# Lecture Notes on Local Analytic Geometry

CIMPA-ICTP-UNESCO-MESR-MICINN-PAKISTAN Research School*

Barbu Berceanu
Institute of Mathematics of the Romanian Academy, Romania
*E-mail address:* barbu.berceanu@imar.ro

Jürgen Herzog
University of Duisburg-Essen, Germany
*E-mail address:* juergen.herzog@uni-essen.de

Olav Arnfinn Laudal
University of Oslo, Norway
*E-mail address:* arnfinnl@math.uio.no

Gerhard Pfister
University of Kaiserslautern, Germany
*E-mail address:* pfister@mathematik.uni-kl.de

Dorin Popescu
Institute of Mathematics of the Romanian Academy, Romania
*E-mail address:* Dorin.Popescu@imar.ro

Peter Schenzel
University of Halle, Germany
*E-mail address:* schenzel@informatik.uni-halle.de

Stefan Steidel
University of Kaiserslautern, Germany
*E-mail address:* steidel@mathematik.uni-kl.de

Marius Vladoiu
University of Bucharest, Romania
*E-mail address:* vladoiu@gta.math.unibuc.ro

February 4 - February 13, 2012

*Abdus Salam School of Mathematical Sciences, GC University Lahore – Pakistan

**Abstract**

The aim of the CIMPA-ICTP-UNESCO-MESR-MICINN-PAKISTAN Research School on Local Analytic Geometry at Abdus Salam School of Mathematical Sciences, GC University Lahore, Pakistan (organized by A. D. Choudary, Alexandru Dimca and Gerhard Pfister) was to introduce the participants to local analytic geometry and related topics. A basis of the course was the book entitled *Local Analytic Geometry* (cf. [JP00]). An important part of the school was to establish computational methods in local algebra and to provide an introduction to the computer algebra system SINGULAR (cf. [DGPS12]). This article includes the lecture notes corresponding to the lectures held during the school and was edited by Gerhard Pfister and Stefan Steidel.

# Contents

# 1 Singular and Applications (G. Pfister)

The introductory talk by G. Pfister about SINGULAR and its applications is online available at

*http://www.sms.edu.pk/downloads/Cimpa/2012/vortragCimpa2012.1.pdf.*

# 2 Basics of Analytic Geometry (O. A. Laudal)

This section contains a short reminder of the theory of complex analytic functions, in one variable, and an introduction to the basics of local analytic geometry, with the purpose of setting the stage for the study of complex analytic singularities. Most of the material is extracted from [JP00, Chapter 3].

## 2.1 Functions of a Complex Variable

Let $\mathbf{C}$ be the field of complex numbers. An element $z \in \mathbf{C}$, may be written $z = x + \imath y$, where $\imath = \sqrt{-1}$, and $x, y \in \mathbf{R}$, are real numbers. $\mathbf{C}$ may be considered as a real plane with coordinates $(x, y)$, with topology induced from the Euclidean metric, and the obvious differential structure. Consider the two vector fields,

$$\frac{\partial}{\partial \overline{z}} := \frac{\partial}{\partial x} + \imath \frac{\partial}{\partial y}, \; \frac{\partial}{\partial z} := \frac{\partial}{\partial x} - \imath \frac{\partial}{\partial y}.$$

Any differentiable function, defined in an open subset $U \subset \mathbf{C}$,

$$f : U \to \mathbf{C},$$

for which the following Cauchy-Riemann equation is satisfied,

$$\frac{\partial f}{\partial \overline{z}} := 0,$$

is called a *complex analytic function.* It can, for every point $o \in U$, be written as a formal power series,

$$f(z) = \sum_0^\infty a_n (z - o)^n,$$

converging absolutely in some neighbourhood of $o$. Moreover we have,

$$\frac{\partial f}{\partial z} = \sum_0^\infty n a_n (z - o)^{n-1},$$

as convergent power series. Conversely, any absolutely converging formal power series in a neighbourhood of a point $o \in \mathbf{C}$, satisfies the Cauchy-Riemann equation in the neighbourhood, and is therefore a *complex analytic function* in that neighbourhood. Since the operator $\frac{\partial}{\partial \overline{z}}$ is a derivation, it follows immediately that the *complex analytic functions*, defined in an open subspace $U \subset \mathbf{C}$, forms a ring, $\Omega(U)$. Recall the following *Identity Theorem*,

**Theorem 2.1.** *Given two analytic functions $f, g \in \Omega(U)$, $U$ a connected open subset of $\mathbf{C}$, if $f = g$ on some neighbourhood of a point in $U$, then $f = g$ in $\Omega(U)$.*

This implies quite easily the very importent *maximum modulus principle*,

**Theorem 2.2.** *Let $f \in \Omega(U)$, $U$ a connected open subset of $\mathbf{C}$. Let $o \in U$, and suppose*

$$|f(o)| \geqslant |f(z)|,$$

*for all $z$ in some neighbourhood of the point $o$. Then $f$ is constant in $U$.*

Let, for $o \in U$, $\mathbf{O}_o$ be the ring of *germs of analytic functions at the point $o$*, i.e. the quotient ring of equivalence classes of analytic functions defined in some neighbourhood of $o$, under the equivalence relation given as,

$$f \sim g,$$

if there exists a neighbourhood $o \in V$, such that $f$ and $g$ are both defined in $V$, and such that their restrictions to $V$, are equal. Clearly, the map,

$$\pi : O_o \to \mathbf{C}$$

defined by, $\pi(f) = f(o)$, is a homomorphism of rings, and one easily shows that $O_o$ is a local ring, with maximal ideal, $\mathbf{m} = ker\pi$. It is clear that $O_o$ may be identified with the ring of converging power series, $\mathbf{C}\{\{z\}\} \subset \mathbf{C}[[z]]$.

## 2.2 Analytic Functions of Several Variables

Now, let $\mathbf{C}[[z]]$ be the $\mathbf{C}$-algebra of formal power series in $n$ variables $z := (z_1, \ldots, z_n)$. Consider the topological space, $\mathbf{C}^n$, with complex coordinates, $(z_1, \ldots, z_n)$, and denote by $o$ the point $[0, ..., 0] \in \mathbf{C}^n$. Let $r = (r_1, \ldots, r_n) \in \mathbf{R}^n$ be positive real numbers, and consider the *polycylindre*,

$$\mathbf{P_r} := \{z \in \mathbf{C}^n |\ |z_i| < r_i, i = 1, ..., n\}$$

**Lemma 2.3.** *Let $p = (p_1, \ldots, p_n) \in \mathbf{C}^n$, with $p_i \neq 0,\ i = 1, \ldots, n$, and assume a formal power series $f \in \mathbf{C}[[z]]$, converges for $z = p$. Then there is polycylindre $\mathbf{P_r}$, such that $f$ is absolutely and uniformly convergent on all compact subsets of $\mathbf{P_r}$.*

*Proof.* Since $f = \sum_{\underline{l}, |\underline{l}|=0}^{\infty} a_{\underline{l}} z^{\underline{l}}$, where we have put, $\underline{l} := (l_1, \ldots, l_n), |\underline{l}| := \sum_i l_i,\ a_{\underline{l}} := a_{l_1,\ldots,l_n},\ z^{\underline{l}} := z_1^{l_1} \cdots z_n^{l_n}$, converges for $z = p$, the real numbers $|a_{\underline{l}} p^{\underline{l}}|$ must be bounded by some positive number $M$. Let $0 < t_i < 1, i = 1, \ldots, n$ be real numbers, then we must have,

$$|a_{\underline{l}} p^{\underline{l}} t^{\underline{l}}| \leqslant M t^{\underline{l}},$$

implying that, for any $z \in \mathbf{P_r}$, with $r_i = |p_i| t_i, i = 0, \ldots, n$,

$$\sum_0^\infty |a_{\underline{l}} z^{\underline{l}}| \leqslant \sum_0^\infty M t^{\underline{l}} = M(\prod_1^n \frac{1}{1 - t_i}),$$

proving the contention. □

Let $U \subset \mathbf{C}^n$ be an open subset, and consider a map,

$$\varphi : U \to \mathbf{C}.$$

**Definition 2.4.** $\varphi$ is called analytic if for all $o \in U$, there is a formal power series $f$, in an appropriate coordinate system, converging in a neighbourhood, $V$ of $o$, such that, restricted to V, we have $\varphi = f$. The set of analytic functions in the open subset, $U$, is denoted by $\mathbf{O}(U)$.

Since absolutely and uniformly convergent power series form a ring, we observe that $\mathbf{O}$ is a sheaf of rings defined on the (strong) topology of $\mathbf{C}^n$. If the coordinate system $(z_1, ..., z_n)$, and its origin $o$ is as above, we find that the germ of analytic functions $\mathbf{O}_o$defined at $o$ may be identified with the subring,

$$\mathbf{C}\{\{z\}\} \subset \mathbf{C}[[z]],$$

of convergent power series at $o$. Clearly, this ring of germs of analytic functions in $n$ variables, is local with maximal ideal $\mathbf{m} \subset \mathbf{C}\{\{z\}\}$ generated by the *coordinate functions*, $z_i, i = 1, \dots, n$.

## 2.3 Analytic Subsets, Analytic Geometry

Since 1637, when René Descartes wrote *La Géométrie*, we are used to call a subset $X$ of $\mathbf{C}^n$, (or at least, of $\mathbf{R}^2$), an algebraic subset, if it can be *can be cut out* by a set of *algebraic equations.* The analogous notion of *analytic subset* of $\mathbf{C}^n$, is defined in the same way.

**Definition 2.5.** A subset $X \subseteq \mathbf{C}^n$ is called *locally analytic*, if for any $p \in X$, there exists an open neighbourhood $V$of $p$ in $\mathbf{C}^n$, and analytic functions, $f_1, \dots, f_m \in \mathbf{O}(V)$, such that,

$$X \cap V = \{z \in V \mid f_i(z) = 0, i = 1, \dots, m\}.$$

$X$ is called *an analytic subset* of the open subset $U \subseteq \mathbf{C}^n$ if it is both locally analytic, and closed in $U$.

We need several very important results, the proofs of which we shall omit, referring to the text [JP00, Chapter 3]. First, a theorem saying that a non-trivial analytic subset of $\mathbf{C}^n$ is nowhere dense,

**Theorem 2.6.** *Let $U \subseteq \mathbf{C}^n$ be an open subset, and let $X \subset U$ be analytic, then,*

$$\overline{U - X} = U.$$

Next, we need the following,

**Theorem 2.7** (Riemann Extension Theorem)**.** *With the notations above, let*

$$f : U - X \to \mathbf{C}$$

*be an analytic function, locally bounded in $X$, then $f$ is the restriction of an analytic function defined in $U$.*

## 2.4 Tools for studying Local Analytic Geometry

Let $f \in \mathbf{C}\,[[z]]$, be a formal power series. We shall say that $f$ is of *order* $b$ (at $o$) if,

$$f = \sum_{|\underline{l}| \geqslant b} a_{\underline{l}} z^{\underline{l}},$$

and if,

$$\sum_{|\underline{l}|=b} a_{\underline{l}} z^{\underline{l}}, \neq 0.$$

The formal power series is said to be *regular* of order $b$ in the coordinate $z_n$, if

$$\sum_{|\underline{l}|=b} a_{\underline{l}} z^{\underline{l}} = z_n^b,$$

and we call it a *Weierstrass polynomia* in $z_n$, if it has the form,

$$f = z_n^b + c_1 z_n^{b-1} + \ldots + c_b,\ c_j \in \mathbf{C}\left[[z_1, \ldots, z_{n-1}]\right], c_i(o) = 0, i = 1, \ldots, b.$$

These definitions makes it possible to state a number of essential theorems for the study of local analytic geometry. First, the *Noether's Normalisation Theorem for Hypersurfaces*,

**Theorem 2.8.** *Assume the formal power series $f$ is of order $b$, then there exists a coordinate change with respect to which, $f$ becomes regular of order $b$ in $z_n$.*

*Proof.* Suppose, $\sum_{|\underline{l}|=b} a_{\underline{l}} \neq 0$, then, put,

$$z_i = z_n - z_i', i < n, z_n' = z_n,$$

and check that the coordinate system $(z_1', \ldots, z_n')$, does the job.

If $\sum_{|\underline{l}|=b} a_{\underline{l}} = 0$, we may pic $\beta_i, i = 1, \ldots, n-1$, and put,

$$z_i = \beta_i z_n - z_i', i < n, z_n' = z_n,$$

such that, in the new coordinate system, $\sum_{|\underline{l}|=b} a_{\underline{l}} \neq 0$, completing the proof. □

The next theorem, along these lines, is the weak *Weierstrass Division Theorem* (WDT),

**Theorem 2.9.** *Let $f, g \in \mathbf{C}\{\{z_1, \ldots, z_n\}\}$. Assume $f$ is regular of order $b$ in $z_n$. Then there are unique elements, $q \in \mathbf{C}\{\{z_1, \ldots, z_n\}\}, r \in \mathbf{C}\{\{z_1, \ldots, z_{n-1}\}\}[z_n]$ with degree, as polynomial in $z_n$, $deg(r) < b$, such that,*

$$f = qg + r.$$

This result is a consequence of a more general theorem which we will prove later. It is, however, useful here for the proof of the *Weierstrass Preparation Theorem.*

**Theorem 2.10.** *Let $f \in \mathbf{C}\{\{z_1, \ldots, z_n\}\}$ be regular of order $b$ in $z_n$. Then there exists a unique unit, $u \in \mathbf{C}\{\{z_1, \ldots, z_n\}\}$, and a Weierstrass polynomial $q$ in $z_n$, such that,*

$$f = uq$$

*Proof.* Use the Weierstrass Division Theorem, with $f$ as given, and $g = z_n^n$. Then we find,

$$z_n^b = qf + r,$$

where $r$ is a polynomial in $z_n$ of degree less than $b$. It follows that $q$ is a unit, and that,

$$f = q^{-1}(z_n^b - r),$$

and because of the unicity of $q, r$ it is clear that $(z_n^b - r)$, is a Weierstrass polynomial. □

From this we easily deduce the following,

**Corollary 2.11.** *Let* $f \in \mathbf{C}\{\{z_1, \dots, z_n\}\}$ *be regular in* $z_n$*, of order* $b$*, then the* $\mathbf{C}\{\{z_1, \dots, z_{n-1}\}\}$*-module,* $\mathbf{C}\{\{z_1, \dots, z_n\}\}/(f)$ *is finitely generated of rank* $b$*.*

Now we are ready for the *General Weierstrass Division Theorem* (GWDT). Recall that the maximal ideal of the local algebra, $R := \mathbf{C}\{\{y_1, \dots, y_m\}\}$, denoted $\mathbf{m}_R$ is generated by the coordinate functions, i.e. by $(y_1, \dots, y_m)$.

**Theorem 2.12.** *Let,*

$$\phi : R := \mathbf{C}\{\{y_1, \dots, y_m\}\} \to S := \mathbf{C}\{\{z_1, \dots, z_n\}\},$$

*be a homomorphism of algebras. Then the following two conditions are equivalent,*

1. *$S$ is a finitely generated $R$-module*
2. *$dim_{\mathbf{C}} S/\mathbf{m}S < \infty$*

An elementary, but long proof is found in [JP00, Chapter 3]. Let us see how to deduce the Weak Weierstrass Division Theorem, from this (GWDT). Put $m = n$, and $\phi(y_i) = z_i, i = 1, \dots, n-1, \phi(y_n) = f$. If $f$ is regular in $z_n$, of order $b$, then obviously $dim_{\mathbf{C}} S/\mathbf{m}S = b$, and so $S$ is finitely generated as $R$-module, that is $\mathbf{C}\{\{z_1, \dots, z_n\}\}/(f)$ is a finitely generated $\mathbf{C}\{\{z_1, \dots, z_{n-1}\}\}$-module. By Nakayamas Lemma, is generated by, $\{1, z_n, \dots, z_n^b\}$, which is exactly the contention of the (WDT).

Now, one of the most essential emerging from these tools, is the *Implicit Function Theorem,*

**Theorem 2.13.** *Let* $f \in \mathbf{C}\{\{z_1, \dots, z_n, y\}\}$*, and assume,*

$$f(o) = 0, \ \frac{\partial f}{\partial y} \neq 0,$$

*then there exists a unique element,* $\varphi \in \mathbf{C}\{\{z_1, \dots, z_n\}\}$ *such that,*

$$f(z_1, \dots, z_n, y) = 0$$

*if and only if,*

$$y = \varphi(z_1, \dots, z_n).$$

*Proof.* The conditions imply that $f$ is regular of order 1 in $y$. WPT then tells us that there are a unit $u$ and a unique $\varphi(z_1, \dots, z_n) \in \mathbf{C}\{\{z_1, \dots, z_n\}\}$ such that,

$$f = u(y - \varphi(z_1, \dots, z_n)),$$

which is what we want. □

We end this introduction with the following.

**Theorem 2.14.** *The local ring* $\mathbf{O}_n := \mathbf{C}\{\{z_1, \dots, z_n\}\}$ *is Noetherian.*

*Proof.* By induction on the dimension $n$. If $\mathbf{O}_{n-1}$ is Noetherian, then by Hilbert's Basis Satz, $\mathbf{O}_{n-1}[z_n]$ is Noetherian. If $\alpha \subset \mathbf{O}_n$ is an ideal, and $g \in \alpha$, then by WPT there exists a unit $u$ and a Weierstrass polynomial, $h$, such that $g = uh$, implying that $h \in \alpha$. We know that the ideal $\alpha \cap \mathbf{O}_{n-1}[z_n]$ is finitely generated, by say, $(g_1, \dots, g_r)$. This implies that $\alpha = ((g_1, \dots, g_r), g)$, since any $f \in \alpha$, by WDT can be written as $f = qg + s$, with $s \in \alpha \cap \mathbf{O}_{n-1}[z_n]$. □

## 3 Basics of Local Algebra (B. Berceanu)

No contribution by the author. We refer to [JP00, Chapter 1].

## 4 Standard Bases (S. Steidel)

The aim of this section is to develop the algorithm to compute a standard basis respectively Gröbner basis of a given ideal due to Buchberger (cf. [B65]). Moreover, we illustrate in some examples how to use SINGULAR (cf. [DGPS12]) for this demand.

### 4.1 Motivation

Consider the ideal $I = \langle xy + y, y^2 + x \rangle \subseteq \mathbb{Q}[x, y]$. At the the end of this section we will answer the following questions:

1. $f = x^4 + x^2y^2 \in I$ ?
2. $\dim_{\mathbb{Q}} \big(\mathbb{Q}[x, y]/I\big) =$ ?
3. $I \cap \mathbb{Q}[y] =$ ?

### 4.2 Notation

Let $K$ be a field and $X = \{x_1, \ldots, x_n\}$ be a set of variables. We denote by $\text{Mon}(X) = \{X^\alpha \mid \alpha \in \mathbb{N}^n\} = \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha_i \in \mathbb{N}\}$ the set of monomials, and by $K[X]$ the polynomial ring over $K$ in these $n$ variables.

### 4.3 Monomial orderings & leading data

**Definition 4.1.** A *monomial ordering* is a total ordering $>$ on $\text{Mon}(X)$ such that

$$X^\alpha > X^\beta \implies X^\gamma X^\alpha > X^\gamma X^\beta \quad \forall\, \alpha, \beta, \gamma \in \mathbb{N}^n.$$

**Definition 4.2.** Let $>$ be a fixed monomial ordering on $\text{Mon}(X)$ and let $f \in K[X] \setminus \{0\}$ be a polynomial. Then we obtain a unique representation of $f$ as a sum of non-zero terms

$$f = c_\alpha X^\alpha + c_\beta X^\beta + \ldots + c_\gamma X^\gamma$$

with $X^\alpha > X^\beta > \ldots > X^\gamma$ and $c_\alpha, c_\beta, \ldots, c_\gamma \in K$. Then we define

1. the *leading monomial* of $f$ by $\text{LM}(f) := X^\alpha$,
2. the *leading coefficient* of $f$ by $\text{LC}(f) := c_\alpha$,
3. the *leading term* of $f$ by $\text{LT}(f) := \text{LC}(f) \cdot \text{LM}(f) = c_\alpha X^\alpha$,
4. the *tail* of $f$ by $\text{tail}(f) := f - \text{LT}(f) = c_\beta X^\beta + \ldots + c_\gamma X^\gamma$.

There is an important distinction between *global* and *local monomial orderings* which especially influences the uniqueness of a standard or Gröbner basis.

**Definition 4.3.** Let $>$ be a monomial ordering on $\text{Mon}(X)$.

1. $>$ is called *global* if $X^\alpha > 1$ for all $\alpha \in \mathbb{N}^n \smallsetminus \{0\}$.
2. $>$ is called *local* if $X^\alpha < 1$ for all $\alpha \in \mathbb{N}^n \smallsetminus \{0\}$.
3. $>$ is called *mixed* if it is neither global nor local.

Some of the most important monomial orderings are introduced in the following definition.

**Definition 4.4.** Let $X^\alpha, X^\beta \in \mathrm{Mon}(X)$. The following are monomial orderings on $\mathrm{Mon}(X)$.

1. The *lexicographical ordering* $>_{lp}$:

$$X^\alpha >_{lp} X^\beta \quad :\Longleftrightarrow \quad \begin{array}{l} \exists\, 1 \le i \le n:\ \alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}, \\ \alpha_i > \beta_i. \end{array}$$

2. The *degree reverse lexicographical ordering* $>_{dp}$:

$$X^\alpha >_{dp} X^\beta \quad :\Longleftrightarrow \quad \begin{array}{l} \deg(X^\alpha) > \deg(X^\beta) \text{ or} \\ \big(\deg(X^\alpha) = \deg(X^\beta) \text{ and } \exists\, 1 \le i \le n: \\ \alpha_n = \beta_n, \ldots, \alpha_{i+1} = \beta_{i+1},\, \alpha_i < \beta_i\big). \end{array}$$

3. The *negative degree reverse lexicographical ordering* $>_{ds}$:

$$X^\alpha >_{ds} X^\beta \quad :\Longleftrightarrow \quad \begin{array}{l} \deg(X^\alpha) < \deg(X^\beta) \text{ or} \\ \big(\deg(X^\alpha) = \deg(X^\beta) \text{ and } \exists\, 1 \le i \le n: \\ \alpha_n = \beta_n, \ldots, \alpha_{i+1} = \beta_{i+1},\, \alpha_i < \beta_i\big). \end{array}$$

Note that $>_{lp}$ and $>_{dp}$ are global orderings, whereas $>_{ds}$ is a local ordering.

*Example* 4.5. Consider $x^3y^2z, x^2y^4, z^8 \in \mathrm{Mon}(x, y, z)$.

1. $x^3y^2z >_{lp} x^2y^4 >_{lp} z^8$.
2. $z^8 >_{dp} x^2y^4 >_{dp} x^3y^2z$.
3. $x^2y^4 >_{ds} x^3y^2z >_{ds} z^8$.

The localization of $K[X]$ with respect to the maximal ideal $\langle X\rangle = \langle x_1, \ldots, x_n\rangle$ is defined by

$$K[X]_{\langle X\rangle} = \left\{ \frac{f}{g} \;\middle|\; f, g \in K[X],\, g(0, \ldots, 0) \neq 0 \right\}.$$

Now, we define the localization of $K[X]$ with respect to any monomial ordering. The set

$$S_> := \{u \in K[X] \smallsetminus \{0\} \mid \mathrm{LM}(u) = 1\}$$

is a multiplicatively closed set for any monomial ordering $>$.

**Definition 4.6.** Let $>$ be any monomial ordering on $\mathrm{Mon}(X)$. The ring

$$K[X]_> := S_>^{-1}K[X] = \left\{ \frac{f}{u} \;\middle|\; f, u \in K[X],\, u \neq 0,\, \mathrm{LM}(u) = 1 \right\}$$

is called the *localization of* $K[X]$ *with respect to* $S_>$.

Note that there are canonical inclusions $K[X] \subseteq K[X]_> \subseteq K[X]_{\langle X \rangle}$ and the following properties are quite obvious.

1. $K[X]_> = K[X] \iff S_> = K \setminus \{0\} \iff >$ is global.
2. $K[X]_> = K[X]_{\langle X \rangle} \iff S_> = K[X] \setminus \langle X \rangle \iff >$ is local.

We extend the leading data defined in Definition 4.2 to $K[X]_>$.

**Definition 4.7.** Let $>$ be any monomial ordering on $\mathrm{Mon}(X)$. For $f \in K[X]_> \setminus \{0\}$ choose $u \in K[X] \setminus \{0\}$ such that $\mathrm{LT}(u) = 1$ and $uf \in K[X]$. Then we define

1. the *leading monomial* of $f$ by $\mathrm{LM}(f) := \mathrm{LM}(uf)$,
2. the *leading coefficient* of $f$ by $\mathrm{LC}(f) := \mathrm{LC}(uf)$,
3. the *leading term* of $f$ by $\mathrm{LT}(f) := \mathrm{LT}(uf)$,
4. the *tail* of $f$ by $\mathrm{tail}(f) := f - \mathrm{LT}(f)$.

