $*$. To determine these, we choose some linear form h, e.g. given by the polynomial x. As the next step, we need to determine the divisor of $\pi^*(h)$ in terms of the E_i in each of the chart to make sure that we do not miss out any contribution.

chart 2: $\pi^*(h) = 1 \cdot E_{1,1} + 1 \cdot E_{1,2} + V(x, y^2 + z)$

chart 3: $\pi^*(h) = 1 \cdot E_{1,1} + 1 \cdot E_{1,2} + V(1)$

This allows us to determine the self-intersection numbers as

$$\begin{aligned} 0 = \pi^*(h).E_{1,1} &= 1 \cdot E_{1,1}.E_{1,1} + 1 \cdot E_{1,2}.E_{1,1} + H.E_{1,1} \\ &= E_{1,1}^2 + 1 + 1 \\ 0 = \pi^*(h).E_{1,2} &= 1 \cdot E_{1,1}.E_{1,2} + 1 \cdot E_{1,2}.E_{1,2} + H.E_{1,1} \\ &= 1 + E_{1,2}^2 + 1 \end{aligned}$$

Therefore the complete intersection matrix is

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

5 Genus of Exceptional Curves on a Blown-up Surface

As soon as we have identified the $\mathbb{C}$-components of the exceptional divisors on the surface (as described in the preceding section), we are able to compute the genus of one $\mathbb{C}$-component. To this end, we first observe that the geometric genus is a birational invariant and therefore already determined by a single chart. Considering a $\mathbb{Q}$-irreducible exceptional divisor E and its decomposition $E = \cup_{i=1}^a E_i$ over $\mathbb{C}$ then we have

$$g(E) = ag(E_i) - a + 1 \quad \forall 1 \leq i \leq a.$$

Therefore the genus of the E_i can be computed from the genus of E, which, in turn, is already available in SINGULAR.

Example 5.1. *In the example of the previous section (figure 2), we choose chart 2 to determine the genus of $E_{1,1}$ and $E_{1,2}$. To this end, we compute the genus of E: $g(E) = -1$ and conclude by the above formula that both $E_{1,i}$ are of genus zero - which is exactly what we expected.*

Using SINGULAR, the computations of 4.3 and 5.1 look as follows:

```
LIB "zeta.lib";
ring R=0,(x,y,z),dp;
ideal I=x2+y2+z3;
list L=resolve(I);
intersectionDiv(L);
[1]:
   -2, 1
    1,-2
[2]:
    0,0
[3]:
   [1]:
      [1]:
          2,1,1
      [2]:
          3,1,1
```

6 Spectrum of a Hypersurface Singularity

In this section, we explain how to compute the negative part of the spectrum of an isolated hypersurface singularity using the resolution of the singularity. For a definition of the spectrum see e.g. [11],[12] Note that (denoting by μ the Milnor number) the spectrum $S = \{\alpha_1, \ldots, \alpha_\mu\}$ is a set of rational numbers in the open interval $(-1, n-1)$, where n the dimension of the ambient space. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a non-zero polynomial defining a hypersurface V with one isolated singularity at the origin and let $\pi : X \longrightarrow \mathbb{C}^n$ be an embedded resolution V. Let $E_i, 1 \leq i \leq s$ be the ($\mathbb{C}$-irreducible) exceptional divisors. For each $1 \leq j \leq s$, we denote by $N(E_j)$ the multiplicity of E_j in the divisor of $f \circ \pi$ and by $\nu(\omega, E_j) - 1$ the multiplicity of E_j in the divisor of $\pi^*(\omega)$, where ω is a holomorphic n-form in $\mathbb{C}^n$. In particular, $\nu(E_j) = \nu(dx_1 \wedge \ldots \wedge dx_n, E_j)$.
For each holomorphic n-form ω in $\mathbb{C}^n$ the geometric weight $g(\omega)$ with respect to the resolution is defined by

$$g(\omega) = min_{i \in I}\{\nu(\omega, E_i)/N(E_i)\} \ .$$

The following lemma is the basis for the computation of the negative part of the spectrum:

