EVALUATING CYCLICITY OF CUBIC SYSTEMS WITH ALGORITHMS OF COMPUTATIONAL ALGEBRA

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ABSTRACT. We describe an algorithmic approach to studying limit cycle bifurcations in a neighborhood of an elementary center or focus of a polynomial system. Using it we obtain an upper bound for cyclicity of a family of cubic systems. Then using a theorem by Christopher [3] we study bifurcation of limit cycles from each component of the center variety. We obtain also the sharp bound for the cyclicity of a generic time-reversible cubic system.

1. INTRODUCTION

Consider systems of ordinary differential equations on $\mathbb{R}^2$ of the form

$$\dot{u} = P(u, v), \qquad \dot{v} = Q(u, v) \tag{1}$$

where P and Q are polynomials, $\max\{\deg P, \deg Q\} = n$. We view (1) as defining a family of systems parametrized by the coefficients of P and Q. The parameter space denoted by $\mathcal{E}$ is a Euclidean $(n + 1)(n + 2)$-space, every point E of which corresponds to a system of the form (1). A singular point $(u_0, v_0) \in \mathbb{R}^2$ of a system $E \in \mathcal{E}$ is said to have *cyclicity* k with respect to $\mathcal{E}$ if and only if any sufficiently small perturbation of E in $\mathcal{E}$ has at most k limit cycles in a sufficiently small neighborhood of (u_0, v_0), and k is the smallest number with this property. The problem of the cyclicity of a center or a focus of a system of the form (1), which we always assume to be located at the origin, is known as the *local 16th Hilbert problem* ([8]), based on its connection to Hilbert's still unresolved 16th problem, which in part asks for a bound on the number of limit cycles anywhere in the phase portrait of a system of the form (1) in terms of n alone.

The concept of cyclicity was introduced by Bautin in his seminal paper [1], where he showed that the cyclicity problem in the case of an elementary focus or center can be reduced to the problem of finding a basis for the ideal of focus quantities (the so-called Bautin ideal) in the ring of polynomials in the coefficients of the system.

Bautin's approach is described in details and further developed in [14, 23, 24]. The cyclicity problem for some families of polynomial systems was treated also in [8, 13, 25, 27, 28, 26].

Following Bautin's method the cyclicity problem can be easily solved in the case when the Bautin ideal of the system is a radical one (see e.g. [23, 27, 28]). A method to treat the cyclicity problem with a Bautin ideal which is a non-radical ideal in the polynomial ring of the coefficient of system (1) but still a radical one in a certain coordinate ring has been recently proposed in [16]. In [20] it was generalized to the case when the Bautin ideal is non-radical also in the coordinate ring, but has a primary decomposition

Key words and phrases. Bautin ideal, cyclicity, polynomial system.

of the form $\cap_{i=1}^{s} Q_i$, where $\sqrt{Q_i} = Q_i$ for $i = 1, \ldots, s$, $\sqrt{Q_s} \neq Q_s$ and $\sqrt{Q_s}$ is a maximal ideal.

In the present paper we extend the method to the case when the Bautin ideal is of a general form in the coordinate ring, that is, it is equal to $\cap_{i=1}^{s} Q_i$, where for some $1 \leq k \leq s$ the

$Q_1, \ldots, Q_k$ are radical ideals while $Q_{k+1}, \ldots Q_s$ are not radical. We believe that the described approach can be applied to evaluate the cyclicity of many other systems for which the variety of $\cap_{i=k+1}^{s} Q_i$ is "much less" than the variety of $\cap_{i=1}^{k} Q_i$.

At present two problems of a major interest for the theory of plane polynomial systems of ODEs are the cyclicity problem for the general real cubic system

$$\dot{x} = \lambda x + ix(1 - a_{10}x - a_{01}y - a_{-12}x^{-1}y^2 - a_{20}x^2 - a_{11}xy - a_{02}y^2 - a_{-13}x^{-1}y^3) \tag{2}$$

and the center problem for the associated complex system

$$\begin{aligned} \dot{x} =& x(1 - a_{10}x - a_{01}y - a_{-12}x^{-1}y^2 - a_{20}x^2 - a_{11}xy - a_{02}y^2 - a_{-13}x^{-1}y^3), \\ \dot{y} =& -y(1 - b_{2,-1}x^2y^{-1} - b_{10}x - b_{01}y - b_{3,-1}x^3y^{-1} - b_{20}x^2 - b_{11}xy - b_{02}y^2). \end{aligned} \tag{3}$$

However the study of these systems involves tremendous computations which cannot be completed even with powerful computers, so in recent years many works have been devoted to investigation of different subfamilies of (2) and (3).

In the present paper we study the cyclicity of the polynomial family considered in [19],

$$\dot{x} = \lambda x + i(x - a_{10}x^2 - a_{-12}\bar{x}^2 - a_{11}x^2\bar{x} - a_{-1,3}\bar{x}^3). \tag{4}$$

We first obtain an upper bound for the cyclicity of the system for "almost all" values of parameters. Then using the approach of Christopher [3] we study bifurcations of limit cycles from each component of the center variety. In the last section the cyclicity of generic time-reversible systems in the family (2) is investigated.