For a given set of polynomials we consider some special ideal, the so-called *leading ideal* which is the central notion when defining a standard basis of an ideal.

**Definition 4.8.** Let $>$ be any monomial ordering on $\mathrm{Mon}(X)$ and $G \subseteq K[X]_>$ be a set of polynomials. Then we define the *set of leading monomials* of $G$ by $\mathrm{LM}(G) := \{\mathrm{LM}(g) \mid g \in G\}$, and the *leading ideal* of $G$ by $L(G) := \langle \mathrm{LM}(g) \mid g \in G \setminus \{0\} \rangle_{K[X]}$.

## 4.4 Normal forms

**Definition 4.9.** Let $f \in K[X]_>$ and $G \in K[X]_>$.

1. $f$ is called *reduced with respect to $G$* if no monomial of $f$ is contained in $L(G)$.
2. $G$ is called *reduced* if
   (a) $0 \notin G$,
   (b) $\mathrm{LM}(g) \nmid \mathrm{LM}(f)$ for all $f, g \in G$,
   (c) $\mathrm{LC}(g) = 1$ for all $g \in G$,
   (d) $\mathrm{tail}(g)$ is reduced with respect to $G$ for all $g \in G$.

*Remark* 4.10. If $>$ is global then each finite set can be transformed into a reduced one generating the same ideal.

If $>$ is local this is general not achievable. For instance, consider $>_{ds}$ and $f = x - x^2$. Then $f$ is not reduced with respect to $\{f\}$. But also the reduction $f + xf = x - x^3$ is not reduced with respect to $\{f\}$, and this process does not end in finitely many steps.

**Definition 4.11.** Let $\mathcal{G}$ denote the set of all finite ordered sets $G \subseteq K[X]_>$. A map

$$\mathrm{NF} : K[X]_> \times \mathcal{G} \longrightarrow K[X]_>,\ (f, G) \longmapsto \mathrm{NF}(f, G)$$

is called a *normal form* on $K[X]_>$ if, for all $f \in K[X]_>$ and all $G \in \mathcal{G}$, the following hold.

(a) $\mathrm{NF}(0, G) = 0$.

(b) $\mathrm{NF}(f, G) \neq 0$ implies $\mathrm{LM}(\mathrm{NF}(f, G)) \notin L(G)$.

(c) If $G$ is finite, then either $f - \mathrm{NF}(f, G) = 0$, or there exists a representation

$$f - \mathrm{NF}(f, G) = \sum_{g \in G} c_g g, \quad c_g \in K[X]_>$$

such that

$$\mathrm{LM}(f) \geq \max\{\mathrm{LM}(c_g g) \mid g \in G,\ c_g g \neq 0\}.$$

This representation is called the *standard representation* of $f - \mathrm{NF}(f, G)$ with respect to $G$.

Moreover, NF is called a *reduced nornal form* if $\mathrm{NF}(f, G)$ is reduced with respect to $G$ for all $f \in K[X]_>$ and all $G \in \mathcal{G}$.

*Remark* 4.12. Analogously to Remark 4.17, in the case of a global ordering there exists a normal form, and even a reduced normal form, due to B. Buchberger. In the case of an arbitrary ordering there exists a *polynomial weak normal form* (cf. [GP07, Definition 1.6.5, Algorithm 1.7.6]), due to T. Mora.

Hence, for simplification, in the remaining considerations we mainly restrict ourselves to the case that $>$ is a global monomial ordering.

To provide an algorithm to compute a normal form, we make use of the following definition.

**Definition 4.13.** Let $f, g \in K[X]_> \setminus \{0\}$ with $\mathrm{LM}(f) = X^\alpha$ and $\mathrm{LM}(g) = X^\beta$. Set $\gamma = \mathrm{lcm}(\alpha, \beta) = (\max(\alpha_1, \beta_1), \ldots, \max(\alpha_n, \beta_n) \in \mathbb{N}^n$, and let $X^\gamma := \mathrm{lcm}(X^\alpha, X^\beta)$ be the *least common multiple* of $X^\alpha$ and $X^\beta$. We define the *s-polynomial* of $f$ and $g$ by

$$\mathrm{spoly}(f, g) := X^{\gamma - \alpha} \cdot f - \frac{\mathrm{LC}(f)}{\mathrm{LC}(g)} \cdot X^{\gamma - \beta} \cdot g.$$

**Algorithm 1** Computing Normal Form (NF, `reduce`)

Assume that $>$ is a global monomial ordering.
**Input:** $f \in K[X]$, $G \in \mathcal{G}$.
**Output:** $\mathrm{NF}(f, G)$.

$h = f$;
**while** $h \neq 0$ & $G_h = \{g \in G \mid \mathrm{LM}(g) \text{ divides } \mathrm{LM}(h)\} \neq \emptyset$ **do**
  choose $g \in G_h$;
  $h = \mathrm{spoly}(h, g)$;
**return** $h$;

*Example* 4.14.

1. Consider $>_{dp}$, $f = x^4 + y^3 + 3z^2 + x + y - 3 \in \mathbb{Q}[x, y, z]$, and $G = \{x, y\} \subseteq \mathbb{Q}[x, y, z]$. Then we compute the normal form of $f$ with respect to $G$ in the following way:

$h_0 = f$;
$\mathrm{LM}(h_0) = x^4$, $G_{h_0} = \{x\}$;
$h_1 = \mathrm{spoly}(h_0, x) = h_0 - x^3 \cdot x = y^3 + 3z^2 + x + y - 3$;
$\mathrm{LM}(h_1) = y^3$, $G_{h_1} = \{y\}$;
$h_2 = \mathrm{spoly}(h_1, y) = h_1 - y^2 \cdot y = 3z^2 + x + y - 3$;
$\mathrm{LM}(h_2) = z^2$, $G_{h_2} = \emptyset$;

Hence, we obtain $\mathrm{NF}(f, G) = 3z^2 + x + y - 3$.

Moreover, we compute a reduced normal form of $f$ with respect to $G$ in the following way:

$h_0 = 0$, $g_0 = f$;
$\tilde{g}_1 = \mathrm{NF}(g_0, G) = 3z^2 + x + y - 3$;
$h_1 = h_0 + \mathrm{LT}(\tilde{g}_1) = 3z^2$, $g_1 = \mathrm{tail}(\tilde{g}_1) = x + y - 3$; $\tilde{g}_2 = \mathrm{NF}(g_1, G) = -3$;
$h_2 = h_1 + \mathrm{LT}(\tilde{g}_2) = 3z^2 - 3$, $g_2 = \mathrm{tail}(\tilde{g}_2) = 0$;

Hence, we obtain $\mathrm{redNF}(f, G) = z^2 - 1$.

The corresponding Singular code is as follows:

```
ring R = 0,(x,y,z),dp;
poly f = x4+y3+3z2+x+y-3;
ideal G = x,y;
reduce(f,G,1);
// ** G is no standard basis
//--> 3z2+x+y-3

interred(reduce(f,G));
// ** G is no standard basis
//--> _[1]=z2-1
```

2. Consider $>_{lp}$, $f = x^2yz + z^3 \in \mathbb{Q}[x, y, z]$, and $G = \{xy, xy+z\} \subseteq \mathbb{Q}[x, y, z]$. Then we may compute, due to Algorithm 1, on the one hand

$$\mathrm{NF}(f, G) = \mathrm{spoly}(f, xy) = f - xz \cdot xy = z^3,$$

and on the other hand

$$\mathrm{NF}(f, G) = \mathrm{spoly}(f, xy + z) = f - xz \cdot (xy + z) = -xz^2 + z^3.$$

Consequently, in this case the normal form is not unique but we will later on see under which condition this can be achieved.

## 4.5 Standard & Gröbner bases

**Definition 4.15.** Let $>$ be any monomial ordering on $\mathrm{Mon}(X)$ and $I \subseteq K[X]_>$ be an ideal.

1. A finite set $G \subseteq K[X]_>$ of polynomials is called a *standard basis* of the ideal $I$ if

$$G \subseteq I \quad \text{and} \quad L(G) = L(I).$$

2. If $>$ is global, a standard basis is also called a *Gröbner basis*.

3. By saying that $G$ is a standard basis, we mean that $G$ is a standard basis of the ideal $\langle G\rangle_{K[X]_>}$ generated by $G$.

The following basic proposition deals with existence and uniqueness of standard respectively Gröbner bases.

**Proposition 4.16.** *Let $>$ be any monomial ordering on* $\mathrm{Mon}(X)$*, and $I \subseteq K[X]_>$ be a non-zero ideal. Then the following hold.*

1. *There exists a standard basis G of $I$ and $I = \langle G\rangle_{K[X]_>}$, that is, the standard basis G generates $I$ as $K[X]_>$-ideal.*
2. *A reduced standard basis G of $I$ is unique.*

*Remark* 4.17. Reduced Gröbner bases can always be computed but, in contrast, reduced standard bases are, in general, not computable.

**Lemma 4.18.** *Let $>$ ba any monomial ordering on* $\mathrm{Mon}(X)$*, $I \subseteq K[X]_>$ an ideal, $G \subseteq I$ a standard basis of $I$, $f \in K[X]_>$, and* NF *be a normal form on $K[X]_>$ with respect to $G$. Then the following hold:*

1. *$f \in I$ if and only if* $\mathrm{NF}(f, G) = 0$.
2. *If* $\mathrm{NF}(\_, G)$ *is a reduced normal form then it is unique.*

*Proof.* Let $f \in I$ and assume that $\mathrm{NF}(f, g) \neq 0$. Due to Definition 4.11 and the fact that $G$ is a standard basis of $I$ this implies $\mathrm{LM}(\mathrm{NF}(f, G)) \notin L(G) = L(I)$ which contradicts $f \in I$. Hence, it follows $\mathrm{NF}(f, G) = 0$. On the other hand, let $\mathrm{NF}(f, G) = 0$. Then, again due to Definition 4.11, $f - \mathrm{NF}(f, G)$ has a standard representation, i.e. it follows $f = f - \mathrm{NF}(f, G) = \sum_{g \in G} c_g g \in I$ for suitable $c_g \in K[X]$.

Now, assume that $h_1, h_2$ are two reduced normal forms of $f$ with respect to $G$. Then $h_1 - h_2 \neq 0$, and $h_1 - h_2 = (f - h_2) - (f - h_1) \in I$ due to the standard representation of $h_1$ and $h_2$. Moreover, it follows $\mathrm{LM}(h_1 - h_2) \in L(I) = L(G)$, i.e. there is a monomial of either $h_1$ or $h_2$ that is contained in $L(G)$. This contradicts the fact that $\mathrm{NF}(\_, G)$ is a reduced normal form. □

The following criterion by Buchberger is essential for the computation of Gröbner bases.

**Theorem 4.19** (Buchberger's Criterion)**.** *Let $>$ be any monomial ordering on* $\mathrm{Mon}(X)$*, $I \subseteq K[X]_>$ be an ideal, and let $G \subseteq I$. Moreover, let* $\mathrm{NF}(\_, G)$ *be a weak normal form on $K[X]_>$ with respect to $G$. Then the following are equivalent:*

1. *$G$ is a standard basis of $I$.*
2. $\mathrm{NF}(f, G) = 0$ *for all $f \in I$.*
3. *$I = \langle G\rangle_{K[X]_>}$ and* $\mathrm{NF}(\mathrm{spoly}(g, g'), G) = 0$ *for all $g, g' \in G$.*

*Example* 4.20. Consider $>_{dp}$, $F = \{f_1, f_2\} = \{xy + y, y^2 + x\}$, and NF = redNF. Then we compute a Gröbner basis of $I = \langle F\rangle$ in the following way:

1. $G = F$, $P = \{(f_1, f_2\}$;

**Algorithm 2** Computing Gröbner Bases (`std`)

Assume that $>$ is a global monomial ordering.

**Input:** $F = \{f_1, \ldots, f_r\} \subseteq K[X]$ and NF, a normal form.
**Output:** $G$, a Gröbner basis of $\langle F \rangle$.

$G = F$;
$P = \{(f, g) \mid f, g \in G,\ f \neq g\}$;
**while** $P \neq \emptyset$ **do**
  choose $(f, g) \in P$;
  $P = P \smallsetminus \{(f, g)\}$;
  $h = \mathrm{NF}(\mathrm{spoly}(f, g), G)$;
  **if** $h \neq 0$ **then**
    $P = P \cup \{(h, f) \mid f \in G\}$;
    $G = G \cup \{h\}$;
**return** $G$;

2. $P = \emptyset$;
   $\mathrm{spoly}(f_1, f_2) = yf_1 - xf_2 = y^2 - x^2 = -x^2 + y^2 \xrightarrow{-f_2} -x^2 - x \longrightarrow x^2 + x$;
   $h = \mathrm{NF}(\mathrm{spoly}\,/f_1, f_2), G) \neq 0$;
   $f_3 = x^2 + x$;
   $P = \{(f_1, f_3), (f_2, f_3)\}$;
   $G = \{f_1, f_2, f_3\}$;
3. $P = \{(f_2, f_3)\}$;
   $\mathrm{spoly}(f_1, f_3) = xf_1 - yf_3 = xy - xy = 0$;
   $h = \mathrm{NF}(\mathrm{spoly}(f_1, f_3), G) = 0$;
4. $P = \emptyset$;
   $\mathrm{spoly}(f_2, f_3) = x^2 f_2 - y^2 f_3 = x^3 - xy^2 \xrightarrow{-xf_3} -xy^2 - x^2 \xrightarrow{+yf_1} -x^2 + y^2 \xrightarrow{+f_3} y^2 + x \xrightarrow{-f_2} 0$;
   $h = \mathrm{NF}(\mathrm{spoly}(f_2, f_3), G) = 0$;

Hence, $G = \{f_1, f_2, f_3\}$ is a Gröbner basis of $I = \langle F \rangle$.

The corresponding Singular code is as follows:

```
ring R = 0,(x,y),dp;
ideal I = xy+y, y2+x;
ideal G = std(I);
G;
//--> G[1]=y2+x
//--> G[2]=xy+y
//--> G[3]=x2+x
```

## 4.6 Answer to motivation questions

Remember that we consider the ideal $I = \left\langle xy + y, y^2 + x \right\rangle \subseteq \mathbb{Q}[x, y]$. Moreover, we already computed a Gröbner basis $G = \{xy + y, y^2 + x, x^2 + x\}$ of $I$ with respect to $>_{dp}$ in Example 4.20.

### 4.6.1 Ideal membership

Let $f = x^4 + x^2y^2 \in \mathbb{Q}[x, y]$. We want to decide whether $f$ is an element of the ideal $I$. Due to Lemma 4.18 we just have to compute the normal form of $f$ with respect to $G$.

$$\begin{aligned} x^4 + x^2y^2 &\xrightarrow{-x^2 f_3} x^2y^2 - x^3 \\ &\xrightarrow{-xy f_1} -x^3 - xy^2 \\ &\xrightarrow{+x f_3} -xy^2 + x^2 \\ &\xrightarrow{+y f_1} x^2 + y^2 \\ &\xrightarrow{-f_3} y^2 - x \\ &\xrightarrow{-f_2} -x - x = -2x \end{aligned}$$

Consequently, we obtain $\mathrm{NF}(f, G) = -2x \neq 0$ and conclude $f \notin I$.

The corresponding Singular code is as follows:

```
ring R = 0,(x,y),dp;
ideal I = xy+y, y2+x;
poly f = x4+x2y2;
reduce(f,std(I));
//--> -2x
```

### 4.6.2 Vector space dimension of quotient ring

We want to detect the vector space dimension $\dim_{\mathbb{Q}}\big(\mathbb{Q}[x, y]/I\big)$. The following lemma is helpful in this direction.

**Lemma 4.21.** *Let $>$ be a degree ordering on $\mathrm{Mon}(X)$, and $I \subseteq K[X]$ an ideal. Then the following holds:*

$$\dim_K \big(K[X]/I\big) = \dim_K \big(K[X]/L(I)\big).$$

Since $G$ is a Gröbner basis of $I$, we obtain $L(I) = L(G) = \langle xy, y^2, x^2\rangle$ so that $\{\overline{1}, \overline{x}, \overline{y}\}$ is a $\mathbb{Q}$-basis of $\mathbb{Q}[x, y]/L(I)$. Hence, it follows $\dim_{\mathbb{Q}}\big(\mathbb{Q}[x, y]/I\big) = \dim_{\mathbb{Q}}\big(\mathbb{Q}[x, y]/L(I)\big) = 3$.

The corresponding Singular code is as follows:

```
ring R = 0,(x,y),dp;
ideal I = xy+y, y2+x;
vdim(std(I));
//--> 3
```

### 4.6.3 Elimination of variables

We want to determine the ideal $I \cap \mathbb{Q}[y]$. The following concept of an elimination ordering is helpful in this direction.

**Definition 4.22.** A monomial ordering $>$ on $\mathrm{Mon}(x, y)$ is called an *elimination ordering for $y$* if for $f \in K[x, y]$ the condition $\mathrm{LM}(f) \in K[y]$ implies $f \in K[y]$.

**Lemma 4.23.** *Let $>$ be an elimination ordering for $y$, $I \subseteq K[x,y]$ be an ideal, and $G$ be a standard basis of $I$. Then $G' = \{g \in G \mid \mathrm{LM}(g) \in K[y]\}$ is a standard basis of $I' = I \cap \mathbb{Q}[y]$.*

Since $>_{lp}$ on $\mathrm{Mon}(x,y)$ is an elimination for $y$ we have to compute a Gröbner basis of $I = \langle xy + y, x + y^2 \rangle \subseteq \mathbb{Q}[x,y]$ via Algorithm 2. We compute $G = \{x + y^2, y^3 - y\}$ so that $G' = \{y^3 - y\}$ is a Gröbner basis of $I' = I \cap \mathbb{Q}[y]$.

The corresponding Singular code is as follows:

```
ring R1 = 0,(x,y),lp;
ideal I = xy+y, x+y2;
std(I);
//--> _[1]=y3-y
//--> _[2]=x+y2
```

or alternatively

```
ring R2 = 0,(x,y),dp;
ideal I = xy+y, y2+x;
eliminate(I,x);
//--> _[1]=y3-y
```

# 5 Local Cohomology (P. Schenzel)

## 5.1 What is local cohomology?

The aim of this section is to provide the basic notions of local cohomology. To this end let $R$ denote a commutative Noetherian ring. By $(R, \mathfrak{m})$ we denote a local ring with ist unique maximal ideal $\mathfrak{m}$ and its residue field $k = R/\mathfrak{m}$. Let $I \subset R$ denote an ideal of $R$.

### 5.1.1 Definitions

**Definition 5.1.** For an ideal $I$ of $R$ let $\Gamma_I$ denote the section functor with respect to $I$. That is, $\Gamma_I$ is the subfunctor of the identity functor given by

$$\Gamma_I(M) = \{m \in M | \operatorname{Supp} Rm \subseteq V(I)\}.$$

The right derived functors of $\Gamma_I$ are denoted by $H^i_I(-)$, $i \in \mathbb{N}$. They are called the local cohomology functors with respect to $I$.

### 5.1.2 The mapping cone

Let $C$ denote a complex of $R$-modules. For an integer $k \in \mathbb{Z}$ let $C[k]$ denote the complex $C$ shifted $k$ places to the left and the sign of differentials changed to $(-1)^k$, i.e.

$$(C[k])^n = C^{k+n} \quad \text{and} \quad d_{C[k]} = (-1)^k d_C.$$

Moreover note that $H^n(C[k]) = H^{n+k}(C)$.

For a homomorphism $f : C \to D$ of two complexes of $R$-modules let us consider the mapping cone $M(f)$. This is the complex $C \oplus D[-1]$ with the boundary map $d_{M(f)}$ given by the following matrix

$$\begin{pmatrix} d_C & 0 \\ -f & -d_D \end{pmatrix}$$

where $d_C$ resp. $d_D$ denote the boundary maps of $C$ and $D$ resp. Note that $(M(f), d_{M(f)})$ forms indeed a complex.

There is a natural short exact sequence of complexes

$$0 \to D[-1] \xrightarrow{i} M(f) \xrightarrow{p} C \to 0,$$

where $i(b) = (0, -b)$ and $p(a, b) = a$. Clearly these homomorphisms make $i$ and $p$ into homomorphisms of complexes. Because $H^{n+1}(D[-1]) = H^n(D)$ the connecting homomorphism $\delta$ provides a map $\delta^\cdot : H^\cdot(C) \to H^\cdot(D)$. By an obvious observation it follows that $\delta^\cdot = H^\cdot(f)$. Note that $f : C \to D$ induces an isomorphism on cohomology if and only if $M(f)$ is an exact complex.

### 5.1.3 Koszul complexes

For a complex $C$ and $x \in R$ let $C \xrightarrow{x} C$ denote the multiplication map induced by $x$, i.e. the map on $C^n$ is given by multiplication with $x$. Furthermore let $C \to C \otimes_R R_x$ denote the natural map induced by the localization, i.e. the map on $C^n$ is given by $C^n \xrightarrow{i} C^n \otimes_R R_x$, where for an $R$-module $M$ the map $i$ is the natural map $i : M \to M \otimes_R R_x$.

Let us construct the Koszul and Čech complexes with respect to a system of elements $\underline{x} = x_1, \ldots, x_r$ of $R$. To this end we consider the ring $R$ as a complex concentrated in degree zero. Then define

$$K^\cdot(x; R) = M(R \xrightarrow{x} R) \text{ and } K^\cdot_x(R) = M(R \to R_x).$$

Note that both of these complexes are bounded in degree 0 and 1. Inductively put

$$\begin{aligned} K^\cdot(\underline{x}; R) &= M(K^\cdot(\underline{y}; R) \xrightarrow{x} K^\cdot(\underline{y}; R)) \quad \text{and} \\ K^\cdot_{\underline{x}}(R) &= M(K^\cdot_{\underline{y}}(R) \to K^\cdot_{\underline{y}}(R) \otimes_R R_x), \end{aligned}$$

where $\underline{y} = x_1, \ldots, x_{r-1}$ and $x = x_r$. For an $R$-module $M$ finally define

$$K^\cdot(\underline{x}; M) = K^\cdot(\underline{x}; R) \otimes_R M \text{ and } K^\cdot_{\underline{x}}(M) = K^\cdot_{\underline{x}}(R) \otimes_R M.$$

Call them (co-) Koszul complex resp. Čech complex of $\underline{x}$ with respect to $M$. Obviously the Čech complex is bounded. It has the following structure

$$K_{\underline{x}}(M) : 0 \to M \to \oplus_i M_{x_i} \to \oplus_{i<j} M_{x_i x_j} \to \ldots \to M_{x_1 \cdots x_r} \to 0$$

with the corresponding boundary maps. With the previous notation define $D_{\underline{x}}(M)$ the following complex

$$D_{\underline{x}}(M) : 0 \to \oplus_i M_{x_i} \to \oplus_{i<j} M_{x_i x_j} \to \ldots \to M_{x_1 \cdots x_r} \to 0,$$

where we start with $D_{\underline{x}}(M)^0 = \oplus_i M_{x_i}$ in homological degree 0 and the sign of the differentials is changed. That is, $D_{\underline{x}}(M)$ is the truncated and shifted Čech complex of $M$ with respect to $\underline{x}$.

**Lemma 5.2.** *There is the following short exact sequence of complexes*

$$0 \to D_{\underline{x}}(M)[-1] \to K_{\underline{x}}(M) \to M \to 0,$$

*where $M$ is considered as a complex concentrated in degree 0.*

*Proof.* The proof follows easily since $D_{\underline{x}}(M)^{i-1} = K_{\underline{x}}(M)^i$ for all $i \geq 1$ and the natural morphism $K_{\underline{x}}(M) \to M$. ☐

## 5.2 A sketch on sheaf cohomology

### 5.2.1 Global transforms

Let $I \subset R$ denote an ideal. Let $M$ be an $R$-module. The natural inclusions $I^{n+1} \subseteq I^n$ define the direct system $\{\mathrm{Hom}_R(I^n, M)\}_{n \in \mathbb{N}}$.

**Definition 5.3.** The direct limit of the system $\{\mathrm{Hom}_R(I^n, M)\}_{n \in \mathbb{N}}$ is called the global (or ideal) transform of $M$ with respect to $I$.

In the next we collect a few properties for of global transforms. For some of the details we refer to [BS98] and [S98].