Lemma 6.1 (Varchenko). *A non-positive rational number s is in the spectrum S if and only if there exsist a holomorphic n-form ω in $\mathbb{C}^n$ such that $s = g(\omega) - 1$.*

Let us consider the following example:

```
LIB"zeta.lib";
ring R=0,(x,y,z),dp;
ideal I=(xz+y2)*(xz+y2+x2)+z5;
list L=resolve(I);

spectralNeg(L);

[1]:
```

```
   -11/40
[2]:
   -3/40
[3]:
   -1/20
[4]:
   -1/40
```

Remark 6.2. *In practice, all computations can be done over $\mathbb{Q}$:*
If D is a $\mathbb{Q}$- irreducible divisor in the resolution then D is a **disjoint** *union of some of the E_i from this resolution and $\nu(\omega, E_i) = \nu(\omega, D)$ resp.$N(E_i) = N(D)$.*

7 ζ-Function

In this section, we will describe how to compute the Denef-Loeser ζ - function for a given hypersurface singularity [4]. The Denef-Loeser ζ - function $Z_{top}^{(d)}(f,s)$ is defined as follows:
Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a non-zero polynomial defining a hypersurface V and let $\pi : X \longrightarrow \mathbb{C}^n$ be an embedded resolution of V. Denote by E_i , $i \in I$, the irreducible components of the divisor $\pi^{-1}(f^{-1}(0))$. For each subset $J \subset I$ we define

$$E_J := \cap_{j\in J} E_j \text{ and } E_J^* := E_J \setminus \cup_{j\notin J} E_{J\cup\{j\}}\ .$$

For each $j \in I$ let $N(E_j)$ and $\nu(E_j)$ be defined as in the previous section. The global Denef-Loeser ζ - function of f is

$$Z_{top}^{(d)}(f,s) := \sum_{\substack{J\subset I \text{s.th.} \\ d|N(E_j)\forall j\in J}} \chi(E_J^*) \prod_{j\in J} (\nu(E_j) + N(E_j)s)^{-1} \in \mathbb{Q}(s).$$

Intersecting the E_J^* with the preimage of zero in the above formula leads to the local Denef-Loeser zeta function

$$Z_{top,0}^{(d)}(f,s) := \sum_{\substack{J\subset I \text{s.th.} \\ d|N(E_j)\forall j\in J}} \chi(E_J^* \cap \pi^{-1}(0)) \prod_{j\in J} (\nu(E_j) + N(E_j)s)^{-1} \in \mathbb{Q}(s).$$

Lemma 7.1. *Let D_l , $l \in L$, be the $\mathbb{Q}$-irreducible components of the divisor $\pi^{-1}(f^{-1}(0))$. For each subset $J \subset L$ define D_J and D_J^* as above. Then*

$$Z_{top}^{(d)}(f,s) := \sum_{\substack{J\subset L \text{s.th.} \\ d|N(D_j)\forall j\in J}} \chi(D_J^*) \prod_{j\in J} (\nu(D_j) + N(D_j)s)^{-1} \in \mathbb{Q}(s), resp.$$

$$Z_{top,0}^{(d)}(f,s) := \sum_{\substack{J\subset I \text{s.th.} \\ d|N(D_j)\forall j\in J}} \chi(D_J^* \cap \pi^{-1}(0)) \prod_{j\in J} (\nu(D_j) + N(D_j)s)^{-1} \in \mathbb{Q}(s).$$

[4]In practice, only the case of dimension 2 is implemented

Proof. The D_l are smooth and therefore disjoint unions of some of the E_j. This implies that for $M = \{l_1, \dots, l_q\} \subset L$ we obtain $\chi(D_M^*) = \sum \chi(E_J^*)$, the sum is on all subsets $J = \{j_1, \dots, j_q\} \subset I$ such that E_{j_i} is a component of D_{l_i}. Moreover, it is easy to see that for a component E_j of D_l always $N(E_j) = N(D_l)$ and $\nu(E_j) = \nu(D_l)$. □

In the case of surfaces (n=3) the Euler characteristic $\chi(D_J^*)$ of the D_J^* can be computed as follows:

(i) $\chi(D_j^*) = 0$ for each $\mathbb{Q}$-component of the strict transform

(ii) For an exceptional divisor D_j the Euler characteristic $\chi(D_j)$ can be computed via the resolution tree using the following facts: Let $\sigma : Y \longrightarrow Z$ be a single blow-up step with center C in the resolution tree leading to the exceptional divisor E and let D be a divisor in Z intersecting C transversally with strict transform $\tilde{D}$.