**Theorem 5.4.** *With the previous notations and definitions there are the following results:*

*(a) There is a natural homomorphism $i_M : M \to D_I(M)$ and an exact sequence*
$$0 \to H_I^0(M) \to M \stackrel{i_M}{\to} D_I(M) \to H_I^1(M) \to 0.$$

*(b) The homomorphism $i_M$ is an isomorphisms if and only if* $\mathrm{grade}(I, M) \geq 2$. *Moreover, $i_{D_I(M)}$ is an isomorphism.*

*(c) There is an isomorphism $D_I(M) \cong H^0(D_{\underline{x}}(M))$, where $\underline{x} = x_1, \ldots, x_r$ is a system of elements such that $(\underline{x})R = I$.*

*(d) $D_I(R)$ is a commutative ring. In the case of $R$ a domain it follows that*

$$D_I(R) = \{q \in Q(R) | I^n q \subseteq R \textit{ for some } n \in \mathbb{N}\},$$

*where $Q(R)$ denotes the field of fractions of $R$.*

### 5.2.2 Čech cohomology and local cohomology

The following result was proved by Grothendieck (see [G67]). It allows to compute the local cohomology by the aid of the Čech complexes.

**Theorem 5.5.** *Let $M$ denote an $R$-module. Let $\underline{x} = x_1, \ldots, x_r$ be a system of elements of $R$ such that $(\underline{x})R = I$. Then there are functorial isomorphisms $H_I^i(M) \cong H^i(K_{\underline{x}}(M))$ for all $i \in \mathbb{N}$.*

In particular, the Čech cohomology depends only on the radical of the ideal $I$. This result has various applications for the local cohomology modules (see e.g. [S98]).

### 5.2.3 Sheaf cohomology

Let $X = \operatorname{Spec} R$ be the set of prime ideals of the commutative Noetherian ring $R$. Let $I \subset R$ be an ideal. Define $U = X \setminus V(I)$, where $V(I) = \{\mathfrak{p} \in X | I \subseteq \mathfrak{p}\}$. Then $X$ becomes becomes a topological space with the $U$'s as the set of open subsets. This is called the Zariski topology.

**Definition 5.6.** Let $U \subset X$ denote an open subset, that is $U = X \setminus V(I)$ for an ideal $I \subset R$. With the definition of

$$\mathcal{O}_X(U) = D_I(R)$$

$\mathcal{O}_X$ becomes a sheaf and the pair $(X, \mathcal{O}_X)$ is an affine scheme (see [H77]). Moreover for an $R$-module $M$ define

$$\mathcal{F}_X(U) = D_I(M).$$

Then $\mathcal{F}$ is a sheaf on $(X, \mathcal{O}_X)$.

*Remark* 5.7. There is a one-to-one correspondance between $R$-modules $M$ and sheafs $\mathcal{F}$ on $(X, \mathcal{O}_X)$ by $M \mapsto \mathcal{F}_X$ (as defined above) and $\mathcal{F} \mapsto M = \mathcal{F}(X)$. For the last map remember that $X = X \setminus V(I)$ with the unit ideal $I = R$. Therefore $D_R(M) = M$ as easily seen. In particular we get $\mathcal{O}_X(X) = R$.

**Definition 5.8.** Let $\mathcal{F}$ denote a sheaf on $(X, \mathcal{O}_X)$. For an open subset $U \subseteq X$ we define $H^i(U, \mathcal{F}) = H^i(D_{\underline{x}}(M)), i \in \mathbb{N}$. Here $(\underline{x})R = I$ is the ideal given by $U = X \setminus V(I)$ and $M = \mathcal{F}(X)$. There is an exact sequence

$$0 \to H^0_I(M) \to M \to H^0(U, \mathcal{F}) \to H^1_I(M) \to 0$$

and isomorphisms $H^i(U, \mathcal{F}) \cong H^{i+1}_I(M)$ for $i \geq 1$. It is clear that $H^i(D_{\underline{x}}(M)$ does not depend on the particular choice of the generators $\underline{x}$ of $I$.

Note that the exact sequence and the isomorphisms follow easily by the definition of the complex $D_{\underline{x}}(M)$ and the results in Theorem 5.4.

### 5.2.4 Proj

Here we want to motify some of our investigations to the case of graded rings. To let $S = \oplus_{n \geq 0} S_n$ denote an $\mathbb{N}$-graded ring, where $S_0$ is a commutative ring and $S = S_0[S_1]$. Moreover we write $\mathfrak{m} = \oplus_{n>0} S_n$ for the irrelevant maximal ideal.

Then we define $X = \operatorname{Proj} S = \{\mathfrak{p} \in \operatorname{Spec} S | \mathfrak{p} \text{ is homogeneous and } \mathfrak{m} \not\subseteq \mathfrak{p}\}$ and $V_+(I) = \{\mathfrak{p} \in \operatorname{Proj} S | I \subseteq \mathfrak{p}\}$. With $U = X \setminus V_+(I)$ for homogeneous ideals $\operatorname{Proj} S$ becomes a topological space.

**Definition 5.9.** Let $U \subset X$ denote an open subset, that is $U = X \setminus V(I)$ for a homogeneous ideal $I \subset R$. With the definition of

$$\mathcal{O}_X(U) = D_I(S)_0, \text{the degree zero component of } D_I(S),$$

$\mathcal{O}_X$ becomes a sheaf and the pair $(X, \mathcal{O}_X)$ is a projective scheme (see [H77]). Moreover for a graded $S$-module $M$ define

$$\mathcal{F}_X(U) = D_I(M)_0.$$

Then $\mathcal{F}$ is a sheaf on $(X, \mathcal{O}_X)$. We write also $\tilde{M}$ instead of $\mathcal{F}$.

The sheafification correspondance $M \mapsto \tilde{M}$ is functorial, exact and commutes with tensor products.

**Definition 5.10.** Let $n \in \mathbb{N}$ denote an integer. Then we define $\mathcal{O}_X(n) = \widetilde{S(n)}$ and $\mathcal{O}_X(-n) = \operatorname{Hom}_S(\widetilde{S(n)}, S)$. For a sheaf $\mathcal{F}$ on $X$ we put $\mathcal{F}(n) = \mathcal{F} \otimes \mathcal{O}(n)$. Let $U \subset X = \operatorname{Proj} X$ denote an open subset. We denote by $\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$ the sections of $\mathcal{F}$ in $U$. Furthermore, put $\Gamma_\star(\mathcal{F}) = \oplus_{n\in\mathbb{Z}}\Gamma(X, \mathcal{F}(n))$.

*Remark* 5.11. (A) $\Gamma_\star(\mathcal{F})$ admitts the structure of a graded $S$-module. For a graded $S$-module $M$ there is a natural homomorphism of graded $S$-modules $M \to \Gamma_\star(\tilde{M})$ which is in general neither injective nor surjective.
(B) For the polynomial ring $S = k[x_0, \ldots, x_r]$ with its natural grading it follows that $S \cong \Gamma_\star(\mathcal{O}_X)$ (see [H77, II 5.13]).
(C) Let $S = k[x, y, z, w]/(xw - yz, y^3 - x^2z, y^2w - xz^2, yw^2 - z^3)$. Then $S$ is a domain. Let $q = yw/z \in Q(S)$, where $Q(S)$ denotes the quotient field. Then $xq = y^2, yq = xz, zq = yw, wq = z^2$ and therefore $q \in \Gamma(X, \mathcal{O}(1) \setminus S_1$. That is $S \neq \Gamma_\star(\mathcal{O}_X)$.
(D) The sheafification functor $M \mapsto \tilde{M}$ allows to consider the coherent sheaf $\mathcal{F}$ on $X$ as an equivalence class of finitely generated graded $S$-modules, where two such modules $M$ and $N$ are called equivalent if the truncated modules $M_{\geq r}$ and $N_{\geq r}$ are isomorphic for some $r \in \mathbb{Z}$.
(E) It follows that $\tilde{M} = 0$ if and only if $M$ is a graded $S$-module with $\operatorname{Supp}_S M \subseteq V(\mathfrak{m})$.

### 5.2.5 Sheaf cohomology on projective varieties

Let $I \subset S$ denote a homogeneous ideal with generators $I = (\underline{x})$ for some homogeneous elements $\underline{x} = x_1, \ldots, x_r$. Let $M$ denote a graded $S$-module. Then the complexes $K_{\underline{x}}(M)$ and $D_{\underline{x}}(M)$ are complexes of graded $S$-modules with homogeneous boundary maps of degree zero. Therefore their cohomology modules are also graded $S$-modules. Now let us consider the particular situation of $I = \mathfrak{m}$, the irrelevant ideal of the graded ring $S$, generated by $\underline{x} = x_0, \ldots, x_r$.

**Definition 5.12.** For an integer $n \in \mathbb{Z}$ and a coherent sheaf $\mathcal{F}$ on $(X, \mathcal{O}_X) = \operatorname{Proj} S$ we define $H^i(X, \mathcal{F}(n)) = H^i(D_{\underline{x}}(M))_n$, where $M$ denotes a graded $S$-module of the equivalence class defining $\mathcal{F}$. In fact it follows easily that this definition is independent on the choice of $M$. Moreover we put $H^i_\star(X, \mathcal{F}) = \oplus_{n\in\mathbb{Z}}H^i(X, \mathcal{F}(n))$ for all $i \geq 0$.

It follows by the definitions that $\Gamma_\star(X, \mathcal{F}) = H^0_\star(X, \mathcal{F})$.

## 5.3 Computational aspects

### 5.3.1 Relation to local cohomology

As above let $S$ denote a graded ring with $\mathfrak{m} = \oplus_{n>0}S_n$. Let $M$ denote a graded $S$-module and $\mathcal{F}$ its accociated sheaf on $(X, \mathcal{O}_X) = \operatorname{Proj} S$.

**Theorem 5.13.** *There is an exact sequence of graded $S$-modules*

$$0 \to H^0_{\mathfrak{m}}(M) \to M \to H^0_\star(X, \mathcal{F}) \to H^1_{\mathfrak{m}}(M) \to 0$$

*and isomorphisms $H^{i+1}_{\mathfrak{m}}(M) \cong H^i_\star(X, \mathcal{F})$ for all $i > 0$. Moreover all the homomorphisms are homgogeneous of degree 0.*

The proof follows by the definition of the sheaf cohomology as the cohomology of the truncated complex $D_{\underline{x}}(M)$ and its relation to the Čech complex that carries as cohomology modules the local cohomology $H^i_{\mathfrak{m}}(M), i \in \mathbb{Z}$. In fact, the above exact sequence measures the deviation of the natural map $M \to H^0_\star(X, \mathcal{F})$ from being an isomorphism. In the example (C) above we get $H^1_{\mathfrak{m}}(S) \cong k(-1)$.

### 5.3.2 Local Duality

Now we assume that $S = k[x_0, \ldots, x_r]$ a polynomial ring over a field $k$ with its natural grading. Then $(X, \mathcal{O}_X)$ for $X = \operatorname{Proj} S$ is called the projective $r$-space $(\mathbb{P}^r_k, \mathcal{O}_{\mathbb{P}^r_k})$. We write also $\mathfrak{m} = (x_0, \ldots, x_r)$ for the irrelevant maximal ideal of $S$.

*Remark* 5.14. (A) Let $M$ denote a finitely generated $S$-module. Then $\mathcal{F} = \tilde{M}$ is a coherent sheaf and

$$\dim_k H^i(X, \mathcal{F}(n)) < \infty$$

for all $i, n \in \mathbb{N}$. Let $i > 0$ be a positive integer. Then

$$H^i(X, \mathcal{F}(n)) = 0 \text{ for all } n \gg 0.$$

(B) It is known that $H^i(X, \mathcal{F}(n)) = 0$ for $i > \dim X$ and all $n \in \mathbb{Z}$.
(C) The Euler characteristic of $\mathcal{F}$ is defined by

$$\chi(\mathcal{F}, n) = \sum_{i \geq 0} (-1)^i \dim_k H^i(X, \mathcal{F}(n)).$$

It is equal to the Hilbert polynomial.

For a graded $S$-module $M$ we write $M^\dagger$ for the graded $k$-vector space dual $\oplus_{n \in \mathbb{Z}} \operatorname{Hom}_k(M_{-n}, k)$ equipped with the natural structure as an $S$-module. Note that $M \to (M^\dagger)^\dagger$ is an isomorphism for $M$ a finitely generated graded $S$-module $M$.

**Theorem 5.15.** ***(Local Duality)*** *Let $M$ denote a finitely generated $S$-module. Then there are natural isomorphisms homogeneous of degree zero*

$$H^i_{\mathfrak{m}}(M) \cong \operatorname{Ext}^{r+1-i}_S(M, S(-r-1))^\dagger$$

*for all $i \in \mathbb{N}$.*

*Proof.* Since $S$ is a polynomial ring the Čech complex $C_{\underline{x}}(S)$ provides a flat resolution of $(S(-r-1))^\dagger = \operatorname{Hom}_k(S(-r-1), k)$, Macaulay's inverse system. Therefore

$$H^i_{\mathfrak{m}}(M) \cong H^i(C_{\underline{x}}(S) \otimes_S M) \cong \operatorname{Tor}^{r+1-i}_S(M, \operatorname{Hom}_k(S(-r-1), k))$$

for all $i \in \mathbb{N}$. Since $M$ is a finitely generated $R$-module it follows that

$$H^i_{\mathfrak{m}}(M) \cong \operatorname{Hom}_k(\operatorname{Ext}^{r+1-i}_S(M, S(-r-1)), k) = \operatorname{Ext}^{r+1-i}_S(M, S(-r-1))^\dagger$$

as required. □

As a consequence for the sheaf cohomoogy on $(\mathbb{P}^r_k, \mathcal{O}_{\mathbb{P}^r_k})$ there is the following result. We write $h^i(X, \mathcal{F}) = \dim_k H^i(X, \mathcal{F})$ for all $i \in \mathbb{Z}$.

**Corollary 5.16.** *Let $M$ denote a finitely generated graded $S$-module and $\mathcal{F} = \tilde{M}$. Let $n \in \mathbb{Z}$ denote an integer. Then there are an exact sequence of $k$-vectorspaces*

$$0 \to \operatorname{Ext}_S^{r+1}(M,S)^\vee_{-n-r-1} \to M_n \to H^0(\mathcal{O}_{\mathbb{P}^r_k}, \mathcal{F}(n)) \to \operatorname{Ext}_S^r(M,S)^\vee_{-n-r-1} \to 0$$

*and isomorphisms of $k$-vectorspaces*

$$H^i(\mathcal{O}_{\mathbb{P}^r_k}, \mathcal{F}(n)) \cong \operatorname{Ext}_S^{r-i}(M,S)^\vee_{-n-r-1}, \text{ for } i \geq 1,$$

*where* $\operatorname{Hom}_k(-,k) = -^\vee$ *denotes the dual of $k$-vector spaces.*

As an application we get the sheaf cohomology of the structure sheaf of the projective $r$-space.

**Corollary 5.17.** *For the sheaf cohomology of the the projective $r$-space it follows:*

(a) $H^i(\mathbb{P}^r_k, \mathcal{O}_{\mathbb{P}^r_k}(n)) = 0$ *for all* $i \neq 0, r$ *and all* $n \in \mathbb{Z}$.

(b) $h^0(\mathbb{P}^r_k, \mathcal{O}_{\mathbb{P}^r_k}(n)) = \binom{r+n}{r}$ *for* $n \geq 0$ *and* $h^0(\mathbb{P}^r_k, \mathcal{O}_{\mathbb{P}^r_k}(n)) = 0$ *for all* $n < 0$.

(c) $h^r(\mathbb{P}^r_k, \mathcal{O}_{\mathbb{P}^r_k}(n)) = 0$ *for* $n \geq -r$ *and* $h^r(\mathbb{P}^r_k, \mathcal{O}_{\mathbb{P}^r_k}(n)) = \binom{-n-1}{r}$ *for all* $n \leq -r-1$.

### 5.3.3 Computer algebra systems

Let $X \subset \mathbb{P}^r_k$ a projective variety. How to compute its sheaf cohomology

$$H^i(X, \mathcal{O}_X(n)) = H^i(\mathbb{P}^r_k, i_\star \mathcal{O}_X(n))$$

or at least its dimensions $h^i(X, \mathcal{O}_X(n))$ for various $i, n \in \mathbb{Z}$?

The solution is related to the result in Corollary 5.16. For instance with Singular it is possible to compute the modules $\operatorname{Ext}_S^i(M,S)$ for a finitely generated $S$-module $M$. Then one might use Corollary 5.16 in order to get the expressions for $h^i(X, \mathcal{O}_X(n))$. In fact, this is implemented in Singular in the library `sheafcoh.lib` (see [DGPS12]). We refer to the Singular Manual for the details and examples.

In the following we want to illustrate a direct approach to the calculation of a certain sheaf cohomology.

**Definition 5.18.** For an irreducible, integral projective curve $C \subset \mathbb{P}^r_k$ let $\mathcal{J}_C$ denote the ideal sheaf of $C$. Then $H^1_\star(\mathcal{J}_C) = \oplus_{n \in \mathbb{Z}} H^1(\mathbb{P}^r_k, \mathcal{J}_C(n))$ is called the Hartshorne-Rao module. It is a graded $S$-module of finite length. Note that the vanishing of its $n$-th graded component characterizes $n$-normality of the curve $C$. Note that $H^1_\star(\mathcal{J}_C) = 0$ if and only if $C \subset \mathbb{P}^r_k$ is arithmetically Cohen-Macaulay.

*Example* 5.19. Let $C \subset \mathbb{P}^6_k$ the rational curve given parametrically by

$$x_0 = s^{13}, x_1 = s^{12}t^2, x_2 = s^{11}t^3, x_3 = s^{10}t^3, x_4 = s^9t^4, x_5 = st^{12}, x_6 = t^{13}.$$

This is a projection of the rational normal curve $C(13) \subset \mathbb{P}^{13}_k$. It admitts an (extremal) 9-secant. Note that all curves with an extremal secant line are projections of rational normal curves. The Hartshorne-Rao module is

$$H^1_\star(\mathcal{J}_C) \cong \operatorname{Ext}^6_S(S/I, S(-7))^\dagger,$$

where $I \subset S = k[x_0, \ldots, x_6]$ denotes the defining ideal of $C$.

The Singular code is as follows:

```
ring R = 0,(s,t,x(0..6)),Dp;
ideal I = x(0)-s13,x(1)-s12t,x(2)-s11t2,x(3)-s10t3,x(4)-s9t4,
          x(5)-st12,x(6)-t13;
ideal J = eliminate(I,s*t);
ring S = 0,x(0..6),Dp;
ideal I = imap(R,J);
resolution Ir = mres(I,0);
print(betti(Ir),"betti");
           0     1     2     3     4     5     6
------------------------------------------------
    0:     1     -     -     -     -     -     -
    1:     -    10    20    15     4     -     -
    2:     -     1     -     -     -     -     -
    3:     -     -    10    20    15     4     -
    4:     -     -     -     -     -     -     -
    5:     -     1     5    10    10     5     1
    6:     -     -     -     -     -     -     -
    7:     -     -     -     -     -     -     -
    8:     -     1     5    10    10     5     1
------------------------------------------------
total:     1    13    40    55    39    14     2
module M = transpose(Ir[6]);
hilb(std(M));
//         1 t^0
//         2 t^1
//         3 t^2
//         5 t^3
//         7 t^4
//         9 t^5
//         7 t^6
// dimension (affine) = 0
// degree (affine)  = 34
```

For the computation note that $M = \operatorname{Ext}^6_S(S/I, S) \cong F/\operatorname{Im}\psi^t$, where $F$ is the free $S$-module that occurs at homological dimension $r$ and $\psi^t$ is the transpose of the corresponding map.

We get – by reading off the Hilbert function of $\operatorname{Ext}^6_S(S/I, S)$ – the following diagram

| $n$ | $\leq 0$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | $\geq 8$ |
|---|---|---|---|---|---|---|---|---|---|
| $h^1(\mathcal{J}_C(n))$ | 0 | 7 | 9 | 7 | 5 | 3 | 2 | 1 | 0 |

Therefore for the Castelnuovo-Mumford regularity we have $\operatorname{reg}\mathcal{J} = 9$.

# 6 Approximation Theorems (D. Popescu)

The aim of this section is to present some results of Artin approximation and to give some ideas of how these could be applied in different algebraic problems. We start with some preliminaries.

A local ring $(A, m)$ is called *Henselian* if the following property holds:

"Let $f$ be a polynomial in one variable $Y$ over $A$. If $\tilde{y} \in A$ satisfies $f(\tilde{y}) \equiv 0 \mod m$ and $(\partial f/\partial Y)(\tilde{y}) \notin m$ then there exists a solution $y \in A$ of $f$ in $A$ such that $y \equiv \tilde{y} \mod m$."

One can show that this property is equivalent with the following somehow stronger property:

"Let $f = (f_1, \ldots, f_n)$ be a system of polynomials in $Y = (Y_1, \ldots, Y_n)$ over $A$, and $D$ the determinant of the Jacobian matrix $(\partial f_i/\partial Y_j)$. If $\tilde{y} \in A^n$ satisfies $f(\tilde{y}) \equiv 0 \mod m$ and $D(\tilde{y}) \notin m$ then there exists a solution $y \in A^n$ of $f$ in $A$ such that $y \equiv \tilde{y} \mod m$."

Moreover, we may write the above property even in another stronger form:

"Let $f = (f_1, \ldots, f_r)$ be a system of polynomials in $Y = (Y_1, \ldots, Y_n)$ over $A$, $r \leq n$ and $M$ be an $r \times r$-minor of the Jacobian matrix $(\partial f_i/\partial Y_j)$. If $\tilde{y} \in A^n$ satisfies $f(\tilde{y}) \equiv 0 \mod m$ and $M(\tilde{y}) \notin m$ then there exists a solution $y \in A^n$ of $f$ in $A$ such that $y \equiv \tilde{y} \mod m$."

Actually the above property follows easily from the previous one because if for example $M$ is the determinant of $(\partial f_i/\partial Y_j)_{1 \leq i,j \leq r}$ then we may add the polynomials $f_i = Y_i - \tilde{y}_i$ for $r < i \leq n$ obtaining a system of $n$ polynomials. Now the determinant of the Jacobian matrix of the new bigger system is $M$ and we may apply the second property.

These are properties which you studied in a previous lecture here, I will remind you the Implicit Function Theorem from Differential or Analytic Geometry.

**Theorem 6.1.** *Let $f_i(x_1, \ldots, x_n, Y_1, \ldots, Y_n) = 0$, $1 \leq i \leq n$ be some analytic equations, where $f_i : \mathbf{C}^{2n} \to \mathbf{C}$ are analytic maps in a neighborhood of $(0,0) \in \mathbf{C}^{2n}$ with the property that $f_i(0,0) = 0$, $1 \leq i \leq n$ , and $det(\partial f_i/\partial Y_j)(0,0) \neq 0$. Then there exist some unique maps $y_j : \mathbf{C}^n \to \mathbf{C}$, $1 \leq i \leq n$, which are analytic in a neighborhood of $0 \in \mathbf{C}^n$ such that $y_j(0) = 0$ and $f(x,y) = 0$ in a neighborhood of $0$.*

The above theorem says in particular that the local ring of all analytic germs of maps defined in a neighborhood of $0 \in \mathbf{C}^n$ in other words the local ring $\mathbf{C}\{x\}$ of convergent power series in some variables $x$ over $\mathbf{C}$, $x = (x_1, \ldots, x_n)$ is Henselian. Other important Henselian rings are the formal power series ring $K[[x]]$, $x = (x_1, \ldots, x_n)$ over a field K, or its subring the algebraic power series ring $K < x >$, whose elements are those formal power series, which satisfy polynomial equations in one variable over $K[x]$.

The Henselian property is very important. For example with its help we may find in $\mathbf{C}[[x]]$ a root of the polynomial $f = Y^s - u$ for all $s$ and all formal power series $u$ with a free term, that is $u \notin (x)$, of $\mathbf{C}[[x]]$. Indeed, let $a \in \mathbf{C}$ be such that $a^s = u(0)$, ($\mathbf{C}$ is algebraically closed field). Then we have $f(a) \in (x)$ and $(\partial f/\partial Y)(a) = sa^{s-1} \notin (x)$ and so $f$ must have a solution in $\mathbf{C}[[x]]$. In particular you may see that the polynomial $g = y^2 - x^2 - x^3$ is irreducible in $\mathbf{C}[x,y]$ but reducible in $\mathbf{C}[[x,y]]$ since there $g = (y - x\rho)(y + x\rho)$, where $\rho$ is the square root of $1 + x$.

A very important property of Henselian rings is the following lemma.

**Lemma 6.2** (Newton Lemma). *Let $(A, m)$ be a Henselian ring, $c$ a positive integer, $f = (f_1, \ldots, f_r)$ be a system of polynomials in $Y = (Y_1, \ldots, Y_n)$ over $A$, $r \leq n$, and $M$ an $r \times r$-minor of the Jacobian matrix $J = (\partial f_i/\partial Y_j)$. If $\tilde{y} \in A^n$*

*satisfies* $f(\tilde{y}) \equiv 0$ *mod* $M(\tilde{y})^2 m^c$ *then there exists a solution* $y \in A^n$ *of* $f$ *in* $A$ *such that* $y \equiv \tilde{y}$ *mod* $M(\tilde{y})m^c$.

*Proof.* As above we may reduce our problem to the case $r = n$ adding some new polynomials of the form $Y_i - \tilde{y}_i$. Just to get the idea we consider the case $r = n = 1$, $M = \partial f/\partial Y$. We try to find $z \in A$ such that $f(\tilde{y} + M(\tilde{y})z) = 0$. We may suppose $M(\tilde{y}) \neq 0$, because otherwise $\tilde{y}$ is already a solution of $f$. Using Taylor's formula we have

$$f(\tilde{y} + M(\tilde{y})Z) = f(\tilde{y}) + (\partial f/\partial Y)(\tilde{y})(M(\tilde{y})Z) + \sum_{j>1} (\partial^j f/\partial Y^j)(\tilde{y})(M(\tilde{y})Z)^j = 0.$$

By hypothesis we have $f(\tilde{y}) = M(\tilde{y})^2 a$ for some $a \in m^c$. Dividing the above equations by $M(\tilde{y})^2$ we get

$$a + Z + \sum_{j>1} (\partial^j f/\partial Y^j)(\tilde{y})(M(\tilde{y})^{j-2} Z^j) = 0,$$

where we may apply the Henselian property. Thus there exists $z \in m$ such that

$$a + z + \sum_{j>1} (\partial^j f/\partial Y^j)(\tilde{y})(M(\tilde{y})^{j-2} z^j) = 0,$$

and so $y = \tilde{y} + M(\tilde{y})z$ is a solution of $f$. Remains to show that $z$ is in fact in $m^c$. Suppose that $z \in m^j$ for some $j \geq 1$. Then from the above equation we get

$$z = -a - \sum_{j>1} (\partial^j f/\partial Y^j)(\tilde{y})(M(\tilde{y})^{j-2} z^j) \in m^{\min\{c, 2j\}}$$

and it follows by recurrence $z \in m^c$.

Next we give an idea about the case $r = n > 1$. In this case by Taylor's formula we get for $z \in A^n$ as above the system of equations

$$f_k(\tilde{y} + M(\tilde{y})Z) = f_k(\tilde{y}) + \sum_{i=1}^{n} (\partial f_k/\partial Y_i(\tilde{y})(M(\tilde{y})Z_i) +$$

$$\sum_{j>1} ( \sum_{j_1,\ldots,j_n \geq 0, j = j_1 + \ldots + j_n} (\partial^j f_k/\partial Y_{j_1} \cdots \partial Y_{j_n})(\tilde{y})(M(\tilde{y})^j Z_1^{j_1} \cdots Z_n^{j_n}) = 0,$$

$k = 1, \ldots n$. By hypothesis we have $f(\tilde{y}) = M(\tilde{y})^2 a$ for some $a \in m^c A^n$. Using linear algebra there exists a $n \times n$ matrix $C$ over $A[Y]$ such that $JC = MI_n$, $I_n$ being the $n \times n$ unit matrix. Then the above system of equations becomes

$$M(\tilde{y})J(\tilde{y})[C(\tilde{y})a + Z + (\text{terms in } Z \text{ of degree } \geq 2)] = 0.$$

Then it is enough to find $z$ satisfying

$$C(\tilde{y})a + Z + (\text{terms in } Z \text{ of degree } \geq 2) = 0,$$

which follows from the Henselian property. □

**Lemma 6.3.** *Let* $A = K < x >$, $x = (x_1, \ldots, x_n)$, $f = (f_1, \ldots, f_r)$ *be a system of polynomials in* $Y = (Y_1, \ldots, Y_n)$ *over* $A$, $r \leq n$, *and* $M$ *an* $r \times r$*-minor of the Jacobian matrix* $(\partial f_i/\partial Y_j)$. *Let* $\hat{y}$ *be a solution of* $f$ *in* $\hat{A} = K[[x]]$ *such that* $M(\hat{y}) \not\equiv 0$ *mod* $(x)$. *Then for any* $c \in \mathbf{N}$ *there exists a solution* $y^{(c)}$ *in* $A$ *such that* $y^{(c)} \equiv \hat{y}$ *mod* $(x)^c$.

*Proof.* Choose $c$ and an element $\tilde{y}$ in $A^n$ such that $\tilde{y} \equiv \hat{y} \bmod (x)^c$. Then $f(\tilde{y} \equiv f(\hat{y}) = 0 \bmod (x)^c$, $M(\tilde{y}) \equiv M(\hat{y}) \neq 0 \bmod (x)$. It follows $M(\tilde{y})$ invertible and so $f(\tilde{y}) \equiv 0 \bmod M^2(\tilde{y})(x)^c$. Now it is enough to apply Newton Lemma. □

The above lemma says in fact that a special solution $\hat{y}$ of a special system of polynomial equations $f$ can be approximated as well we want in the $(x)$-adic topology of $\hat{A}$ by solutions in $A$, that is the algebraic ones. This is a preliminary form of Artin approximation.

The following lemma is a consequence of the so called Jacobian criterion.

**Lemma 6.4.** *If $A$ is a domain, $q \subset A[Y]$ a prime ideal, and the field extension $Q(A) \subset Q(A[Y]/q)$ is separable then there exist some polynomials $f = (f_1, \ldots, f_r)$ of $q$ such that $qA[Y]_q = (f_1, \ldots, f_r)A[Y]_q$ and the Jacobian matrix $(\partial f_i/\partial Y_j)$ has an $r \times r$-minor which is not in $q$.*

Since $Q(A) \subset Q(A[Y]/q)$ is separable of finite type one can find $u_1, \ldots, u_s \in A[Y]$ algebraically independent over $Q(A)$ and $v \in A[Y]$ such that $Q(A[Y]/q) = Q(A)(u_1, \ldots, u_s, v)$ and $v$ is algebraic separable over $Q(A)(u_1, \ldots, u_s)$. The irreducible polynomial associated to $v$ over $Q(A)(u_1, \ldots, u_s)$ should have the derivation non-zero in $Q(A[Y]/q)$. This is the idea behind the above lemma.

A Noetherian local ring $(A, m)$ has the *property of approximation* if every finite system of polynomial equations $f$ over $A$ in $Y = (Y_1, \ldots, Y_N)$ has its solutions in $A$ dense with respect to the $m$-adic topology in the set of its solutions in the completion $\hat{A}$ of $A$; that is, for every solution $\hat{y}$ of $f$ in $\hat{A}$ and every positive integer $c$ there exists a solution $y$ of $f$ in $A$ such that $y \equiv \hat{y} \mod m^c\hat{A}$.

*Example* 6.5. *Let $k := \mathbf{F}_p(T_1^p, \ldots, T_s^p, \ldots)$, where $\mathbf{F}_p$ is the field with $p$ elements, and $K := k(T_1, \ldots, T_s, \ldots)$, $p$ being a prime number and $(T)_i$ a countable set of variables. Then $[K : k] = \infty$ and the discrete valuation ring $R := k[[x]][K]$ is Henselian but not complete because for example the formal power series $f := \sum_{i=1}^{\infty} T_i x^i$ in the variable $x$ over $K$ is not in $R$. In fact one can show that a formal power series $g = \sum_i g_i x^i$ in $x$ over $K$ is in $R$ if and only if $[k((g_i)_i) : k] < \infty$. Since the polynomial $Y^p - f^p$ has no solutions in $R$ it follows that $R$ has not the property of approximation.*

Next lemma shows how one can apply the property of approximation in an easy algebraic problem.

**Lemma 6.6.** *Suppose that $A$ is a domain and it has the property of approximation. Then $\hat{A}$ is a domain too.*

*Proof.* Suppose that $\hat{A}$ is not a domain, that is there exist two nonzero elements $\hat{y}, \hat{z} \in \hat{A}$ such that $\hat{y}\hat{z} = 0$. Choose a positive integer $c$ such that $\hat{y}, \hat{z} \notin m^c\hat{A}$. Take $f = YZ$. Then there exists $y, z \in A$ a solution of $f$ such that $y \equiv \hat{y} \mod m^c\hat{A}$, $z \equiv \hat{z} \mod m^c\hat{A}$. It follows $yz = 0$ and $y, z \notin m^c$. Contradiction! □

As in a lemma above we can see that if $A$ has the property of approximation then it share with its completion $\hat{A}$ many algebraic properties. The following example is an exception.

*Example* 6.7. Let $\hat{R} = \mathbf{C}[[x]]$ and $h = e^{(e^x-1)}$, where $e^x = \sum_{i=0}^{\infty} x^i/(i!)$. Let $R$ be the algebraic closure of $\mathbf{C}[x,h]$ in $\hat{R}$. Then $R$ is an excellent Henselian discrete valuation ring. Moreover, by Artin's Theorem it has the property of approximation. One can show that the identity is the only $\mathbf{C}$-automorphism of $R$ by G. Pfister, which is not the case of $\hat{R}$.

*Example* 6.8. *Let $K$ be a field of characteristic $p > 0$ and $g_0, \dots, g_{p-1} \in \hat{R} = K[[x]]$ be some formal power series which are algebraically independent over $K[x]$. Set $g = \sum_{i=0}^{p-1} g_i^p x^i$ and let $R$ be the algebraic closure of $K[x,g]$ in $\hat{R}$. Then $R$ is a Henselian discrete valuation ring but it has not the property of approximation because the polynomial $g - \sum_{i=0}^{p-1} Y_i^p x^i$ has a solution in $\hat{R}$ but none in $R$ since the transcendental degree of $R$ over $K[x]$ is 1.*

Let $K$ be a field and $S = K[[x]]$, $x = (x_1, \dots, x_n)$. A formal power series $f \in S$ is called *regular* in $x_n$ of order $t$ if $f(0, \dots, 0, x_n) \neq 0$ has the order $t$.

**Theorem 6.9** (Weierstrass Preparation Theorem). *If $f$ is regular in $x_n$ of order $t$ then there exist a unique polynomial $g = x_n^t + \sum_{i=0}^{t-1} a_i x_n^i$, with $a_i \in (x_1, \dots, x_{n-1})K[[x_1, \dots, x_{n-1}]]$ and a unique unit $u \in S$ such that $f = ug$ ($g$ is called the* Weierstrass polynomial *of $f$) .*

**Theorem 6.10** (Weierstrass Division Theorem). *Let $f, h \in S$ be two formal power series. If $f$ is regular of order $t$ then there exist unique formal power series $q \in S$, $w_0, \dots, w_{t-1} \in K[[x_1, \dots, x_{n-1}]]$ such that $h = qf + \sum_{i=1}^{t-1} w_i x_n^i$.*

**Lemma 6.11.** *If $K$ is infinite and $f \in S$ is non-zero then there exists a $K$-automorphism $\tau$ of $S$ of the form $x_i \to x_i + c_i x_n$, for $i < n$, $x_n \to x_n$, $c_i \in K$ such that $\tau(f)$ is regular in $x_n$ of a certain order $t$.*

*Example* 6.12. *Let $h$ be a polynomial of degree $>> 0$ in $Y_1$ over $\mathbf{C}$ and $w = e^x = \sum_{i=o}^{\infty} x^i/(i!)$. Let $f = h - h(x) + (Y_1 - x)Y_2$ be a polynomial equation in $Y = (Y_1, Y_2)$ over $R := \mathbf{C} < x >$. Set $y_1 = x + w$. By Taylor's formula $h(y_1) - h(x) \in (y_1 - x) = (w)$ and so there exists $y_2 \in \hat{R} = \mathbf{C}[[x]]$ such that $(y_1, y_2)$ is a solution of $f$ in $\hat{R}$. Let $c \in \mathbf{N}$ and set $w_c = \sum_{i=o}^{c} x^i/(i!)$. Then for $\tilde{y}_1 = x + w_c$ there exists $\tilde{y}_2 \in R$ using Taylor's formula such that $(\tilde{y}_1, \tilde{y}_2)$ is a solution of $f$ with $\tilde{y}_1 \equiv y_1$ mod $x^c$. It follows that $w_c y_2 \equiv w y_2 = h(y_1) - h(x) \equiv h(\tilde{y}_1) - h(x) = w_c \tilde{y}_2$ mod $x^c$. Thus $\tilde{y}_2 \equiv y_2$ mod $x^c$ and so the solution $(\tilde{y}_1, \tilde{y}_2)$ of $f$ in $R$ coincides modulo $x^c$ with the previous one. This is in fact what our next theorem states.*

Let $R = K < x >$, $x = (x_1, \dots, x_n)$ be the ring of algebraic power series in $x$ over $K$, that is the algebraic closure of the polynomial ring $K[x]$ in the formal power series ring $\hat{R} = K[[x]]$. Let $f = (f_1, \dots, f_q)$ in $Y = (Y_1, \dots, Y_N)$ over $R$ and $\hat{y}$ a solution of $f$ in $\hat{R}$.

**Theorem 6.13** (M. Artin [A69]). *For any $c \in \mathbf{N}$ there exists a solution $y^{(c)}$ in $R$ such that $y^{(c)} \equiv \hat{y}$ mod $(x)^c$.*

*Proof.* We restrict to the case when the characteristic of $K$ is zero, because we do not want to have separability problems and we believe it is enough for the purpose of our lectures. Apply induction on $n$, the case $n = 0$ being trivial. Let $h : R[Y] \to \hat{R}$ be the morphism of $R$-algebras given by $Y \to \hat{y}$. Since $\hat{R}$ is a domain we see that Ker $h$ is a prime ideal. It is enough to consider the case when $f$ generates Ker $h$. There is an argument to reduce the problem to the case when $f$ generates $P :=\text{Ker } h \cap K[x, Y]$ but we prefer to skip it. However we will suppose that $f$ generates $P$.

Set $r = \text{height}(P)$. As the fraction field extension $Q(R) \subset Q(\hat{R})$ is separable it follows that $Q(R) \subset Q(R[Y]/P)$ is separable and we may suppose after renumbering of $(f_i)$ that there exists an $r \times r$-minor $M$ of the Jacobian matrix $(\partial f_i/\partial Y_j)_{i\in[r],j\in[N]}$ which is not in $P$, that is $M(\hat{y}) \neq 0$.

Applying an automorphism of $R$ of type $x_i \to x_i + a_i x_n$ for $i < n$ and $x_n \to x_n$ for some $a_i \in K$ we may suppose that $M(\hat{y})(0, \dots, 0, x_n) \neq 0$, and so $M^2(\hat{y})$ is regular in $x_n$ of order a certain $u \in \mathbf{N}$. Set $R' = K < x_1, \dots, x_{n-1} >$, $\hat{R}' = K[[x_1, \dots, x_{n-1}]]$ and $m' = (x_1, \dots, x_{n-1})$. By Weierstrass Preparation Theorem (see [JP00, Section 3.2]) there exist $\hat{a}_i$, $0 \leq i < u$ in $m'\hat{R}'$ such that $M^2(\hat{y})$ is associated in divisibility with the polynomial

$$\hat{a} = x_n^u + \hat{a}_{u-1} x_n^{u-1} + \ldots + \hat{a}_0$$

from $\hat{R}'[x_n]$.

Let $Y_{sj}$, $A_k$, $1 \leq s \leq N$, $0 \leq j, k < u$ be some new variables. Substitute in $f$, $M^2$ the variable $Y_s$ by

$$Y_s^+ = \sum_{j=0}^{u-1} Y_{sj} x_n^j$$

and divide the result by the monic polynomial

$$A = x_n^u + \sum_{j=0}^{u-1} A_j x_n^j.$$

We obtain

$$M^2(Y^+) = AH((Y_{sj}), (A_k)) + \sum_{j=0}^{u-1} G_j((Y_{sj}), (A_k)) x_n^j,$$

$$f_i(Y^+) = AH_i((Y_{sj}), (A_k)) + \sum_{j=0}^{u-1} F_{ij}((Y_{sj}), (A_k)) x_n^j,$$

for some polynomials $H, H_i, G_j, F_{ij} \in K[x_1, \dots, x_{n-1}][(Y_{sj}), A_k]$, $1 \leq i \leq r$, $0 \leq j, k < u$. Using Weierstrass Division Theorem we get

$$\hat{y}_s = \sum_{j=0}^{u-1} \hat{y}_{sj} x_n^j + \hat{a}\hat{b}_s,$$

where $\hat{y}_{sj} \in \hat{R}'$, $\hat{b}_s \in \hat{R}$. Set $\hat{y}_s^+ = \sum_{j=0}^{u-1} \hat{y}_{sj} x_n^j$. Substituting $Y^+$ by $\hat{y}^+$ above we obtain

$$M^2(\hat{y}^+) = \hat{a}H((\hat{y}_{sj}), (\hat{a}_k)) + \sum_{j=0}^{u-1} G_j((\hat{y}_{sj}), (\hat{a}_k)) x_n^j,$$

$$f_i(\hat{y}^+) = \hat{a}H_i((\hat{y}_{sj}), (\hat{a}_k)) + \sum_{j=0}^{u-1} F_{ij}((\hat{y}_{sj}), (\hat{a}_k)) x_n^j,$$

$1 \le i \le r$. But

$$M^2(\hat{y}^+) \equiv M^2(\hat{y}) \equiv 0, \quad f_i(\hat{y}^+) \equiv f_i(\hat{y}) = 0 \mod \hat{a}$$

and using the unicity from Weierstrass Division Theorem, it follows from the above system of equations that

$$G_j((\hat{y}_{sj}), (\hat{a}_k)) = 0, \quad F_{ij}((\hat{y}_{sj}), (\hat{a}_k)) = 0,$$

$1 \le i \le r$, $0 \le j < u$, that is $(\hat{y}_{sj}), (\hat{a}_k)$ is a solution of the system of polynomials $G = (G_j)$, $F = (F_{ij})$ in $\hat{R}'$. Then by induction hypothesis for $c' = c + u + 1$ there exists a solution $(y_{sj}^{(c')}, a_k^{(c')})$ of $F, G$ in $R'$ such that

$$y_{sj}^{(c')} \equiv \hat{y}_{sj}, \quad a_k^{(c')} \equiv \hat{a}_k \mod m'^{c'} \hat{R}'.$$

Choose $b_s \in K[x]$ such that $b_s \equiv \hat{b}_s \mod (x)^{c'}$ and set $\tilde{a}^{(c')} = x_n^u + \sum_{k=0}^{u-1} a_k^{(c')} x_n^k$, $\tilde{y}_s^{(c')} = \tilde{a}^{(c')} b_s + \sum_{j=0}^{u-1} y_{sj}^{(c')} x_n^j$. Clearly $\tilde{y}_s^{(c')} \equiv \hat{y}_s \mod (x)^{c'}$ and

$$M^2(\tilde{y}^{(c')}) \equiv 0, \quad f_i(\tilde{y}^{(c')}) \equiv 0 \mod (\tilde{a}^{(c')}).$$

On the other hand $M^2(\tilde{y}^{(c')}) \equiv M^2(\hat{y}) \mod (x)^{(c')}$. As $c' > u$ it follows that $M^2(\tilde{y}_s^{(c')})$ is regular in $x_n$ of order $u$ and so $M^2(\tilde{y}_s^{(c')})$ is associated in divisibility with $\tilde{a}^{(c')}$ by Weierstrass Preparation Theorem. Thus $f_i(\tilde{y}^{(c')}) \equiv 0 \mod M^2(\tilde{y}_s^{(c')})$, $1 \le i \le r$.

Now note that $M^2(\tilde{y}_s^{(c')})$ does not belong to $(x)^{u+1}$ because it is regular of order $u$, but $f_i(\tilde{y}^{(c')}) \equiv f_i(\hat{y}) = 0 \mod (x)^{c'}$ and we get $f_i(\tilde{y}^{(c')}) \equiv 0 \mod M^2(\tilde{y}_s^{(c')})(x)^c$. By Newton Lemma there exists a solution $y^{(c)}$ in $R$ of $f_i$, $1 \le i \le r$ such that $y^{(c)} \equiv \tilde{y}^{(c')} \equiv \hat{y} \mod (x)^c$.

It remains to show that the solution $y^{(c)}$ is a solution also of $f_i$ with $r < i \le q$ if $c$ is big enough. In other words, a solution of $f_i$, $1 \le i \le r$ which is closed to $\hat{y}$ is a solution of $f_i$ for all $i > r$. Let $I := \sqrt{(f_1, \ldots, f_r)} = \cap_{i=1}^e p_i$ be the irreducible primary decomposition of $I$, $p_i$ being prime ideals of $K[x, Y]$ and $h_c : K[x, Y] \to \hat{R}$ be the morphism given by $Y \to y^{(c)}$. Clearly $\operatorname{Ker} h_c \supset I$. Since $P \supset (f_1, \ldots, f_r)$ we see that $P \supset p_i$ for some $i$, let us say $P \supset p_1$. But height $p_1 \ge r$ since the jacobian matrix $(\partial f_i / \partial Y_j)_{i \in [r], j \in [N]}$ has a $r \times r$-minor $M$ which is not in $P$ and so not in $p_1$. Thus $P = p_1$, which ends the problem when $e = 1$.

Suppose that $e > 1$. Choose a polynomial $g \in (\cap_{i=2}^e p_i) \setminus p_1$. Then $g(\hat{y}) \neq 0$ and for $c$ big enough we get $g(y^{(c)}) \neq 0$, that is $\cap_{i=2}^e p_i \not\subset \operatorname{Ker} h_c$. As $(\cap_{i=2}^e p_i) p_1 \subset I \subset \operatorname{Ker} h_c$ we get $P = p_1 \subset \operatorname{Ker} h_c$, which is enough. □

Let $(A, m)$ be a Noetherian local ring. $A$ has the property of *strong approximation* if for every finite system of polynomial equations $f$ in $Y = (Y_1, \ldots, Y_N)$ over $A$ there exist a map $\nu : \mathbf{N} \to \mathbf{N}$ with the following property: If $\tilde{y} \in A^N$ satisfies $f(\tilde{y}) \equiv 0 \mod m^{\nu(c)}$, $c \in \mathbf{N}$ then there exists a solution $y \in A^N$ of $f$ in $A$ with $y \equiv \tilde{y} \mod m^c$. The function $\nu$ is called the *Artin function* of $A$. By Artin the Henselization of a local ring, which is essentially of finite type over a field, has the property of strong approximation. If $A$ is complete then it has the property of strong approximation by G. Pfister and myself.

Next lemma shows how one can apply the property of strong approximation in an easy algebraic problem.

**Lemma 6.14.** *Suppose that $A$ is a domain and it has the property of strong approximation. Let $(a_n)_{n\in\mathbf{N}}$ be a sequence of elements of $A$, which converges in the $m$-adic topology to an element $a \in A$. If $a$ is irreducible then there exists a positive integer $t$ such that $a_n$ is irreducible for all $n \geq t$.*

*Proof.* Let $\nu$ be the Artin function associated to the polynomial $f = YZ - a$ over $A$. Let $t > \nu(1)$ be such that $a_n \equiv a \mod m^{\nu(1)}$ for all $n \geq t$. If $a_n$ is reducible for some $n \geq t$ then there exist $\tilde{y}$, $\tilde{z}$ in $m$ such that $\tilde{y}\tilde{z} = a_n \equiv a \mod m^{\nu(1)}$. In particular, $f(\tilde{y}, \tilde{z}) \equiv 0 \mod m^{\nu(1)}$ and so there exists $y, z \in A$ such that $f(y, z) = 0$ and $y \equiv \tilde{y},\ z \equiv \tilde{z} \mod m$. Thus $a = yz$ and $y, z \in m$ which is impossible. □

**Proposition 6.15.** *A Noetherian local ring has the property of approximation if and only if it has the property of strong approximation.*

*Proof.* Suppose that $(A, m)$ has the property of strong approximation. Let $f$ be a finite system of polynomial equations in $Y = (Y_1, \ldots, Y_N)$ over $A$ and $\hat{y}$ a solution of $f$ in the completion $\hat{A}$. Let $\nu$ be the Artin function associated to $f$, $c$ be a positive integer and choose $y \in A^N$ such that $y \equiv \hat{y} \mod m^{\nu(c)}\hat{A}$. Then $f(y) \equiv f(\hat{y}) = 0 \mod m^{\nu(c)}\hat{A}$ and so there exists a solution $\tilde{y}$ of $f$ in $A$ such that $\tilde{y} \equiv y \mod m^c$. Clearly, $\tilde{y} \equiv \hat{y} \mod m^c\hat{A}$.

Conversely, suppose that $A$ has the property of approximation. Let $f$, $c$ be as above. Since the completion $\hat{A}$ has the strong approximation, let $\nu$ be the Artin function of $f$ over $\hat{A}$. We claim that this function works for $f$ over $A$ too. Indeed, let $\tilde{y}$ be in $A^N$ such that $f(\tilde{y}) \equiv 0 \mod m^{\nu(c)}$. Then there exists a solution $\hat{y}$ of $f$ in $\hat{A}$ such that $\hat{y} \equiv \tilde{y} \mod m^c\hat{A}$. Since $A$ has the property of approximation there exists a solution $y$ of $f$ in $A$ such that $y \equiv \hat{y} \equiv \tilde{y} \mod m^c\hat{A}$. □

# 7 Algorithms and Computations in Local Algebra (M. Vladoiu)

The aim of this section is to develop the algorithm to compute the Hilbert function, Hilbert series, Hilbert polynomial and Krull dimension respectively for a graded module over the positively graded polynomial ring in $n$ indeterminates over a field $K$. Moreover, we illustrate in the case of graded $K$–algebra $K[x_1, \ldots, x_r]/I$, where $I$ is homogeneous ideal how to use Singular (cf. [DGPS12]) for this demand.