  (a) $\chi(E) = k \cdot \chi(\mathbb{P}^2) = 3k$ if C is a $\mathbb{Q}$-irreducible zero-dimensional center corresponding to k points over $\mathbb{C}$.

  (b) $\chi(E) = \chi(\mathbb{P}^1 \times C) = 4 - 4g(C)$ if C is a curve.

  (c) $\chi(\tilde{D}) - \chi(E \cap \tilde{D}) = \chi(D) - \chi(C \cap D)$

(iii) If $J = \{i, j, k\}$ then $\chi(D_J^*) = \#(D_i \cap D_j \cap D_k)$.

(iv) $\chi(D_i \cap D_j) = 2 - 2g(D_i \cap D_j)$.

(v) If $J = \{i, j\}$ then $\chi(D_J^*) = \chi(D_i \cap D_j) - \sum_{k, k \neq i,j} \#(D_i \cap D_j \cap D_k)$.

(vi) $\chi(D_i^*) = \chi(D_i) - \sum_{i \neq j} \chi(D_i \cap D_j) + \sum_{j,k \neq i} \#(D_i \cap D_j \cap D_k)$.

Example 7.2. *Let us consider the same example as before, an A_2 surface singularity: As already illustrated in figure 2, the tree of blow-ups of the resolution process consists of 9 charts and we have 2 exceptional divisors arising from blow-ups whose center is a point and 1 from a blow-up whose center is 1-dimensional. The strict transform of the original V is irreducible and will be denoted by E_4. Since the surface only has one singular point at the origin, the global and the local Denef-Loeser zeta function coincide in this case.*

As can easily be checked by direct calculation, we have

J	$\chi(E_J^*)$	J	$\chi(E_J^*)$	J	$\chi(E_J^*)$
1	0	1,2	0	1,2,3	0
2	1	1,3	0	1,2,4	0
3	0	1,4	2	1,3,4	2
4	0	2,3	2	2,3,4	0
		2,4	0		
		3,4	0		

For the multiplicities, we obtain

	E_1	E_2	E_3	E_4
N	2	3	6	1
ν	3	5	8	1

Hence, we obtain the Denef-Loeser zeta-function

$$\begin{aligned} \frac{1}{3s+5} &+ \frac{2}{(2s+3)(s+1)} + \frac{2}{(3s+5)(6s+8)} + \frac{2}{(2s+3)(6s+8)(s+1)} \\ &= \frac{s+4}{(3s+4)(s+1)} \end{aligned}$$

Using SINGULAR, the computation looks as follows:

```
LIB"zeta.lib";
ring R=0,(x,y,z),dp;
ideal I=x2+y2+z3;
list re=resolve(I);
zetaDL(re,1);
[1]:
(s+4)/(3s2+7s+4)
```

8 Implementation and Timings

All algorithms described in this article have been implemented and are available as part of the SINGULAR software package [10] (versions 2.2 and higher). All implementations have been carried out in the programming language of SINGULAR. The following timings have been taken on an Intel Celeron 2 GHz, 256 MB RAM (all timings are specified in seconds):

Example	equation	identify divisors	intersection matrix and genus of the E_i	Denef-Loeser zeta-funct.
A_{10}	$x^{11}+y^2-z^2$	2	11	15
A_{20}	$x^{21}+y^2-z^2$	5	26	41
E_6	$x^3+y^4+z^2$	3	13	24
W_{17}	$y^7+xy^5+x^4+z^2$	14	29	121
	$x^2y^2z^2+x^7+y^8+z^8$	68	105	694

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