First we recall the definitions of a graded ring and graded module together with some basic constructions and elementary properties.

**Definition 7.1.** A *graded ring* A is a ring together with a direct sum decomposition $A = \bigoplus_{\nu \geq 0} A_\nu$, where the $A_\nu$ are abelian groups satisfying $A_\nu A_\mu \subset A_{\nu+\mu}$ for all $\nu, \mu \geq 0$.

A *graded $K$-algebra*, $K$ a field, is a $K$-algebra which is a graded ring such that $A_\nu$ is a $K$-vector space for all $\nu \geq 0$, and $A_0 = K$.

The $A_\nu$ are called *homogeneous components* and the elements of $A_\nu$ are called *homogeneous* elements of *degree* $\nu$.

*Example* 7.2. Let $A := K[x_1, \ldots, x_r]$ be the polynomial ring over the field $K$ in $r$ indeterminates. Then $S$ is a graded $K$–algebra with the following direct sum decomposition $A = \bigoplus_{\nu \geq 0} A_\nu$, where $A_\nu$ is the $K$–vector space generated by all monomials of degree $\nu$ (the degree of the monomial $x_1^{i_1} \cdots x_r^{i_r}$ is $i_1 + \cdots + i_r$).

**Definition 7.3.** Let $A = \bigoplus_{\nu \geq 0} A_\nu$ be a graded ring. An $A$–module $M$, together with a direct sum decomposition $M = \bigoplus_{\nu \in \mathbb{Z}} M_\nu$ into abelian groups is called a *graded $A$–module* if $A_\nu M_\mu \subset M_{\nu+\mu}$ for all $\nu \geq 0$, $\mu \in \mathbb{Z}$.

The elements from $M_\nu$ are called *homogeneous* of *degree $\nu$*.

**Definition 7.4.** Let $M = \bigoplus_{\nu \in \mathbb{Z}} M_\nu$ be a graded $A$–module and define $M(d) := \bigoplus_{\nu \in \mathbb{Z}} M(d)_\nu$ with $M(d)_\nu := M_{\nu+d}$. Then $M(d)$ is a graded $A$–module and, in particular $A(d)$ is a graded $A$–module. $M(d)$ is called the *$d$–th shift* of $M$.

**Lemma 7.5.** *Let $M = \bigoplus_{\nu \in \mathbb{Z}} M_\nu$ be a graded $A$–module and $\subset M$ a submodule. The following conditions are equivalent:*

*(1) $N$ is graded with the induced grading, that is, $N = \bigoplus_{\nu \in \mathbb{Z}} (M_\nu \cap N)$.*

*(2) $N$ is generated by homogeneous elements.*

*(3) Let $m = \sum m_\nu$, $m_\nu \in M_\nu$. Then $m \in N$ if and only if $m_\nu \in N$ for all $\nu$.*

**Definition 7.6.** Let $A = \bigoplus_{\nu \geq 0} A_\nu$ be a graded ring and $M = \bigoplus_{\nu \in \mathbb{Z}} M_\nu$, $N = \bigoplus_{\nu \in \mathbb{Z}} N_\nu$ be graded $A$-modules. A homomorphism $\varphi : M \to N$ is called *homogeneous* of degree d if $\varphi(M_\nu) \subset N_{\nu+d}$ for all $\nu$. If $\varphi$ is homogeneous of degree zero we call $\varphi$ just homogeneous.

*Example* 7.7. Let $M$ be a graded $A$-module and $f \in A_d$ then the multiplication with $f$ defines a homogeneous homomorphism $M \to M$ of degree $d$. It also defines a homogeneous homomorphism $M \to M(d)$ of degree 0.

**Lemma 7.8.** *Let $A$ be a graded ring and $M, N$ be graded $A$-modules. Let $\varphi : M \to N$ be a homogeneous $A$-module homomorphism, then* $\mathrm{Ker}(\varphi)$, $\mathrm{Coker}(\varphi)$ *and* $\mathrm{Im}(\varphi)$ *are graded $A$-modules with the induced grading.*

**Lemma 7.9.** *Let $A = \bigoplus_{\nu \geq 0} A_\nu$ be a Noetherian graded $K$-algebra and $M = \bigoplus_{\nu \in \mathbb{Z}} M_\nu$ be a finitely generated $A$-module. Then*

*(1) there exists $m \in \mathbb{Z}$ such that $M_\nu = \langle 0 \rangle$ for $\nu < m$;*

*(2) $dim_K M_\nu < \infty$ for all $\nu$.*

*Proof.* (1) is obvious because $M$ is finitely generated and a graded $A$-module. To prove (2) it is enough to prove that $M_\nu$ is a finitely generated $A_0$-module for all $\nu$.

By assumption $M$ is finitely generated and we may choose finitely many homogeneous elements $m_1, \ldots, m_k$ to generate $M$. Assume that $m_i \in M_{e_i}$ for $i = 1, \ldots, k$, then $\sum_i A_{n-e_i} \cdot m_i = M_n$ (with the convention $A_\nu = 0$ for $\nu < 0$). This implies that $M_n$ is a finitely generated $A_0$-module because the $A_\nu$ are finitely generated $A_0$-modules. □

As a consequence of Lemma 7.9 we may introduce the following definition.

**Definition 7.10.** Let $A = \bigoplus_{\nu \geq 0} A_\nu$ be a Noetherian graded $K$–algebra, $K$ a field, and let $M = \bigoplus_{\nu \in \mathbb{Z}} M_\nu$ be a finitely generated graded $A$–module. The *Hilbert function* $H_M : \mathbb{Z} \to \mathbb{Z}$ of $M$ is defined by

$$H_M(n) := \dim_K(M_n)\,,$$

and the *Hilbert–Poincaré series* $\mathrm{HP}_M$ of $M$ is defined by

$$\mathrm{HP}_M(t) := \sum_{\nu \in \mathbb{Z}} H_M(\nu) \cdot t^\nu \in \mathbb{Z}[[t]][t^{-1}]\,.$$

The following results are elementary properties of $H_M$ and $\mathrm{HP}_M$.

**Lemma 7.11.** *Let $A = \bigoplus_{\nu \geq 0} A_\nu$ be a Noetherian graded $K$–algebra, and let $M$ be a finitely generated graded $A$–module.*

(1) *Let $N \subset M$ be a graded submodule, then*

$$H_M(n) = H_N(n) + H_{M/N}(n)$$

*for all $n$, in particular,* $\mathrm{HP}_M(t) = \mathrm{HP}_N(t) + \mathrm{HP}_{M/N}(t)$.

(2) *Let $d$ be an integer, then*

$$H_{M(d)}(n) = H_M(n + d)$$

*for all $n$, in particular,* $\mathrm{HP}_{M(d)}(t) = t^{-d}\,\mathrm{HP}_M(t)$.

(3) *Let $d$ be a non–negative integer, let $f \in A_d$, and let $\varphi : M(-d) \to M$ be defined by $\varphi(m) := f \cdot m$, then* $\mathrm{Ker}(\varphi)$ *and* $\mathrm{Coker}(\varphi)$ *are graded $A/\langle f \rangle$–modules with the induced gradings and*

$$H_M(n) - H_M(n - d) = H_{\mathrm{Coker}(\varphi)}(n) - H_{\mathrm{Ker}(\varphi)}(n - d)\,,$$

*in particular,* $\mathrm{HP}_M(t) - t^d\,\mathrm{HP}_M(t) = \mathrm{HP}_{\mathrm{Coker}(\varphi)}(t) - t^d\,\mathrm{HP}_{\mathrm{Ker}(\varphi)}(t)$.

*Proof.* (1) holds, because $N_\nu = N \cap M_\nu$ and $(M/N)_\nu = M_\nu / N_\nu$. (2) is an immediate consequence of the definition of $M(d)$, and (3) is a consequence of (1) and (2). ☐

**Theorem 7.12.** *Let $A = \bigoplus_{\nu \geq 0} A_\nu$ be a graded $K$–algebra, and assume that $A$ is generated, as $K$–algebra, by $x_1, \ldots, x_r \in A_1$. Then, for any finitely generated (positively) graded $A$–module $M = \bigoplus_{\nu \geq 0} M_\nu$,*

$$\mathrm{HP}_M(t) = \frac{Q(t)}{(1 - t)^r} \quad \textit{for some } Q(t) \in \mathbb{Z}[t]\,.$$

*Proof.* We prove the theorem using induction on $r$. In the case $r = 0$, $M$ is a finite dimensional $K$–vector space, and, therefore, there exists an integer $n$ such that $M_\nu = \langle 0 \rangle$ for $\nu \geq n$. This implies $\mathrm{HP}_M(t) \in \mathbb{Z}[t]$.

Assume that $r > 0$, and consider the map $\varphi : M(-1) \to M$ defined by multiplication with $x_1$. Using Lemma 7.11 (3), we obtain

$$(1 - t) \cdot \mathrm{HP}_M(t) = \mathrm{HP}_{\mathrm{Coker}(\varphi)}(t) - t\,\mathrm{HP}_{\mathrm{Ker}(\varphi)}(t).$$

Now both $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ are graded $A/\langle x_1 \rangle \cong A_0[\overline{x}_2, \ldots, \overline{x}_r]$–modules, where $\overline{x}_i := x_i \bmod \langle x_1 \rangle$, $i = 2, \ldots, r$. Using the induction hypothesis we obtain $\mathrm{HP}_{\mathrm{Coker}(\varphi)}(t) = Q_1(t)/(1 - t)^{r-1}$ and $\mathrm{HP}_{\mathrm{Ker}(\varphi)}(t) = Q_2(t)/(1 - t)^{r-1}$ for some $Q_1, Q_2 \in \mathbb{Z}[t]$. This implies $\mathrm{HP}_M(t) = \big(Q_1(t) - tQ_2(t)\big)/(1 - t)^r$. ☐

With the notations of Theorem 7.12, we cancel all common factors in the numerator and denominator of $\mathrm{HP}_M(t) = Q(t)/(1-t)^r$, and we obtain

$$\mathrm{HP}_M(t) = \frac{G(t)}{(1-t)^s}\,, \quad 0 \le s \le r\,, \quad G(t) = \sum_{\nu=0}^{d} g_\nu t^\nu \in \mathbb{Z}[t]\,,$$

such that $g_d \neq 0$ and $G(1) \neq 0$, that is, $s$ is the pole order of $\mathrm{HP}_M(t)$ at $t = 1$.[1]

**Definition 7.13.** Let $A = \bigoplus_{\nu \ge 0} A_\nu$ be a Noetherian graded $K$–algebra, and let $M = \bigoplus_{\nu \ge 0} M_\nu$ be a finitely generated (positively) graded $A$–module.

1. The polynomial $Q(t)$, respectively $G(t)$, defined above, is called the *first Hilbert series*, respectively the *second Hilbert series*, of $M$.
2. Let $d$ be the degree of the second Hilbert series $G(t)$, and let $s$ be the pole order of the Hilbert–Poincaré series $\mathrm{HP}_M(t)$ at $t = 1$, then

$$P_M := \sum_{\nu=0}^{d} g_\nu \cdot \binom{s-1+n-\nu}{s-1} \in \mathbb{Q}[n]$$

   is called the *Hilbert polynomial* of $M$ (with $\binom{n}{k} = 0$ for $k < 0$).

**Corollary 7.14.** *With the above assumptions, $P_M$ is a polynomial in $n$ with rational coefficients, of degree $s-1$, and satisfies $P_M(n) = H_M(n)$ for $n \ge d$. Moreover, there exist $a_\nu \in \mathbb{Z}$ such that*

$$P_M = \sum_{\nu=0}^{s-1} a_\nu \cdot \binom{n}{\nu} = \frac{a_{s-1}}{(s-1)!} \cdot n^{s-1} + \textit{ lower terms in } n\,,$$

*where $a_{s-1} = G(1) > 0$.*

*Proof.* The equality $1/(1-t)^s = \sum_{\nu=0}^{\infty} \binom{s-1+\nu}{s-1} \cdot t^\nu$ (see Example 7.18) implies

$$\sum_{\nu=0}^{\infty} H_M(\nu) t^\nu = \mathrm{HP}_M(t) = \left(\sum_{\nu=0}^{d} g_\nu t^\nu\right) \cdot \sum_{\mu=0}^{\infty} \binom{s-1+\mu}{s-1} t^\mu\,.$$

Therefore, for $n \ge d$, we obtain

$$H_M(n) = \sum_{\nu=0}^{d} g_\nu \cdot \binom{s-1+n-\nu}{s-1} = P_M(n)\,.$$

It is easy to see that the leading term of $P_M \in \mathbb{Q}[n]$ is $\sum_{\nu=0}^{d} g_\nu n^{s-1}/(s-1)!$ which equals $G(1) \cdot n^{s-1}/(s-1)!$. In particular, we obtain $\deg P_M = s-1$.

Finally, we have to prove that $P_M = \sum_{\nu=0}^{s-1} a_\nu \binom{n}{\nu}$ for suitable $a_\nu \in \mathbb{Z}$ and $a_{s-1} > 0$. Suppose that we can find such $a_\nu \in \mathbb{Z}$. Then

$$P_M = \frac{a_{s-1}}{(s-1)!} \cdot n^{s-1} + \text{ lower terms in } n\,.$$

Now, $P_M(n) = H_M(n) > 0$ for $n$ sufficiently large implies $a_{s-1} > 0$. Finally, the existence of suitable integer coefficients $a_\nu$ is a consequence of the following general lemma. □

[1] We set $G(t) := 0$, $s := 0$, if $M = 0$. Throughout this lecture the zero–polynomial has degree $-1$, and we set $\binom{n}{-1} := 0$ if $n \ge 0$ and $\binom{-1}{-1} := 1$.

**Lemma 7.15.** *Let $f \in \mathbb{Q}[t]$ be a polynomial of degree $m$ and $n_0 \in \mathbb{N}$ such that $f(n) \in \mathbb{Z}$ for all $n \geq n_0$. Then $f(n) = \sum_{\nu=0}^{m} a_\nu \binom{n}{\nu}$ for suitable $a_\nu \in \mathbb{Z}$.*

*Proof.* Let $g(n) := f(n+1) - f(n)$. Then $g$ is a polynomial of degree $m-1$ and $g(n) \in \mathbb{Z}$ for $n \geq n_0$. By induction on $m$ we may assume that there exist $b_\nu \in \mathbb{Z}$ such that $g(n) = \sum_{\nu=0}^{m-1} b_\nu \binom{n}{\nu}$. Now consider the function

$$h(n) := f(n) - \sum_{\nu=1}^{m} b_{\nu-1} \binom{n}{\nu} .$$

Then

$$\begin{aligned} h(n+1) - h(n) &= g(n) - \sum_{\nu=1}^{m} b_{\nu-1} \left( \binom{n+1}{\nu} - \binom{n}{\nu} \right) \\ &= g(n) - \sum_{\nu=1}^{m} b_{\nu-1} \binom{n}{\nu-1} = 0 . \end{aligned}$$

It follows that $h(n) = h(0)$ for all $n \in \mathbb{N}$, hence $f(n) = h(0) + \sum_{\nu=1}^{m} b_{\nu-1} \binom{n}{\nu}$. This implies $f(n) = \sum_{\nu=0}^{m} a_\nu \binom{n}{\nu}$ with $a_0 = h(0)$ and $a_\nu = b_{\nu-1}$ for $\nu \geq 1$. □

**Definition 7.16.** Let $R$ be an arbitrary ring. The supremum of the lengths $r$, taken over all strictly decreasing chains $P_0 \supsetneq P_1 \supsetneq \cdots \supsetneq P_r$ of prime ideals of $R$, is called the *Krull dimension* of $R$ and denoted by $\dim(R)$. If $M$ is an $R$–module, then the *Krull dimension* of $M$, $\dim(M)$, is defined by $\dim(M) = \dim\bigl(R/\operatorname{Ann}(M)\bigr)$.

*Remark* 7.17. It is a consequence of Hilbert–Serre's Theorem (see [GP07, Exercise 5.3.5]) that the degree $s$ (from Corollary 7.14) of the Hilbert polynomial $P_M$ is just $\dim(M) - 1$.

In the following we want to answer to the following question:

*How do we compute the Hilbert–Poincaré series, Hilbert function, Hilbert polynomial and Krull dimension for $K[x_1, \ldots, x_r]/I$, where $I$ is a homogeneous ideal?*

*Example* 7.18. Let $K[x] := K[x_1, \ldots, x_r]$ be the polynomial ring in $r$ indeterminates, considered as graded $K$–algebra. Then $H_{K[x]}(n) = P_{K[x]}(n) = \binom{n+r-1}{r-1}$ and, therefore,

$$\mathrm{HP}_{K[x]}(t) = \sum_{\nu=0}^{\infty} \binom{r-1+\nu}{r-1} t^\nu = \frac{1}{(1-t)^r} .$$

It follows from Definition 7.13 that the Hilbert polynomial and the Hilbert–Poincaré series determine each other. Hence, it suffices to study and compute the Hilbert–Poincaré series. The following theorem is fundamental for the computation of the Hilbert–Poincaré series for arbitrary Noetherian graded $K$–algebras and is based on the standard bases theory developed in Section 4.

**Theorem 7.19.** *Let $>$ be any monomial ordering on $K[x] := K[x_1, \ldots, x_r]$, and let $I \subset K[x]$ be a homogeneous ideal. Then*

$$\mathrm{HP}_{K[x]/I}(t) = \mathrm{HP}_{K[x]/L(I)}(t) ,$$

*where $L(I)$ is the leading ideal of $I$ with respect to $>$.*

*Proof.* We have to show that $H_{K[x]/I}(n) = H_{K[x]/L(I)}(n)$, or, equivalently, $\dim_K K[x]_n/I_n = \dim_K K[x]_n/L(I)_n$ for all $n$.

Let $S := \{x^\alpha \notin L(I) \mid \deg(x^\alpha) = n\}$. We shall prove that $S$ represents a $K$–basis in $K[x]_n/I_n$ and $K[x]_n/L(I)_n$. To do so, choose a standard basis $G$ of $I$ and let $f \in K[x]_n$. Then we obtain that both, $\mathrm{NF}(f, G)$ and $\mathrm{NF}\big(f, L(G)\big)$, are elements of $K[x]_n$. Iterating this process by computing the Mora normal form of the tail of $\mathrm{NF}(f, G)$, respectively $\mathrm{NF}\big(f, L(G)\big)$, we can assume that $\mathrm{NF}(f, G)$, $\mathrm{NF}\big(f, L(G)\big) \in \sum_{x^\alpha \in S} K \cdot x^\alpha$.

Since $\mathrm{NF}(f, G) = 0$ (respectively $\mathrm{NF}\big(f, L(G)\big) = 0$) if and only if $f \in I$ (respectively $f \in L(I)$), the latter implies that $S$ represents a $K$–basis of $K[x]_n/I_n$ and $K[x]_n/L(I)_n$. This proves the theorem. □

*Remark* 7.20. In particular it follows from the theorem and Remark 7.17 that

$$\dim(K[x_1, \dots, x_r]/I) = \dim(K[x_1, \dots, x_r]/L(I)),$$

for every homogeneous ideal $I$ and every monomial ordering on $K[x_1, \dots, x_r]$. However, this is true for an arbitrary ideal $I$ of $K[x_1, \dots, x_r]$, see [GP07, Corollary 7.5.5].

Note that this theorem implies that the computation of the Hilbert–Poincaré series for $K[x_1, \dots, x_r]/I$ reduces to the computation of $K[x_1, \dots, x_r]/L(I)$ for an arbitrary monomial ordering. The following lemma is the key step for writing down an algorithm to compute the Hilbert–Poincaré series for a monomial ideal.

**Lemma 7.21.** *Let $I \subset K[x] := K[x_1, \dots, x_r]$ be a homogeneous ideal, and let $f \in K[x]$ be a homogeneous polynomial of degree $d$ then*

$$\mathrm{HP}_{K[x]/I}(t) = \mathrm{HP}_{K[x]/\langle I, f\rangle}(t) + t^d\,\mathrm{HP}_{K[x]/(I:\langle f\rangle)}(t)\,.$$

*Proof.* We consider the following exact sequence

$$0 \longrightarrow \big(K[x]/(I : \langle f\rangle)\big)(-d) \xrightarrow{\cdot f} K[x]/I \longrightarrow K[x]/\langle I, f\rangle \longrightarrow 0$$

and use Lemma 7.11. □

*Example* 7.22. Let $I := \langle xz, yz\rangle \subset K[x, y, z]$. Using Lemma 7.21 for $f := z$ we obtain

$$\mathrm{HP}_{K[x,y,z]/\langle xz,yz\rangle}(t) = \mathrm{HP}_{K[x,y]}(t) + t \cdot \mathrm{HP}_{K[z]}(t) = \frac{-t^2 + t + 1}{(1-t)^2}\,,$$

and, therefore,

$$P_{K[x,y,z]/\langle xz,yz\rangle}(n) = \binom{n+1}{1} + \binom{n}{1} - \binom{n-1}{1} = n + 2\,.$$

*Example* 7.23. Let $f_1, \dots, f_s$ be a homogeneous regular sequence in the polynomial ring $S := K[x_1, \dots, x_r]$. If $a_i = \deg(f_i)$, then

$$\mathrm{HP}_{S/\langle f_1,\dots,f_s\rangle}(t) = \frac{\prod_{i=1}^{s}(1 - t^{a_i})}{(1-t)^r}.$$

Indeed, the assertion follows by induction on $s$. The case $s = 1$ is an easy consequence of Lemma 7.11(2) and Lemma 7.21. For the induction step, we obtain from the definition of a regular sequence that the homogeneous homomorphism

$$S/\langle f_1, \ldots, f_{i-1}\rangle(-a_i) \xrightarrow{\cdot f_i} S/\langle f_1, \ldots, f_{i-1}\rangle$$

is injective for all $i = 1, \ldots, r$. Since $\mathrm{Coker}(\xrightarrow{\cdot f_i})$ is $S/\langle f_1, \ldots, f_i\rangle$ we obtain the desired formula from the induction hypothesis.

Using Lemma 7.21 we obtain the following algorithm for computing the Hilbert–Poincaré series for a monomial ideal.

---

**Algorithm 3** MonomialHilbertPoincare $(I)$, the Hilbert–Poincaré series for $K[x]/I$, where $I$ is a monomial ideal

---

**Input:** $I := \langle m_1, \ldots, m_k\rangle \subset K[x]$, $m_i$ monomials in $x = (x_1, \ldots, x_r)$.

**Output:** A polynomial $Q(t) \in \mathbb{Z}[t]$ such that $Q(t)/(1-t)^r$ is the Hilbert–Poincaré series of $K[x]/I$.

- choose $S = \{x^{\alpha_1}, \ldots, x^{\alpha_s}\} \subset \{m_1, \ldots, m_k\}$ to be the minimal set of monomial generators of $I$;
- if $S = \{0\}$ then return 1;
- if $S = \{1\}$ then return 0;
- if all elements of $S$ have degree 1 then return $(1-t)^s$;
- choose $1 \le i \le s$ such that $\deg(x^{\alpha_i}) > 1$ and $1 \le k \le r$ such that $x_k \mid x^{\alpha_i}$;
- return $\big($MonomialHilbertPoincare $(\langle I, x_k\rangle)$

  $+\, t \cdot$ MonomialHilbertPoincare $(I : \langle x_k\rangle)\big)$.

---

This algorithm together with Theorem 7.19 imply the following algorithm for the computation of the Hilbert–Poincaré series of $K[x]/I$.

---

**Algorithm 4** HilbertPoincare$(I)$, the Hilbert–Poincaré series of $K[x]/I$.

---

**Input:** $I := \langle f_1, \ldots, f_k\rangle \subset K[x]$ a homogeneous ideal, $x = (x_1, \ldots, x_r)$.

**Output:** A polynomial $Q(t) \in \mathbb{Z}[t]$ such that $Q(t)/(1-t)^r$ is the Hilbert–Poincaré series of $K[x]/I$.

- compute a standard basis $\{g_1, \ldots, g_s\}$ of $I$ w.r.t. any monomial ordering;
- return MonomialHilbertPoincare$(\langle \mathrm{LM}(g_1), \ldots, \mathrm{LM}(g_s)\rangle)$.

---

*Example* 7.24. We want to compute the Hilbert–Poincaré series, Hilbert polynomial, Hilbert function and Krull dimension for the ring $S := K[x_1, \ldots, x_5]/I$, where $I = \langle f, g\rangle = \langle x_1x_2 - x_3^2, x_3x_4x_5 - 3x_4x_5^2\rangle$.

Since $I$ is a homogeneous ideal, we apply Theorem 7.19 for the monomial ordering $>_{dp}$ (see the definition in Section 4). First we compute a standard basis for $I$ with respect to $>_{dp}$ by Theorem 4.19. Since $\mathrm{LM}(f) = x_1x_2$ and $\mathrm{LM}(g) = x_3x_4x_5$ are relatively prime then $\mathrm{NF}(\mathrm{spoly}(f,g),\{f,g\}) = 0$ (see [GP07, Exercise 1.7.1]). Therefore, $\{f,g\}$ is a standard basis of $I$ with respect to $>_{dp}$, $L(I) = (x_1x_2, x_3x_4x_5)$ and $\mathrm{HP}_{S/I}(t) = \mathrm{HP}_{S/L(I)}(t)$. Note that $x_1x_2, x_3x_4x_5$ is a regular sequence, hence by Example 7.23 one has

$$\mathrm{HP}_{S/I}(t) = \frac{(1-t^2)(1-t^3)}{(1-t)^5} = \frac{1+2t+2t^2+t^3}{(1-t)^3}.$$

Following the notations of Definition 7.13 we have $G(t) = 1+2t+2t^2+t^3 = \sum_{\nu=0}^{3} g_\nu t^\nu$, $d = \deg(G(t)) = 3$, $s = 3$ and consequently the Hilbert polynomial

$$P_{S/I}(n) = \sum_{\nu=0}^{3} g_\nu \cdot \binom{3-1+n-\nu}{3-1} = \sum_{\nu=0}^{3} g_\nu \frac{(n-\nu+2)(n-\nu+1)}{2} = 3n^2+2.$$

Moreover, $P_{S/I}(n) = H_{S/I}(n)$ for $n \geq 3$, according to Corollary 7.14. Moreover, we obtain

$$\begin{aligned}\mathrm{HP}_{S/I}(t) &= \frac{1+2t+2t^2+t^3}{(1-t)^3}\\ &= -1 + \frac{5t^2-t+2}{(1-t)^3} = -1 + \frac{6}{(1-t)^3} + \frac{-9}{(1-t)^2} + \frac{5}{1-t},\end{aligned}$$

and by Example 7.18

$$\begin{aligned}\sum_{\nu=1}^{\infty} H_{S/I}(\nu)t^\nu &= \mathrm{HP}_{S/I}(t)\\ &= -1 + \sum_{\nu=0}^{\infty} 6\binom{\nu+2}{2}t^\nu - \sum_{\nu=0}^{\infty} 9\binom{\nu+1}{1}t^\nu + \sum_{\nu=0}^{\infty} 5\binom{\nu}{0}t^\nu,\end{aligned}$$

that is $H_{S/I}(n) = P_{S/I}(n)$ for all $n \geq 1$ and $H_{S/I}(0) = 1$. Finally, by the Remark 7.17 we also have that $\dim(S/I) = s = 3$.

The corresponding Singular code for the above computations is as follows:

```
ring S=0,(x(1..5)),dp;
ideal I=x(1)*x(2),x(3)*x(4)*x(5);
hilb(I);
//-->  1 t^0
//--> -1 t^2
//--> -1 t^3
//-->  1 t^5
//-->
//-->  1 t^0
//-->  2 t^1
//-->  2 t^2
//-->  1 t^3
```

We obtain that $Q(t) = t^5 - t^3 - t^2 + 1$ (first Hilbert series) and $G(t) = t^3 + 2t^2 + 2t + 1$ (second Hilbert series). The same results can be obtained using the Singular commands `hilb(I,1)`,`hilb(I,1)` and `dim(I)`.

```
hilb(I,1);
//--> 1,0,-1,-1,0,1,0
hilb(I,2);
//--> 1,2,2,1,0
dim(I);
//--> 3
```

The output of the commands `hilb(I,1)` and `hilb(I,2)` are integer vectors corresponding to the coefficients of the first Hilbert series, respectively second Hilbert series. More precisely, the vector

$$v = (v_0, \ldots, v_k, 0) = (1, 0, -1, -1, 0, 1, 0)$$

has to be interpreted as follows: $Q(t) = \sum_{i=0}^{k} v_i t^i$. The last output is the Krull dimension of $S/I$. Finally, we compute the Hilbert polynomial of $S/I$.

```
LIB"poly.lib";
hilbPoly(I);
//-> 4,0,6
```

The output is read as follows: the integer vector $v = [v_0, \ldots, v_r]$ gives the formula of the Hilbert polynomial

$$P_{S/I} = \frac{\sum_{i=0}^{r} v_i t^i}{r!}.$$

Therefore, in our concrete case the Hilbert polynomial of $S/I$ is $P_{S/I} = (6t^2 + 4)/2! = 3t^2 + 2$.

# 8 Regularity and Smoothness (J. Herzog)

This is a short introduction to regularity and smoothness. The reader who wants to study more details is referred to the book "Commutative Algebra" by Matsumura [M70]. An exhaustive treatment of the module of differential can be found in the book "Kähler differentials" by Kunz [K86]. The geometric aspects of the theory are described in Hartshorne's book "Algebraic Geometry", [H77].

## 8.1 A motivating example

Let $k$ be an algebraically closed field, $H = Z(f) \subset \mathring{A}^n$ an affine hypersurface with $f = f_m + f_{m+1} + \cdots + f_d$, $m \leq d$, where the $f_i$ are the homogeneous components of $f$ and $f_m \neq 0 \neq f_d$.

Assume the point $P = (0, \ldots, 0)$ belongs to $H$ and let $g = \{t\xi \;\; t \in k\}$ with $\xi = (\xi_1, \cdots, \xi_n) \in k^n \setminus \{0\}$ be a line through $P$. Then $g \cap H$ is the set of solutions of

$$\begin{aligned} 0 = f(t\xi) &= f_m(\xi)t^m + \ldots + f_d(\xi)t^d \\ &= t^m(f_m(\xi) + \ldots + f_d(\xi)t^{d-m}) \end{aligned}$$

For $t = 0$ we have the intersection point $P$. The order of root $t = 0$ of the polynomial $f(\xi T) = f_m(\xi)T^m + \ldots + f_d(\xi)T^d \in K[T]$ is called the *intersection multiplicity* of $H$ and $g$ in $P$. It is equal to $m$ if and only if $f_m(\xi) \neq 0$. One says that $g$ is a *tangent* of $H$ at $P$, if the intersection multiplicity $> m$. This is equivalent to say that $f_m(\xi) = 0$. The union of all tangents is the cone $Z(f_m)$, called the *tangent cone*.

*Example* 8.1.

1. The tangent cone of the curve $y - x^2 = 0$ at $(0,0)$ is the $x$-axis $y = 0$.

2. The tangent cone of the curve $x^2 - y^2 + x^4$ is the union of the two lines $x = y$ and $x = -y$.

**Definition 8.2.** $H$ is *nonsingular (regular)* at $P$ if $m = 1$.

If $H$ is nonsingular at $P$, then the tangent cone at $P$ is given by the linear equation

$$\sum_{i=1}^{n} a_i x_i = 0, \quad \text{where} \quad a_i = \frac{\partial f}{\partial x_i}(P).$$

If $H$ is singular at $P$, then $\partial f / \partial x_i(P) = 0$ for $i = 1, \ldots, n$.

## 8.2 The singular locus of a variety

Let $Y \subset \mathring{A}^n$ be an affine variety. In other words, $Y$ is an irreducible closed subset of $\mathring{A}^n$, i.e. there exists a prime ideal $\wp \subset S = k[x_1, \ldots, x_n]$ such that

$$Y = \{P \in \mathring{A}^n \;\; f(P) = 0 \quad \text{for all} \quad f \in \wp\}.$$

The prime ideal $\wp$ is the *vanishing ideal* $I(Y) = \{f \in S \;\; f(P) = 0 \quad \text{for all} \quad P \in Y\}$ of the affine variety $Y$.

The ring $A(Y) = S/\wp$ is called the *affine coordinate ring*, and the quotient field $K(Y)$ of $A(Y)$ is called the *function field* of $Y$. Its elements can be identified with the rational functions on $Y$. The dimension of $Y$, $\dim Y$, is the Krull dimension of $A(Y)$. The dimension of $Y$ coincides with the transcendence degree of $K(Y)/k$.

Let $P = (a_1, \ldots, a_n) \in Y$ a point. Then $\mathfrak{m}_P = (x_1 - a_1, \ldots, x_n - a_n)/\wp \in A(Y)$ is a maximal of $A(Y)$ and $\mathcal{O}_{P,Y} = A(Y)_{\mathfrak{m}_P}$ is the *local ring* of $Y$ at $P$. It can be identified with the ring of germs of regular functions near $P$. The *embedding dimension* of $Y$ at $P$ is defined to be the number $\text{embdim}\mathcal{O}_{P,Y} = \dim_k \mathfrak{m}_P/\mathfrak{m}_P^2$. Note that the *Krull dimension* of $\mathcal{O}_{P,Y}$ coincides with the dimension of the variety $Y$.

Before we give the definition of the singular locus of a variety we first show

**Theorem 8.3.** *Let $Y \subset \mathring{A}^n$ be an affine variety, $P \in Y$ a point, $f_1, \ldots, f_m$ a system of generators of the vanishing ideal $I(Y)$ of $Y$. Then*

$$rank(\frac{\partial f_i}{\partial x_j}(P))_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} = n - embdim\mathcal{O}_{P,Y} \leq n - \dim \mathcal{O}_{P,Y},$$

*where $\mathcal{O}_{P,Y}$ is the ring of germs regular functions of $Y$ near $P$.*

*Proof.* Let $P = (a_1, \ldots, a_n)$ and $S = k[x_1, \ldots, x_n]$. Then $\mathcal{O}_{P,Y}$ is the local ring $(S/I(Y))_{\mathfrak{n}}$ where $\mathfrak{n} = (x_1 - a_1, \ldots, x_n - a_n)$.

We define a $k$-linear map

$$\Theta \; S \to k^n, \quad f \mapsto (\frac{\partial f}{\partial x_1}(P), \ldots, \frac{\partial f}{\partial x_n}(P)).$$

Then for all $i \leq j$ one has

(i) $\Theta(x_i - a_i) = e_i = (0, \ldots, 1, \ldots, 0)$;

(ii) $\Theta((x_i - a_i)(x_j - a_j)) = 0$.

This implies that $\Theta(\mathfrak{n}) = k^n$ and $\Theta(\mathfrak{n}^2) = 0$. Hence $\Theta$ induces a $k$-linear map

$$\bar{\Theta} \; \mathfrak{n}/\mathfrak{n}^2 \longrightarrow k^n$$

which is an isomorphism, since $\dim_k \mathfrak{n}/\mathfrak{n}^2 = n$.

Let $f \in I(Y) = \wp$; then $f = \sum_i g_i f_i$ with $g_i \in S$. The $j$th component of $\Theta(f)$ is

$$\sum_i \frac{\partial g_i}{\partial x_j}(P) f_i(P) + \sum_i g_i(P) \frac{\partial f_i}{\partial x_j}(P).$$

Hence $\Theta(\wp) \subset k^n$ is spanned by the vectors $((\frac{\partial f_i}{\partial x_1}(P), \ldots, \frac{\partial f_i}{\partial x_n}(P)),\ i = 1, \ldots, m$. It follows that

$$\dim_k \Theta(\wp) = \operatorname{rank}(\frac{\partial f_i}{\partial x_j}(P)). \tag{1}$$

On the other hand, the isomorphism $\bar{\Theta} \; \mathfrak{n}/\mathfrak{n}^2 \to k^n$ induces the isomorphism

$$\Theta(\wp) \cong (\wp + \mathfrak{n}^2)/\mathfrak{n}^2. \tag{2}$$

Indeed, for a generator $f_i \in \wp$ we have $f_i + \mathfrak{n}^2 = \sum_{j=1}^n (\partial f_i/\partial x_j)(P)(x_j - a_j) + \mathfrak{n}^2$, so that $\bar{\Theta}(f_i + \mathfrak{n}^2) = (\frac{\partial f_i}{\partial x_1}(P), \ldots, \frac{\partial f}{\partial x_n}(P))$.

We have $\operatorname{embdim}\mathcal{O}_{P,Y} = \dim_k \mathfrak{m}_P/\mathfrak{m}_P^2 \cong \mathfrak{n}/\wp + \mathfrak{n}^2$. Therefore, the following exact sequence of $k$-vector spaces

$$0 \longrightarrow \wp + \mathfrak{n}^2/\mathfrak{n}^2 \longrightarrow \mathfrak{n}/\mathfrak{n}^2 \longrightarrow \mathfrak{n}/(\wp + \mathfrak{n}^2) \longrightarrow 0$$

together with (1) and (2) yield

$$\begin{aligned} \operatorname{rank}(\frac{\partial f_i}{\partial x_j}(P)) &= \dim_k \wp + \mathfrak{n}^2/\mathfrak{n}^2 = \dim_k \mathfrak{n}/\mathfrak{n}^2 - \mathfrak{n}/(\wp + \mathfrak{n}^2) \\ &= n - \operatorname{embdim}\mathcal{O}_{P,Y}. \end{aligned}$$

The inequality $n - \operatorname{embdim}\mathcal{O}_{P,Y} \leq n - \dim \mathcal{O}_{P,Y}$, follows from the fact that $\operatorname{embdim} R \geq \dim R$ for any Noetherian local ring. □

**Definition 8.4.** $Y$ is *nonsingular* at $P$, if

$$\operatorname{rank}(\frac{\partial f_i}{\partial x_j}(P))_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} = n - \dim \mathcal{O}_{P,Y}.$$

Note that $Y$ is nonsingular at $P$ if and only if $\text{embdim}\mathcal{O}_{P,Y} = \dim \mathcal{O}_{P,Y}$: Note also that Definition 8.4 coincides with our definition in the hypersurface case.

**Definition 8.5.** A Noetherian local ring $(R, \mathfrak{m})$ is called *regular*, if $\text{embdim}R = \dim R$.

Thus $Y$ nonsingular at $P \Leftrightarrow \mathcal{O}_{P,Y}$ is regular. We list a few algebraic properties of regular local rings.

**Theorem 8.6.** *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of dimension $d$. The following conditions are equivalent:*

*(a) $R$ is regular.*

*(b) $G_{\mathfrak{m}}(R) = \bigoplus_{n\geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1} = K[x_1, \dots, x_d]$ is the polynomial ring.*

*(c) Each minimal system of generators of $\mathfrak{m}$ is a regular sequence.*

*(d) $\text{projdim} k < \infty$. (The Koszul complex provides a resolution.)*

*(e) $\text{projdim} M < \infty$ for any finitely generated $R$-module.*

Furthermore, regular local rings localize (i.e. $R_P$ is again regular for all $P \in \text{Spec}(R)$), see [BH93, Corollary 2.2.9]), they are factorial domains ([BH93, Theorem 2.2.19]) and in particular they are normal.

For the proof Theorem 8.6 we refer to [BH93, Proposition 2.2.5] and to [BH93, Theorem 2.2.7].

Let $Y$ be an affine variety. We denote by $\text{Sing}(Y)$ the set of singular points of $Y$.

**Theorem 8.7.** *$\text{Sing}(Y)$ is a proper closed subset of $Y$.*

*Proof.* Let $I(Y) = (f_1, \dots, f_m)$. Then $P \in \text{Sing}(Y) \Leftrightarrow \text{rank}(\frac{\partial f_i}{\partial x_j}(P)) < n - \dim Y$ and $f_i(P) = 0$ for $i = 1, \dots, m \Leftrightarrow$ all $(n - \dim Y)$-minors of $(\frac{\partial f_i}{\partial x_j}(P))$ vanish at $P$ and $f_i(P) = 0$ for $i = 1, \dots, m$. Therefore $\text{Sing}(Y)$ is a closed subset of $Y$.

Let $K = K(Y)$ be the function field of $Y$. Then there exists a transcendence basis $\xi_1, \dots, \xi_d \in K$ over $k$ such that $K/k(\xi_1, \dots, \xi_d)$ is a separable algebraic extension, see Matsumura Chapter 10. Hence there exists $\xi_{d+1} \in K$ which is algebraic over $k(\xi_1, \dots, \xi_d)$ with $K = k(\xi_1, \dots, \xi_d, \xi_{d+1})$. The element $\xi_{d+1}$ satisfies an algebraic equation over $k(\xi_1, \dots, \xi_d)$. Clearing denominators, we find an irreducible polynomial $f \in k[x_1, \dots, x_{d+1}]$ with $f(\xi_1, \dots, xi_d, \xi_{d+1}) = 0$. Let $H$ be the hypersurface defined by $f$. Then $K(H) = K(Y)$. Therefore $H$ and $Y$ are birationally equivalent. Hence $H$ and $Y$ have isomorphic open subsets, and we may therefore assume that $Y$ is a hypersurface. Then

$$\text{Sing}(Y) = \{P \in Y \ \frac{\partial f}{\partial x_i}(P) = 0 \quad \text{for} \quad i = 1, \dots, n\}.$$

Suppose $\text{Sing}(Y) = Y$. Then, by Hilbert's Nullstellensatz, $\frac{\partial f}{\partial x_i} \in I(Y) = (f)$ for $i = 1, \dots, n$. Since $\deg \frac{\partial f}{\partial x_i} \leq \deg f - 1$, it follows that $\frac{\partial f}{\partial x_i} = 0$ for $i = 1, \dots, n$.

If $\text{chara}k = 0$, then $f$ is constant, a contradiction. If $\text{chara}k = p > 0$, then $\frac{\partial f}{\partial x_i} = 0$ implies that $f$ is a polynomial in $x_i^p$ for each $i$. Therefore $f = g^p$ for some polynomial $g$, a contradiction, since $f$ is irreducible. □

## 8.3 The module of differentials

Let $R$ be a $k$-algebra. Here $k$ is not necessarily a field. Let $M$ be an $R$-module.

**Definition 8.8.** A *$k$-derivation* is a map $d\ R \longrightarrow M$ such that for all $a, b \in R$ and $x, y \in k$ one has

(a) $d(ab) = adb + bda$,

(b) $d(xa + yb) = xda + ydb$.

The set of $k$-derivations $\mathrm{Der}_k(R, M)$ can be given an obvious $R$-module structure. This module is then call the *module of $k$-derivations* from $R$ to $M$.

The *module of differentials* $\Omega_{R/k}$ is the $R$-module generated by the set $\{da\ \ a \in R\}$, subject to the relations (a) and (b).

The module of differentials $\Omega_{R/k}$ can be characterized by the following universal property: let $\delta\ \ R \to M$ be a $k$-derivation. Then there exists a unique $R$-linear map $\phi\,\Omega_{R/k} \to M$ such that $\delta = \phi \circ d$ where $d\,R \to \Omega_R$ is the canonical derivation with $a \mapsto da$.

It follows that $\mathrm{Der}_k(R, M) \cong \mathrm{Hom}_R(\Omega_{R/k}, M)$.

**Proposition 8.9.** *Let $A = S/I$ an affine $k$-algebra, where $S = k[x_1, \dots, x_n]$ and $I = (f_1, \dots, f_m)$. For $f \in S$ we denote by $\bar{f} \in A$ the residue class of $f$ modulo $I$. Then*

$$\Omega_{A/k} \cong (\bigoplus_{i=1,\dots,n} Adx_i)/U,$$

*where $U$ is generated by the elements $\sum_{i=1}^{n} \overline{\partial f_j/\partial x_i} dx_i$, $j = 1, \dots, m$.*

*Proof.* We denote by $D$ the module $(\bigoplus_{i=1,\dots,n} Adx_i)/U$ and define the map by $d(\bar{f}) = \sum_{i=1}^{n} \overline{\partial f/\partial x_i} dx_i + U$. We claim

(1) $d$ is well defined.

(2) $d$ is a derivation.

(3) If $\delta\ A \to M$ is a derivation, then there exists an $A$-linear map $\varphi\ D \to M$ such that $\delta = \varphi \circ d$.

It follows from (1), (2) and (3) that $D \cong \Omega_{A/k}$.

Proof of (1): Suppose that $\bar{f} = \bar{g}$. Then $f = g + h$ with $h \in I$. It follows that $\partial f/\partial x_i = \partial g/\partial x_i + \partial h/\partial x_i$, and this implies that $d(\bar{f}) - d(\bar{g}) = \sum_{i=1}^{n} \overline{\partial h/\partial x_i} dx_i + U$.

Since $h \in I$, there exist $g_j \in S$ such that $h = \sum_{j=1}^{m} g_j f_j$. It follows that

$$\begin{aligned} \overline{\partial h/\partial x_i} &= \sum_{j=1}^{m} (\overline{\partial g_j/\partial x_i})\bar{f}_j + \bar{g}_j \overline{\partial f_j/\partial x_i} \\ &= \sum_{j=1}^{m} \bar{g}_j \overline{\partial f_j/\partial x_i}. \end{aligned}$$

Therefore, $\sum_{i=1}^{n} \overline{\partial h/\partial x_i} dx_i = \sum_{j=1}^{m} \bar{g}_j (\sum_{i=1}^{n} \overline{\partial f_j/\partial x_i}) dx_i \in U$, and this implies that $d(\bar{f}) = d(\bar{g})$, as desired.

Proof of (2): The rules (a and (b) for derivations follow for $d$ immediately because they are valid for partial derivatives.

Proof of (3): We define the $A$-linear map $\psi \bigoplus_{i=1,\dots,n} Adx_i \to M$ by setting $\psi(dx_i) = \delta(x_i)$ for all $i$. Since $\delta(\bar{f}) = \sum_{i=1}^{n} \overline{\partial f/\partial x_i} \delta(x_i)$ for all $\bar{f} \in A$ it follows that $\psi(U) = 0$. This implies that $\psi$ induces the linear map $\phi : D \to M$ with $\phi(m + U) = \psi(m)$ for all $m \in \bigoplus_{i=1,\dots,n} Adx_i$. Obviously, $\varphi \circ d = \delta$. □

**Theorem 8.10.** *Let $Y \subset \mathring{A}^n$ be an affine variety, $P \in Y$ and $R = \mathcal{O}_{P,Y}$. Then*

*(a) rank$\Omega_{R/k} = \dim R$.*

*(b) $\dim_k(\Omega_{R/k} \otimes_R k) =$ embdim$R$.*

*In particular, $Y$ is nonsingular at $P \Leftrightarrow \Omega_{R/k}$ is free (of rank $= \dim R$).*

*Proof.* Let $A = k[x_1, \dots, x_n]/I(Y)$. We use that $\Omega_{R/k}$ localizes. Therefore

(i) $\Omega_{R/k} \cong (\Omega_{A/k})_P$.

(ii) If $K = K(Y)$, then $\Omega_{R/k} \otimes_R K \cong \Omega_{K/k}$.

We also need that

$$\text{rank}_K \Omega_{K/k} = \text{trdeg} K/k = \dim R.$$

Therefore, rank$\Omega_{R/k} = \dim R$.

Let $I(Y) = (f_1, \dots, f_m)$. Then $\Omega_{A/k} = \big(\bigoplus_{i=1}^{n} Adx_i\big)/U$, where

$$U = \Big(\sum_{j=1}^{n} \overline{\frac{\partial f_i}{\partial x_j}} dx_i\Big)_{i=1,\dots,m}.$$

Here $\bar{g}$ denotes the residues class of $g$ modulo $I$.

We obtain an exact sequence

$$U_{\mathfrak{m}_P} \otimes_R k \xrightarrow{\ \phi\ } \bigoplus_{i=1}^{n} Rdx_i \otimes_R k \longrightarrow \Omega_R \otimes_R k \longrightarrow 0,$$

with $k = R/\mathfrak{m}_P R$.

Identifying $\bigoplus_{i=1}^{n} Rdx_i \otimes_R k$ with $\bigoplus_{i=1}^{n} kdx_i$, the image of $\phi$ is equal to

$$\Big(\sum_j \partial f_i/\partial x_j(P) dx_i\Big)_{i=1,\dots,m}.$$

Hence we see that

$$\dim_k(\Omega_{R/k} \otimes_R k) = n - \text{rank}\Big(\frac{\partial f_i}{\partial x_j}(P)\Big) = \text{embdim}R.$$

□

If we only require that $k$ is perfect instead of algebraically closed one obtains a similar result.

*Example* 8.11. Let $k = K_0(t)$ where $k_0$ is of characteristic $p > 0$ and $t$ is transcendental over $k_0$. In particular, $k$ is not algebraically closed.

Let $R = k[x,y]/(y^2 - (x^p - t))$. Then $R_P$ is regular for all $P \in \text{Spec}(R)$. However $R_P$ is not free of rank 1 for $P = (y)$. Indeed $R/(y) \cong k[x]/(x^p - t)$ is field, and hence $(y)$ is a maximal ideal in $R$ and $R_P$ is regular. However

$$\Omega_{R_P/k} = \frac{R_P dx \oplus R_p dy}{2ydy}.$$

If $p = 2$, then $\Omega_{R_P/k}$ is free of rank 2, and if $p \neq 2$, then $\Omega_{R_P/k}$ is not free.

**Definition 8.12.** Let $R$ be an affine $k$-algebra. $R$ is called *smooth*, if for all $P \in \text{Spec}(R)$ one has that $\Omega_{R_P/k}$ is a free $R_P$-module of rank equals $\dim_P R (= \dim R_P + \dim R/P)$.

One has (see [K86, Theorem 7.14])

**Theorem 8.13.** *The following conditions are equivalent:*

*(a)* $\Omega_{R_P/k}$ *is free of rank equal to* $\dim_P R$.

*(b)* $R_P$ *is geometrically regular over* $k$*, i.e. for any finite field extension* $\ell/k$*, all local rings of* $\ell \otimes_k R_P$ *are regular.*

In the above example, if we choose $\ell = k(\sqrt[p]{t})$, then for $P = (y, x - \sqrt[p]{t})$ we have that $\ell \otimes_k R_P$ is not regular.

More generally, let $R$ be a Noetherian ring, $S$ an $R$-algebra which is essentially of finite type over $R$. Then one defines smoothness as follows:

**Definition 8.14.** $S$ is smooth over $R$, if

(a) $S/R$ is flat.

(b) $\Omega_{S_P/R_Q}$ is a free $S_P$-module of rank equal to $\dim_P S_Q/QS_Q$ for all $P \in \text{Spec}S$. Here $Q = P \cap R$.

Condition (b) is equivalent to saying that $S_P/QS_P$ is geometrically regular over $k(Q) = R_Q/QR_Q$.

One can show that if $R$ contains a field of characteristic 0, then $S/R$ is smooth if and only if $S/R$ is flat and $\Omega_{S/R}$ is a projective $S$-module.

# 9 Classification of Singularities (G. Pfister)

This talk is devoted to the study of hypersurface singularities. Two functions $f$ and $g$ are called right equivalent if there exists an automorphism $\varphi$ of $\mathbb{C}\{x_1, \ldots, x_n\}$ such that $\varphi(f) = g$. They are called contact equivalent if there exists an automorphism $\varphi$ and a unit $u$ in $\mathbb{C}\{x_1, \ldots, x_n\}$ such that $\varphi(f) = u \cdot g$. The aim of this talk is to give a beginning of the classification of functions up to right equivalence. The first thing is to consider Finite Determinacy Theorems. Let $\mathfrak{m}$ be the maximal ideal of $\mathbb{C}\{x_1, \ldots, x_n\}$. A function $f$ is called (right) $k$–determined if for all elements $g$ with $f - g \in \mathfrak{m}^{k+1}$ the functions $f$ and $g$ are right equivalent. In particular, a $k$–determined $f$ is right equivalent to a polynomial of degree at most $k$. A function is called finitely determined, if there exists a $k \in \mathbb{N}$ such that $f$ is $k$–determined. We prove that, if $\mathfrak{m}^{k+1} \subset \mathfrak{m}^2 J(f)$, then $f$ is $k$–determined. (Here $J(f)$ is the Jacobian ideal, that is, the ideal generated by the partial derivatives of $f$.) This is what is usually called the Finite Determinacy Theorem. The idea of the proof is to connect $f$ and $g$ via the path $f + a(g - f)$. So for $a = 0$ we get $f$, and for $a = 1$ we get $g$. For all $a$, we want to prove that there exists a small neighborhood of $a$ in $\mathbb{C}$ such that for all $b \in U$ the functions $f + b(g - f)$ and $f + a(g - f)$ are right equivalent. In order to reach this goal, we need a criterion of local triviality. So, consider an element $F \in \mathbb{C}\{x_1, \ldots, x_n, t\}$ which we view as a family of functions with parameter $t$. Then we prove that this family is trivial if and only if $\frac{\partial F}{\partial t} \in (x_1, \ldots, x_n)\left(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}\right)$.

As an application of the local triviality one can prove the Mather-Yau Theorem. It says that $f$ and $g$ are contact equivalent, if and only if their Tjurina algebras are isomorphic. We will also give a corresponding statement for right equivalence.

Next, we consider the classification of functions up to right equivalence. We first define what a simple function is. This is done by putting a topology on $\mathbb{C}\{x_1, \ldots, x_n\}$. A function $f$ is then called simple, if there are only finitely many equivalence classes in a sufficiently small neighborhood of $f$. The goal is to prove that the simple singularities are given by the *A*-*D*-*E*–singularities. We first state the Splitting Lemma. This allows us to study functions which lie in $\mathfrak{m}^3$. Then, via a sequence of arguments, it is shown that the *A*-*D*-*E*–singularities are simple, and that for nonsimple singularities $f$ we may assume that either $n \geq 3$ and $f \in \mathfrak{m}^3$, or $n = 2$ and $f \in \mathfrak{m}^4$, or $n = 2$ and $f \in (x, y^2)^3$ or $f$ is nonisolated.

Finally, we then show that functions with one of these properties are not simple. The idea is to use the Finite Determinacy Theorem, to reduce the study of right equivalence classes to the study of orbits under a group action in $\mathbb{C}^N$ for some $N \gg 0$. In fact, we study the group action on the $k$–jets, which are the Taylor series of $f$ up to order $k + 1$. These orbits are orbits under the action of an algebraic group. This means that all the actions are given by polynomial functions. We will see that the tangent space to the orbit is equal to $\mathfrak{m}J(f) + \mathfrak{m}^{k+1}/\mathfrak{m}^{k+1}$. In particular, we get the dimension of the orbit of $f$. We deduce that if $g \notin \mathfrak{m}J(f)$, then in the family $f + t \cdot g$ there are only finitely many $t$ such that $f + tg$ is right equivalent to $f$. The knowledge of the dimension of the orbits is our main tool for showing that certain functions are not simple.

## 9.1 Finite Determinacy of Hypersurface Singularities

**Definition 9.1.** Let $f, g \in \mathfrak{m} \subset \mathbb{C}\{x_1, \ldots, x_n\}$, where $\mathfrak{m}$ is, as usual, the maximal ideal.

1. One says that $f$ is *right equivalent* to $g$, $f \underset{R}{\sim} g$, if there exists an automorphism $\varphi$ of $\mathbb{C}\{x_1, \ldots, x_n\}$ such that $\varphi(f) = g$.

2. One says that $f$ is *contact equivalent* to $g$, $f \underset{C}{\sim} g$, if there exists an automorphism $\varphi$ of $\mathbb{C}\{x_1, \ldots, x_n\}$ such that $(\varphi(f)) = (g)$, that is, there exists a unit $u$ such that $\varphi(f) = ug$. Thus $f$ and $g$ are contact equivalent exactly if the $\mathbb{C}$–algebras $\mathbb{C}\{x_1, \ldots, x_n\}/(f)$ and $\mathbb{C}\{x_1, \ldots, x_n\}/(g)$

are isomorphic. This is exactly the same as saying that the germs of the analytic hypersurfaces defined by $f$ and $g$ are isomorphic.

**Definition 9.2.** Let $f \in \mathfrak{m} \subset \mathbb{C}\{x_1, \ldots, x_n\}$. Then $f$ is called *$k$–determined* if all $g \in \mathbb{C}\{x_1, \ldots, x_n\}$ with $f - g \in \mathfrak{m}^{k+1}$ are right equivalent to $f$. If $f$ is $k$–determined for some $k \in \mathbb{N}$, then $f$ is called *finitely determined.* In particular, a $k$–determined $f$ is right equivalent to a polynomial.

The fact that functions which have an isolated singularity are finitely determined follows quite easily from Newton's Lemma **??**, as we will show now.

**Theorem 9.3.** *Let $f \in \mathfrak{m}^2 \subset \mathbb{C}\{x_1, \ldots, x_n\}$. Suppose $\mathfrak{m}^{k+1} \subset \mathfrak{m}J(f)^2$. Then $f$ is $k$–determined. In particular, functions with isolated singularities are finitely determined.*

*Proof.* Let $g \in \mathfrak{m}$ such that $f - g \in \mathfrak{m}^{k+1}$. Consider the following system of equations, which we want to solve for $y_1, \ldots, y_n \in \mathbb{C}\{x_1, \ldots, x_n\}$:

$$F(x, y) := f(y) - g(x) = 0.$$

We start with the initial approximation $\bar{y}_i(x) = x_i$. Obviously $\frac{\partial F}{\partial y_i}\big(x, \bar{y}(x)\big) = \frac{\partial f}{\partial y_i}\big(\bar{y}(x)\big) = \frac{\partial f}{\partial x_i}$. As by assumption $F(x, \bar{y}) = f(x) - g(x) \in \mathfrak{m}^{k+1} \subset \mathfrak{m}J(f)^2$, we can apply Newton's Lemma 6.2. There exist $y_1, \ldots, y_n \in \mathbb{C}\{x_1, \ldots, x_n\}$ with

- $F(x, y) = 0$,
- $y_i(x) \equiv x_i \mod \mathfrak{m}J(f)$. Note that $\mathfrak{m}J(f) \subset \mathfrak{m}^2$.

This says that the $y$ give a coordinate transformation transforming $f$ into $g$. □

So the fact that germs of holomorphic functions with an isolated singularity are right equivalent to a polynomial is a relatively easy consequence of Newton's Lemma. For our purposes we need, however, a stronger version of the Finite Determinacy Theorem. The statement of the above theorem can be strengthened to the following statement.

**Theorem 9.4** (Finite Determinacy Theorem)**.** *Let $f \in \mathfrak{m}^2 \subset \mathbb{C}\{x_1, \ldots, x_n\}$. Suppose that $\mathfrak{m}^{k+1} \subset \mathfrak{m}^2 J(f)$. Then $f$ is $k$–determined. In particular, if $k \geq \mu(f) + 1$, then $f$ is $k$–determined.*

*Proof.* The proof can be found in [JP00] □

The basic idea of the proof of the Finite Determinacy Theorem is to connect $f$ and $g$ via the path $F_t = f + t \cdot (g - f)$, for $0 \leq t \leq 1$. We need a criterion whether for given $t, s$ the element $F_t$ is right equivalent to $F_s$. The following theorem gives us such a criterion "locally". By this we mean that we give a criterion that $F_t$ is right equivalent to $F_0$, for small $t$.

**Theorem 9.5.** *Let $F \in \mathbb{C}\{x_1, \ldots, x_n, t\}$ and $c \geq 0$ be an integer. The following conditions are equivalent:*

1. $\frac{\partial F}{\partial t} \in (x_1, \ldots, x_n)^c \left(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}\right)$.
2. *There exists a $\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathbb{C}\{x_1, \ldots, x_n, t\}^n$ such that*

*(a)* $\varphi_i(x_1, \ldots, x_n, 0) = x_i$,

*(b)* $\varphi_i - x_i \in (x_1, \ldots, x_n)^c$,

*(c)* $F(\varphi_1, \ldots, \varphi_n, 0) = F(x_1, \ldots, x_n, t)$.

*Proof.* The proof can be found in [JP00] □

**Corollary 9.6.** *Suppose that*

$$\frac{\partial F}{\partial t} \in (x_1, \ldots, x_n) \left( \tfrac{\partial F}{\partial x_1}, \ldots, \tfrac{\partial F}{\partial x_n} \right)$$

*for some $F \in \mathbb{C}\{x_1, \ldots, x_n, t\}$. Then there exists a small neighborhood $U$ of $0$ in $\mathbb{C}$ such that for all fixed $a \in U$ the function $F_a := F(x_1, \ldots, x_n, a)$ is right equivalent to $F_0$.*

*Proof.* Consider the elements $\varphi_1, \ldots, \varphi_n \in \mathbb{C}\{x_1, \ldots, x_n, t\}$ which exist according to 9.5. There exists a small open neighborhood $U$ of 0 in $\mathbb{C}$, so that for all $a \in U$, the $\varphi_i(x, a)$ are convergent and $\det \left( \frac{\partial \varphi_i(x,a)}{\partial x_j}(0) \right) \neq 0$. Thus for fixed $a \in U$, $(\varphi_1, \ldots, \varphi_n)$ is an automorphism of $\mathbb{C}\{x_1, \ldots, x_n\}$. Moreover, formula (c) in Theorem 9.5 says that it transforms $F(x_1, \ldots, x_n, 0)$ into $F(x_1, \ldots, x_n, a)$. □

Theorem 9.5 can be generalized as follows:

**Theorem 9.7** (Characterization of Local Analytic Triviality)**.** *Let $c \geq 0$ be an integer and $f \in \mathbb{C}\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$. The following conditions are equivalent:*

1. $\frac{\partial f}{\partial y_i} \in (x_1, \ldots, x_n)^c \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right) + (f)$ *for* $i = 1, \ldots, m$.
2. *There exist $\varphi_1, \ldots, \varphi_n, u \in \mathbb{C}\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ such that*
   - $u(x_1, \ldots, x_n, 0, \ldots, 0) = 1$
   - $\varphi_i(x_1, \ldots, x_n, 0, \ldots, 0) = x_i$
   - $\varphi_i - x_i \in (x_1, \ldots, x_n)^c$
   - $f(x_1, \ldots, x_n, y_1, \ldots, y_m) = u \cdot f(\varphi_1, \ldots, \varphi_n, 0, \ldots, 0)$.

   *If moreover $\frac{\partial f}{\partial y_i} \in (x_1, \ldots, x_n)^c \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$ for all $i$, then we can choose $u = 1$.*

We now come to the Theorem of Mather and Yau.

**Theorem 9.8** (Mather-Yau)**.** *Let $f$ and $g \in \mathfrak{m} = (x_1, \ldots, x_n) \subset \mathbb{C}\{x_1, \ldots, x_n\}$. Then the following conditions are equivalent:*

1. $\mathbb{C}\{x_1, \ldots, x_n\}/(f) \cong \mathbb{C}\{x_1, \ldots, x_n\}/(g)$, *that is, $f$ and $g$ are contact equivalent.*
2. $\mathbb{C}\{x_1, \ldots, x_n\}/\big((f) + \mathfrak{m}J(f)\big) \cong \mathbb{C}\{x_1, \ldots, x_n\}/\big((g) + \mathfrak{m}J(g)\big)$ *as $\mathbb{C}$–algebras.*

*If $f$ and $g$ define isolated singularities these conditions are equivalent to*

3. $\mathbb{C}\{x_1, \ldots, x_n\}/\big((f) + J(f)\big) \cong \mathbb{C}\{x_1, \ldots, x_n\}/\big((g) + J(g)\big)$ *as $\mathbb{C}$–algebras.*

*To put it in another way, the function $f$ and $g$ are contact equivalent if and only if the Tjurina algebras of $f$ and $g$ are isomorphic.*

*Proof.* The proof can be found in [JP00]. □

To find the correct theorem for the case of right equivalence, we consider the Milnor algebra $\mathbb{C}\{x_1, \dots, x_n\}/J(f)$ as $\mathbb{C}\{t\}$–algebra with multiplication $t \cdot h := fh$ for $h \in \mathbb{C}\{x_1, \dots, x_n\}/J(f)$.

**Theorem 9.9** (Mather-Yau for Right Equivalence)**.** *Let $f, g \in \mathfrak{m}$ with $\mathfrak{m} = (x_1, \dots, x_n) \subset \mathbb{C}\{x_1, \dots, x_n\}$. Then the following conditions are equivalent:*

*1.* $f \underset{R}{\sim} g$.

*2.* $\mathbb{C}\{x_1, \dots, x_n\}/\mathfrak{m}J(f) \cong \mathbb{C}\{x_1, \dots, x_n\}/\mathfrak{m}J(g)$ *as $\mathbb{C}\{t\}$–algebras.*

*If $f$ and $g$ define isolated singularities these conditions are equivalent to*

*3.* $\mathbb{C}\{x_1, \dots, x_n\}/J(f) \cong \mathbb{C}\{x_1, \dots, x_n\}/J(g)$ *as $\mathbb{C}\{t\}$–algebras.*

*Proof.* The proof can be found in [JP00]. □

## 9.2 The $A$-$D$-$E$–singularities are simple.

We consider the automorphism group $\mathscr{G}$ of $\mathcal{O}_n = \mathbb{C}\{x_1, \dots, x_n\}$. Let $f \in \mathbb{C}\{x_1, \dots, x_n\}$. The *right equivalence class* of $f$ is exactly the orbit of $f$ under the following group action.[2]

$$\mathscr{G} \times \mathcal{O}_n \longrightarrow \mathcal{O}_n; \quad (G, f) \mapsto G(f).$$

We denote the orbit of $f$ by $\mathscr{G} \cdot f := \{G(f) : G \in \mathscr{G}\}$[3]. The problem with the study of such orbits is, that $\mathcal{O}_n$ as $\mathbb{C}$–vector space is infinite-dimensional! In case of isolated singularities, this is not a severe problem, as we can use the Finite Determinacy Theorem, which allows us to reduce the study of this problem to the finite-dimensional case.

**Definition 9.10.** Let coordinates $x_1, \dots, x_n$ on $(\mathbb{C}^n, 0)$ be given, and let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_n$. Let $k \geq 1$ be a natural number.

1. We define the *$k$–jet space* by $J_k := \mathcal{O}_n/\mathfrak{m}^{k+1}$. Each element in $J_k$ has a representative of type $\sum_{|\nu| \leq k} a_\nu x^\nu$. Note that $J_k$ is a finite-dimensional $\mathbb{C}$–vector space.

2. Let $f = \sum_\nu a_\nu x^\nu \in \mathcal{O}_n$ be given. The *$k - jet$* of $f$ is defined by $j^k f = \sum_{|\nu| \leq k} a_\nu x^\nu$.

3. We can view $\mathscr{G}$ as a subset of $\oplus_{i=1}^n \mathfrak{m}$. Indeed if $G \in \mathscr{G}$ then put $g_i := G(x_i)$. Then $G$ is uniquely determined by the element $(g_1, \dots, g_n)$. We now define

$$\mathscr{G}_k := \{(g_1 \mod \mathfrak{m}^{k+1}, \dots, g_n \mod \mathfrak{m}^{k+1}) : (g_1, \dots, g_n) \in \mathscr{G}\}.$$

[2] Group action means that $(G_1 \cdot G_2)(f) = G_1(G_2(f))$.

[3] In case of contact equivalence, we have to use the semi-direct product $\mathscr{G} \rtimes \mathcal{O}_n^*$ of $\mathscr{G}$ with $\mathcal{O}_n^*$. Here $\mathcal{O}_n^*$ is the group of units in $\mathcal{O}_n = \mathbb{C}\{x_1, \dots, x_n\}$. As a set the semi-direct product is just the Cartesian product, but the multiplication is given by $(\varphi_1, w_1) \cdot (\varphi_2, w_2) := (\varphi_1\varphi_2, w_1 \cdot \varphi_1(w_2))$. One needs this strange looking product so that the action of $\mathscr{G} \rtimes \mathcal{O}_n^*$ on $\mathbb{C}\{x_1, \dots, x_n\}$ sending $((\varphi, u), f)$ to $u \cdot \varphi(f)$ is a group action.

4. We get group actions
$$\mathscr{G}_k \times J_k \longrightarrow J_k.$$
induced by those of $\mathscr{G}$. Here $G \cdot f := j^k(G(f))$. One immediately checks that this is a group action and is well defined.

**Lemma 9.11.** *Let coordinates $x_1, \ldots, x_n$ on $(\mathbb{C}^n, 0)$ be given. Consider $f \in \mathcal{O}_n$, and $k \in \mathbb{N}$ such that $f$ is $k$–determined. Consider the group operation of $\mathscr{G}_k$ on $J_k$, and let $g \in \mathcal{O}_n$ be given. Then*
$$j^k g \in \mathscr{G}_k(j^k f) \iff g \text{ is right equivalent to } f.$$

*Proof.* The implication $\Longleftarrow$ is obvious. Suppose on the other hand that $j^k g \in \mathscr{G}_k(j^k f)$. Then there exists a $G \in \mathscr{G}$ such that $f \equiv G(g) \mod \mathfrak{m}^{k+1}$. Because $f$ is $k$–determined, and $f$ and $G(g)$ have the same $k$–jet, we have that $f$ and $G(g)$ are right equivalent. By transitivity, $f$ and $g$ are right equivalent. □

**Definition 9.12.** On all jet spaces $J_k$, which are as vector spaces isomorphic to some $\mathbb{C}^N$, we take the usual (Euclidean) $\mathbb{C}$–topology. Consider the projection maps
$$\pi_k : \mathcal{O}_n = \mathbb{C}\{x_1, \ldots, x_n\} \longrightarrow J_k = \mathbb{C}\{x_1, \ldots, x_n\}/\mathfrak{m}^{k+1}.$$
We define a *topology* on $\mathbb{C}\{x_1, \ldots, x_n\}$ as the coarsest topology such that all $\pi_k$ are continuous.

Thus a basis for the topology for $\mathcal{O}_n$ is given by $\pi_k^{-1}(V)$ for $V \subset J_k$. Informally speaking, a power series is near zero when "some of its coefficients are small", and the coefficients of monomials which are not small are in a high power of the maximal ideal $\mathfrak{m}$. On $\mathscr{G}$ we similarly define a topology. It is the topology induced by the product topology of $\mathfrak{m}^n$. The action of $\mathscr{G}$ on $\mathcal{O}_n$ is continuous. We omit the boring proof, as we do not need the result.

**Definition 9.13.** We consider the action of $\mathscr{G}$ on $\mathcal{O}_n$. Let $f \in \mathfrak{m} \subset \mathcal{O}_n$. Then $f$ is called *simple*, if there exists an open neighborhood $U$ of $f$, such that the number of orbits which intersect $U$ is finite.

**Theorem 9.14.** *The tangent space to the orbit of $f \in J_k$ under the action of $\mathscr{G}_k$ is equal to $\frac{\mathfrak{m}J(f)+\mathfrak{m}^{k+1}}{\mathfrak{m}^{k+1}}$.*

*Proof.* As $\mathscr{G}_k$ can be viewed as an open subset of $\oplus_{i=1}^n \mathfrak{m}/\mathfrak{m}^{k+1}$, it follows that the tangent space of $\mathscr{G}_k$ is equal to the tangent space of $\oplus_{i=1}^n \mathfrak{m}/\mathfrak{m}^{k+1}$. The differential of the orbit map
$$\mathscr{G}_k \longrightarrow J_k, \quad h \mapsto f \circ h = h(f),$$
at the identity is given by $(h_1, \ldots, h_n) \in \oplus_{i=1}^n \mathfrak{m}/\mathfrak{m}^{k+1} \mapsto h_1 \frac{\partial f}{\partial x_1} + \ldots + h_n \frac{\partial f}{\partial x_n}$, because $f(x_1 + \varepsilon h_1, \ldots, x_n \varepsilon h_n) - f = \varepsilon\big(h_1 \frac{\partial f}{\partial x_1} + \ldots + h_n \frac{\partial f}{\partial x_n}\big)$, as a simple calculation shows. □

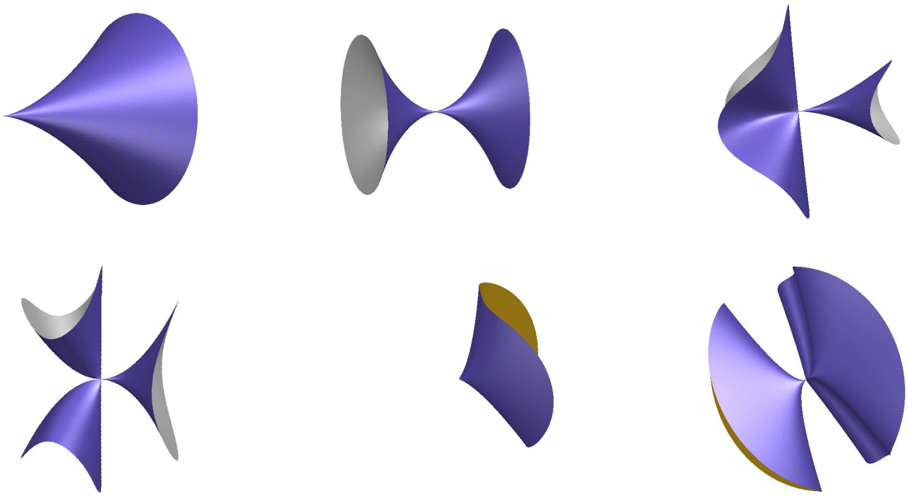
*Remark* 9.15. There is a different way to phrase the definition of simple. Namely, consider families
$$F(x,t) \in \mathbb{C}\{x_1, \ldots, x_n, t\}$$
such that $F(x,t) \in (x_1, \ldots, x_n)$, and $F(x,0) = f$. Families with this property are also called deformations. Then $f$ is called simple, if there exist finitely many

right equivalence classes $G_1, \ldots, G_s$ such that for all families $F(x,t)$ as above and $t$ small and fixed the germ $F_t \in \mathbb{C}\{x_1, \ldots, x_n\}$ is right equivalent to one of the germs in $G_1, \ldots, G_s$.

**Theorem 9.16** (Classification of Simple Singularities)**.** *Suppose that $f \in \mathcal{O}_n$ is simple. Then $f$ is right equivalent to one of the singularities in the following list.*

- $A_k: \; f(x_1, \ldots, x_n) = x_1^{k+1} + x_2^2 + \ldots + x_n^2, \quad k \geq 0,$ [4]
- $D_k: \; f(x_1, \ldots, x_n) = x_1 x_2^2 + x_1^{k-1} + x_3^2 + \ldots + x_n^2, \quad k \geq 4,$
- $E_6: \; f(x_1, \ldots, x_n) = x_1^3 + x_2^4 + x_3^2 + \ldots + x_n^2,$
- $E_7: \; f(x_1, \ldots, x_n) = x_1^3 + x_1 x_2^3 + x_3^2 + \ldots + x_n^2,$
- $E_8: \; f(x_1, \ldots, x_n) = x_1^3 + x_2^5 + x_3^2 + \ldots + x_n^2.$

We give real pictures in dimension two of $A_2$, $A_3$, $D_5$, $D_6$, $E_6$ and $E_7$.



In this section we will show that the $A$-$D$-$E$–singularities of Arnold's list are simple. In the next section we will show that there are no other simple singularities.

We first define the corank of a function. It is the coarsest invariant of right equivalence classes.

**Definition 9.17.** Let $f \in \mathfrak{m}^2 \subset \mathbb{C}\{x_1, \ldots, x_n\}$. The *Hesse matrix* of $f$ is defined by

$$H(f) = \left( \frac{\partial^2 f}{\partial x_i \partial x_j}(0) \right)_{1 \leq i,j \leq n}.$$

The *corank* of $f$ is defined as $\mathrm{Corank}(f) := n - \mathrm{rank}(H(f))$.

The following easy lemma is left as an exercise.

[4]Note that $A_0$ is the germ of a smooth map.

**Lemma 9.18.** *Let $f$ be right equivalent to $g$. Then* $\operatorname{Corank}(f) = \operatorname{Corank}(g)$. *Moreover, let $F_t$ be a holomorphic family of functions with $F_t \in \mathfrak{m}^2$ for all $t$. Then for all $t$ small* $\operatorname{Corank}(F_t) \leq \operatorname{Corank}(F_0)$.

We now state the Splitting Lemma, which is a generalization of the Morse Lemma.

**Lemma 9.19** (Splitting Lemma)**.**

1. *Let $f \in \mathbb{C}\{x_1, \ldots, x_n\}$, with $f \in \mathfrak{m}^2$ and* $\operatorname{Corank}(f)$ *equal to $s$. Suppose that $f$ is finitely determined. Then $f$ is right equivalent to an element of type*

$$g(x_1, \ldots, x_s) + x_{s+1}^2 + \ldots + x_n^2.$$

   *where $g \in \mathbb{C}\{x_1, \ldots, x_s\}$ and $g \in \mathfrak{m}^3$.*

2. *Let $g_1$ and $g_2 \in \mathfrak{m}^3 \subset \mathbb{C}\{x_1, \ldots, x_s\}$ both have isolated singularities. Then $g_1(x_1, \ldots, x_s) + x_{s+1}^2 + \ldots + x_n^2$ is right equivalent to $g_2(x_1, \ldots, x_s) + x_{s+1}^2 + \ldots + x_n^2$ if and only if $g_1$ is right equivalent to $g_2$.*

3. *Let $f \in \mathbb{C}\{x_1, \ldots, x_s\}$ have an isolated singularity. Then $f(x_1, \ldots, x_s) + x_{s+1}^2 + \ldots + x_n^2$ is simple if and only if $f$ is simple.*

*Proof.* The proof can be found in [JP00]. □

**Corollary 9.20.** *Let $f \in \mathfrak{m}^2$ with $\mu(f) < \infty$, and* $\operatorname{Corank}(f) = 1$. *Then there exists a $k \geq 2$ such that $f$ is right equivalent to $A_k$, that is, $x_1^{k+1} + x_2^2 + \ldots + x_n^2$. Moreover, the $A_k$ singularities are simple.*

*Proof.* By the Splitting Lemma, we only have to consider a function in one variable $f(x) \in \mathbb{C}\{x\}$. If $f(x) = 0$ then $\mu(f) = \infty$. As this is not the case, there exists a unique $k$ with $f(x) = x^{k+1}u$, where $u$ is a unit in $\mathbb{C}\{x\}$. By the Implicit Function Theorem, the function $\sqrt[k+1]{u}$ exists, is holomorphic and a unit in $\mathbb{C}\{x\}$. We do the coordinate change $x' = \sqrt[k+1]{u}x$. In this coordinate $f(x') = x'^{k+1}$. This proves that $f$ is right equivalent to $A_k$. If $F_t$ is a deformation of $f$, then $\mu(F_t) \leq \mu(f)$ for all $t$ small. It follows that an $A_k$ singularity only deforms into an $A_l$ singularity with $l \leq k$. In particular, the $A_k$ singularities are simple. □

Now we turn our attention to functions of corank two. By the Splitting Lemma, we only have to consider $f(x, y) \in (x, y)^3 \subset \mathbb{C}\{x, y\}$.

**Proposition 9.21.** *Let $f \in C\{x, y\}$ and $f \in \mathfrak{m}^3$. After a linear coordinate change we may assume that $j^3 f$ is of one of the following types.*

1. *$j^3 f = xy(x + y)$, the zero set of $j^3 f$ consists of three different lines.*
2. *$j^3 f = x^2 y$, two lines coincide.*
3. *$j^3 f = x^3$, the three lines coincide.*
4. *$j^3 f = 0$.*

*Suppose $F_t$ is a holomorphic family with $F_0 = f$ and $F_t \in \mathfrak{m}^2$ for all small $t$. Then for all $t \neq 0$ and small we have for the types (1), (2), resp (3):*

(1) *Either* $\operatorname{Corank}(F_t) = 0$ *or* $1$ *or $j^3 F_t = 0$ consists of three different lines.*

*(2) Either* Corank$(F_t) = 0$ *or* $1$ *or* $j^3F_t = 0$ *consists of three different lines, or two lines coincide.*

*(3) Either* Corank$(F_t) = 0$ *or* $1$ *or* $j^3F_t = 0$ *consists of three different lines, or two lines coincide, or the three lines coincide.*

*Proof.* Write

$$j^3f = ax^3 + bx^2y + cxy^2 + dy^3.$$

*Case 1.* $a = d = 0$. Then $j^3f = xy(bx + cy)$.

(i) If $b = c = 0$ we get *4*.

(ii) Suppose $b = 0$ or $c = 0$, but not both. Without loss of generality, let $c = 0$. Then $j^3f = xy(bx)$. As $b \neq 0$, we can do the coordinate change $y' = by$. Then $j^3f = x^2y'$ which is *2*.

(iii) Both $b$ and $c$ are nonzero. After the coordinate change $x' = bx,\ y' = cy$ we may assume that $b = c$. By putting $x' = \sqrt[3]{b}\cdot x,\ y' = \sqrt[3]{b}\cdot y$ we get $j^3f = x'y'(x' + y')$ which is *1*.

*Case 2.* Suppose $a \neq 0$ or $d \neq 0$. Without loss of generality, we may assume that $a \neq 0$ and, after the coordinate change $x' = \sqrt[3]{a}\cdot x$, even $a = 1$. We factorize $j^3f$:

$$j^3f = (x - \alpha y)(x - \beta y)(x - \gamma y).$$

We do the coordinate change $x' = x - \alpha y$, so that we may assume $\alpha = 0$. Hence

$$j^3f = x\cdot(x - \beta y)(x - \gamma y).$$

(i) If $\beta = \gamma = 0$ we get *3*.

(ii) Suppose that $\beta \neq 0$ or $\gamma \neq 0$. Without loss of generality $\beta \neq 0$. Take $y' = x - \beta y$. Then $j^3f = x'y'(x' - \delta y')$ for some $\delta$, and we are back in *Case 1*.

This proves the first part of the proposition. The second part of the proposition follows from the fact that the corank can only drop, see 9.18, and the fact that the zeros of the polynomial $j^3(F_t(x,1))$ depend continuously on $t$. □

**Proposition 9.22.** *Let* $f \in \mathbb{C}\{x,y\}$, $\mu(f) < \infty$.

1. *Suppose* $j^3f = xy(x + y)$. *Then* $f$ *is right equivalent to* $D_4$.
2. *Suppose* $j^3f = x^2y$. *Then there exists a* $k \geq 5$ *such that* $f$ *is right equivalent to* $D_k$.
3. *The germs* $D_k$ *for* $k \geq 4$ *are simple.*

*Proof.* 1. Suppose that $j^3f = xy(x + y)$. It is an exercise to show that from the Finite Determinacy Theorem it follows that $xy(x+y)$ is 3–determined. It follows without difficulty that $f$ is right equivalent to $D_4$.

2. Suppose $j^3 f = x^2 y$. As we suppose $\mu(f) < \infty$, the element $f$ is $k$–determined for some $k$. Suppose that we have an $s \geq 4$, such that
$$j^s f = x^2 y + a y^s + b x y^{s-1} + x^2 \varphi,$$
for a suitable $\varphi \in \mathfrak{m}^{s-2}$, and $a, b \in \mathbb{C}$. This we certainly have for $s = 4$. We do the coordinate transformation $x' = x + \frac{1}{2} b y^{s-2}$, $y' = y + \varphi$. Then a simple calculation shows that $j^s f = {x'}^2 y' + a {y'}^s$. Renaming $x'$ to $x$ and $y'$ to $y$ we may therefore suppose that $j^s f = x^2 y + a y^s$. There are two cases to consider.

   *Case 1.* We have $a \neq 0$. Do the coordinate change $y' = \sqrt[s]{a} \cdot y$, $x' = \frac{1}{2\sqrt[s]{a}} x$. Then $j^s f = x^2 y + y^s$. As $x^2 y + y^s$ is $s$–determined (exercise), it follows that $f$ is a $D_{s+1}$ singularity.

   *Case 2.* We have $a = 0$. In this case we write down $j^{s+1} f$. Iterate the above coordinate changes until we come in Case 1, or until $s \geq k$. We will show that $s \geq k$ leads to a contradiction. As $f$ is $k$–determined it follows that $f$ is right equivalent to $j^s f = x^2 y + a y^s$. If $a = 0$, then $f$ does not have an isolated singularity, contrary to our assumption.

3. Consider $D_4$, and a holomorphic family $F_t$ with $F_0 = f = xy(x+y)$. We have that $\mu(F_t) < \infty$. If for fixed $t$ small, the corank of $F_t$ is zero, then $F_t$ has an $A_1$–singularity by the Morse Lemma. If the corank is one, then $F_t$ is right equivalent to $A_k$ for some $k$ by 9.20. As $\mu(F_t) \leq \mu(D_4)$, it follows that $k \leq 4$.[5] Hence $D_4$ can only deform in finitely many germs. If $\mathrm{Corank}(F_t) = 2$, it follows that $j^3 F_t = 0$ consists of three different lines by 9.21. Hence there exist coordinates such that $j^3 F_t = xy(x+y)$ for fixed $t$. Hence, by what we just proved, $j^3 F_t$ is right equivalent to $D_4$. This shows that $D_4$ can deform only in finitely many germs.

   The proof that the $D_k$ for $k \geq 5$ are simple, is similar.

□

So in view of Proposition 9.21, the case to consider now are functions $f \in \mathbb{C}\{x, y\}$ whose 3–jet is $x^3$.

**Proposition 9.23.** *Let $f \in \mathbb{C}\{x, y\}$ with $\mu(f) < \infty$. Suppose $j^3 f = x^3$. Then either*

1. *$f$ is right equivalent to $E_6$, $E_7$ or $E_8$ or*
2. *$f$ is right equivalent to a function which lies in the ideal $(x, y^2)^3 = (x^3, x^2 y^2, x y^4, y^6)$.*

*Moreover, $E_6$, $E_7$ and $E_8$ are simple.*

*Proof.* Suppose $j^3 f = x^3$. We write down the four jet.
$$j^4 f = x^3 + a y^4 + b x y^3 + x^2 \varphi,$$
with $a, b \in \mathbb{C}$ and $\varphi \in \mathfrak{m}^2$. We do the coordinate change $x' = x + \frac{1}{3}\varphi$. Hence we may assume
$$j^4 f = x^3 + a y^4 + b x y^3.$$

[5] It is an exercise that even $k \leq 3$.

*Case 1.* Suppose $a \neq 0$. Then by doing the coordinate change $y' = \sqrt[4]{a}y$ we may assume $a = 1$. After doing the coordinate change $y' = y + \frac{1}{4}bx$ we get $j^4 f = x^3 + y^4 + x^2\psi$, with $\psi \in \mathfrak{m}^2$. By the coordinate change $x' = x + \frac{1}{3}\psi$, we can get rid of $\psi$ to obtain $j^4 f = x^3 + y^4$. As $x^3 + y^4$ is 4–determined it follows that $f$ is right equivalent to $E_6$. Redoing this proof for a holomorphic family $F_t$ with $F_0 = x^3 + y^4$ shows that $E_6$ can only deform into singularities of type $A_k$, $D_k$ or $E_6$. As $\mu(F_t) \leq \mu(f)$ it follows that $E_6$ is simple.

*Case 2.* Suppose $a = 0$, and $b \neq 0$. We may assume then that $b = 1$, by the coordinate change $y' = \sqrt[3]{b}y$. From the Finite Determinacy Theorem it follows that $f$ is 5–determined, but it does not follow from this theorem that $f$ is 4–determined. Indeed, one calculates that $(xy^4, x^2y^3, x^3y^2, x^4) \subset \mathfrak{m}^2 J(f)$, but $y^5 \notin \mathfrak{m}^2 J(f)$. But in fact $f$ is 4–determined. To prove this, we write down the 5–jet of $f$

$$j^5 f = x^3 + xy^3 + cy^5 + x\varphi,$$

with $c \in \mathbb{C}$ and $\varphi \in \mathfrak{m}^4$. We do the coordinate change $x' = x + cy^2$, and get

$$j^5 f = {x'}^3 + x'y^3 - 3c{x'}^2y^2 + x'\varphi',$$

with $\varphi' \in \mathfrak{m}^4$. Now we do the coordinate change $y' = y - cx'$. Then

$$j^5 f = {x'}^3 + x'{y'}^3 - 3c{x'}^3y' + \psi,$$

with $\psi \in (x'{y'}^4, {x'}^2{y'}^3, {x'}^3{y'}^2, {x'}^4) \subset \mathfrak{m}^2 J(f)$. Replacing $x'$ by $x'\sqrt[3]{1 - 3cy'}$, and renaming the coordinates we get

$$j^5 f = x^3 + xy^3 + \psi,$$

with $\psi \in (xy^4, x^2y^3, x^3y^2, x^4) \subset \mathfrak{m}^2 J(x^3 + xy^3)$. It follows that $f$ is right equivalent to $x^3 + xy^3$, that is, we have an $E_7$–singularity. Similarly, one shows that $E_7$ is simple, and this proof is left to the reader.

*Case 3.* The final case to consider is $a = b = 0$, that is, $j^4 f = x^3$. So we consider the 5–jet:

$$j^5 f = x^3 + cy^5 + dxy^4 + x^2\varphi,$$

with $c, d \in \mathbb{C}$ and $\varphi \in \mathfrak{m}^3$. Again we can get rid of the $x^2\varphi$ term. The two remaining cases are:

*Case 3a.* Suppose $c \neq 0$. We can after the coordinate change $y' = \sqrt[5]{c}y$ get that $c = 1$. Now put $y' = y + \frac{1}{5}dx$. Then $j^5 f = x^3 + {y'}^5 + x^2\psi$, with $\psi \in \mathfrak{m}^3$. By doing the coordinate change $x' = x + \frac{1}{3}\psi$, we can attain $\psi = 0$. Hence $j^5 f = {x'}^3 + {y'}^5$. By the Finite Determinacy Theorem, it follows that $f$ is right equivalent to an $E_8$–singularity. Similarly, it is proved that $E_8$ is simple.

*Case 3b.* Consider the case $c = 0$. Then $f \in (x^3, xy^4) + (x, y)^6 \subset (x, y^2)^3$, as was to be proved. □

We now succeeded in our goal of proving that the $A$-$D$-$E$-singularities are simple. In the next section, we will show that these are the only ones. Up to now, we already observed the following

*Remark* 9.24. Suppose $f \in \mathcal{O}_n$ is not simple. Then either

1. $f$ has a nonisolated singularity, or

2. Corank($f$) $\geq 3$, or

3. Corank($f$) $= 2$, and we may suppose $f \in \mathbb{C}\{x, y\}$ with $f \in \mathfrak{m}^4$, or

4. Corank($f$) $= 2$, and we may suppose $f \in \mathbb{C}\{x, y\}$ with $f \in (x, y^2)^3$.

*Remark* 9.25. Let $f \in \mathfrak{m}^2 \subset \mathcal{O}_n$ have an isolated singularity, and suppose $k$ satisfies $\mathfrak{m}^{k+1} \subset \mathfrak{m}J(f)$. Then the codimension of the orbit of $f$ under $\mathscr{G}_k$ in $J_k$ is equal to $n + \mu(f)$.

*Proof.* Indeed, applying the previous theorem shows that this codimension is equal to the vector space dimension of $\mathcal{O}_n/\mathfrak{m}J(f)$. Now we have the exact sequence

$$0 \longrightarrow J(f)/\mathfrak{m}J(f) \longrightarrow \mathcal{O}_n/\mathfrak{m}J(f) \longrightarrow \mathcal{O}_n/J(f) \longrightarrow 0. \tag{3}$$

The vector space dimension of $J(f)/\mathfrak{m}J(f)$ is by Nakayama's Lemma the minimal number of generators of $J(f)$. As $f$ has an isolated singularity, $V(J(f)) = \{0\}$, or better, $J(f)$ is an $\mathfrak{m}$–primary ideal. As the dimension of $(\mathbb{C}^n, 0)$ is $n$, it follows that number of generators of $J(f)$ is at least $n$. But then the number of generators of $J(f)$ is exactly $n$, because $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ are $n$ generators of $J(f)$. The statement therefore follows from the exact sequence (3). ☐

**Theorem 9.26.** *Let $f \in \mathfrak{m}^2 \subset \mathbb{C}\{x_1, \ldots, x_n\}$, with finite Milnor number. Let $g \notin \mathfrak{m} \cdot J(f)$. Then there exist only finitely many $t \in \mathbb{C}$ such that $f + t \cdot g$ is right equivalent to $f$.*

*Proof.* Choose $k$ such that $\mathfrak{m}^{k+1} \subset \mathfrak{m}J(f)$. Hence $g$ does not lie in the tangent space to the orbit through $f$ in $J_k$ under the action of $\mathscr{G}_k$. This implies that the line $f + tg$ in the jet space $J_k$ does not lie entirely in the orbit of $f$. Therefore, the line $f + tg$ in $J_k$ intersects the orbit in only finitely many points, as the intersection of the orbit with the line is a constructible subspace of the line. This is exactly the statement of the theorem. ☐

**Proposition 9.27.** *Let $f \in \mathbb{C}\{x_1, \ldots, x_n\}$ be simple. Then $f$ has an isolated singularity.*

*Proof.* Suppose not. Then the Milnor number $\mu(f)$ is not finite. As $\mathfrak{m}J(f) \subset J(f)$, it follows that in particular $\dim_{\mathbb{C}}\big(\mathbb{C}\{x\}/\mathfrak{m}J(f)\big) = \infty$, that is, for all $k > 0$, there exists a $g_k \in \mathfrak{m}^k \setminus \mathfrak{m}J(f) + \mathfrak{m}^{k+1}$, as otherwise $\mathfrak{m}^k \subset \mathfrak{m}J(f)$ by Nakayama. Hence for all sufficiently small $t \neq 0$ the germ $f_k := f + tg_k$ is not right equivalent to $f$. This is because otherwise, $g_k$ had to be in the tangent space of the orbit of $f$ under $\mathscr{G}_k$, but by assumption it is not. Thus even $f_k$ and $f$ are not in the same orbit under $\mathscr{G}_k$. Moreover for $k < s$ also $f_k$ is not right equivalent to $f_s$. Indeed otherwise the classes of $f_s$ and $f_k$ in the jet space $J_k$ would be $\mathscr{G}_k$ equivalent. But they are not, as in $J_k$ the class of $f_s$ is equal to the class of $f$. Hence there are infinitely many nonequivalent function germs in any small neighborhood of $f$, and therefore $f$ is not simple. ☐

We now can reach our goal of proving that there are no other simple singularities except for the *A*-*D*-*E*–singularities.

**Theorem 9.28.** *Germs $f \in \mathcal{O}_n$ with one of the following properties are not simple.*

1. *Nonisolated singularities.*
2. $n = 2$ *and* $f \in \mathfrak{m}^4$.
3. $n = 2$ *and* $f \in (x, y^2)^3$.
4. $f$ *with* $\mathrm{Corank}(f) \geq 3$.

*In particular, Arnold's Classification Theorem of Simple Singularities 9.16 holds.*

*Proof.* It was shown in the previous section, see 9.24 that either $f$ is right equivalent to an $A$-$D$-$E$–singularity, or right equivalent to a germ with one of the above listed properties. So if we show that singularities with one of the above properties are not simple, Arnold's Classification Theorem follows.

1. The fact that nonisolated singularities are not simple was proved in 9.27.
2. We calculate modulo $\mathfrak{m}^5$. If $f$ is simple, then $j^4 f \in J_4$ has the property that in a neighborhood $U$ of $j^4 f$ there are only finitely many orbits under the action of $\mathscr{G}_4$. We look at the tangent space at the orbit. As $f \in \mathfrak{m}^4$, the tangent space $\mathfrak{m}J(j^4 f) + \mathfrak{m}^5/\mathfrak{m}^5$ is a subset of $\mathfrak{m}^4/\mathfrak{m}^5$. This tangent space has at most dimension 4, as $\mathfrak{m}J(j^4 f)$ has four generators which all lie in $\mathfrak{m}^4$. It follows that the orbit of $j^4 f$ has dimension at most 4. This holds for all $f \in \mathfrak{m}^4$. Note that $\mathfrak{m}^4/\mathfrak{m}^5$ is $\mathscr{G}_4$–stable, and has dimension 5. As a finite union of subvarieties of dimension four never can fill up an open subset of a vector space of dimension 5, it follows that all neighborhoods of $j^4 f$ intersect infinitely many orbits. In particular $f$ is not simple.
3. In this case the argument is somewhat more subtle.We refer to [JP00].
4. The argument is similar to 2., but now we calculate modulo $\mathfrak{m}^4$. We know that $\mathfrak{m}J(f)/\mathfrak{m}^4$ has dimension at most $n^2$, but $\mathfrak{m}^3/\mathfrak{m}^4$ has dimension $\binom{n+2}{3}$, which is bigger.

□

# References


[A68] Artin, M.: *On the solution of analytic equations.* Inventiones Mathematicae 5, 277–291 (1968).

[A69] Artin, M.: *Algebraic approximation of structures over complete local rings.* Publ. Math. IHES 36, 23–58 (1969).

[BS98] Brodmann, M.; Sharp, R. Y.: *Local Cohomology. An Algebraic Introduction with Geometric Applications.* Cambr. Stud. in Advanced Math., No. 60, Cambridge University Press (1998).

[BH93] Bruns, W.; Herzog, J.: *Cohen-Macaulay rings.* Cambridge University Press, Cambridge (1993).

[B65] Buchberger, B.: *Ein Algorithmus zum Auffinden der Basiselemente des Restklassenringes nach einem nulldimensionalen Polynomideal.* Dissertation, Innsbruck, Austria (1965).

[DGPS12] Decker, W.; Greuel, G.-M.; Pfister, G.; Schönemann, H.: SINGULAR *3-1-4 — A computer algebra system for polynomial computations.* http://www.singular.uni-kl.de (2012).

[GP07] Greuel, G.-M.; Pfister, G.: *A* SINGULAR *Introduction to Commutative Algebra.* Second edition, Springer (2007).

[G67] Grothendieck, A.: *Local Cohomology.* Notes by R. Hartshorne, Lecture Notes in Math. vol. 41, Springer (1967).

[H77] Hartshorne, R.: *Algebraic Geometry.* Graduate Texts in Mathematics 52, Springer (1977).

[JP00] de Jong, T.; Pfister, G.: *Local Analytic Geometry.* Vieweg (2000).

[K86] Kunz, E.: *Kähler differentials.* Vieweg, Braunschweig/Wiesbaden (1986).

[M70] Matsumura, H.: *Commutative Algebra.* Benjamin, New York (1970).

[S98] Schenzel, P.: *On The Use of Local Cohomology in Algebra and Geometry.* In: Six Lectures in Commutative Algebra, Proceed. Summer School on Commutative Algebra at Centre de Recerca Matemàtica, (Ed.: J. Elias, J. M. Giral, R. M. Miró-Roig, S. Zarzuela), Progr. Math. 166, Birkhäuser, 241–292 (1998).