2. Preliminaries

We recall briefly the approach of [16, 23]. Any polynomial system with an elementary antisaddle at the origin can be written as one complex differential equation

$$\dot{x} = \lambda x + ix - \sum_{(p,q)\in S} a_{p,q}x^{p+1}\bar{x}^q, \tag{5}$$

where

$$S = \{(p_j, q_j) : p_j + q_j \geq 1, j = 1, \ldots, \ell\} \subset (\{-1\} \cup \mathbb{N}_0) \times \mathbb{N}_0,$$

and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. The equation (5) is embedded in a natural way into the two-dimensional complex system

$$\begin{aligned} \dot{x} &= \lambda x + i(x - \sum_{(p,q)\in S} a_{pq}x^{p+1}y^q) = P(x, y), \\ \dot{y} &= \lambda y - i(y + \sum_{(p,q)\in S} b_{qp}x^q y^{p+1}) = Q(x, y). \end{aligned} \tag{6}$$

In the case of a weak focus or a center, that is when $\lambda = 0$, system (6) is written as

$$\begin{aligned} \dot{x} &= i(x - \sum_{(p,q)\in S} a_{pq} x^{p+1} y^q) = P(x,y), \\ \dot{y} &= -i(y + \sum_{(p,q)\in S} b_{qp} x^q y^{p+1}) = Q(x,y). \end{aligned} \tag{7}$$

We denote by $(a,b) = (a_{p_1,q_1}, \ldots, a_{p_\ell,q_\ell}, b_{q_\ell,p_\ell}, \ldots, b_{q_1,p_1})$ the ordered vector of coefficients and by $E(a,b) = \mathbb{C}^{2\ell}$ (resp., $E(\lambda,(a,b))$) the parameter space of (7) (resp., of (6)), and by $\mathbb{C}[a,b]$ the polynomial ring in the coefficients a_{ij}, b_{ji} of system (7) over the field of the complex numbers. For system (7) one can always find (see, for example, [23]) a function Ψ of the form

$$\Psi(x,y) = xy + \sum_{s=3}^{\infty} \sum_{j=0}^{s} v_{j,s-j} x^j y^{s-j}, \tag{8}$$

where the $v_{j,s-j}$ are polynomials in the coefficients of P and Q, such that

$$\frac{\partial \Psi}{\partial x} P(x,y) + \frac{\partial \Psi}{\partial y} Q(x,y) = g_{11} \cdot (xy)^2 + g_{22} \cdot (xy)^3 + g_{33} \cdot (xy)^4 + \cdots . \tag{9}$$

The g_{kk} are polynomials in the coefficients of (7) called the *focus quantities*. A system of the form (7) on $\mathbb{C}^2$ is said to have a center at the origin if it admits a local first integral of the form (8). That is, system (7) with coefficients $(a^*,b^*) \in E(a,b)$ has a center at the origin if and only if $(a^*,b^*) \in \mathbf{V}(g_{11}, g_{22}, g_{33}, \ldots)$, where here and below we denote by $\mathbf{V}(f_1,\ldots,f_s)$ the variety of the ideal $\langle f_1,\ldots,f_s\rangle$. The ideal

$$\mathcal{B} := \langle g_{kk} : k \in \mathbb{N} \rangle \subset \mathbb{C}[a,b]$$

is called the *Bautin ideal* and its variety $\mathbf{V}(\mathcal{B})$ is called the *center variety* of family (7). We will also use the notation $\mathcal{B}_k$ for the ideal $\langle g_{11}, g_{22}, \ldots, g_{kk}\rangle$.

For parameters (λ, a) let $n_{(\lambda,a),\epsilon}$ denote the number of limit cycles of the corresponding system (5) that lie wholly within an ϵ-neighborhood of the origin. To define more precisely the notion of cyclicity for a singular point we say that the singularity at the origin for system (5) with fixed coefficients $(\lambda^*, a^*) \in E(\lambda, a)$ has *cyclicity c with respect to the space* $E(\lambda,a)$ if there exist positive constants δ_0 and ϵ_0 such that for every pair ϵ and δ satisfying $0 < \epsilon < \epsilon_0$ and $0 < \delta < \delta_0$

$$\max\{n_{(\lambda,a),\epsilon} : |(\lambda,a) - (\lambda^*,a^*)| < \delta\} = c\,.$$

Denote by $\mathcal{G}_{(a^*,b^*)}$ the ring of germs of complex analytic functions at (a^*,b^*). The following statement is a reformulation of Theorem 6.2.9 of [23].

Theorem 1. *Suppose that for $(a^*,b^*) \in E(a,b)$ $\mathcal{B} = \mathcal{B}_m$. Then the cyclicity of the origin of the system of the form* (5) *with parameters* $(0,(a^*,b^*)) \in E(\lambda,(a,b))$ *is at most m.*

3. The center variety of a cubic system

The center problem for the real system (4) has been solved in [19]. However to apply our approach we need to know the center variety of the associated complex system of the type (7) which is written as

$$\begin{aligned} \dot{x} &= i(x - a_{10}x^2 - a_{-12}y^2 - a_{11}x^2y - a_{-1,3}y^3), \\ \dot{y} &= -i(y - b_{2,-1}x^2 - b_{01}y^2 - b_{3,-1}x^3 - b_{11}xy^2). \end{aligned} \tag{10}$$

Thus, first we solve the center problem for (10).

Theorem 2. *For system* (10) $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_5)$ *and the minimal associate primes of* $\mathcal{B}$ *are:*
1) $J_1 = \langle a_{-13}, b_{3,-1}, a_{11}, b_{11}\rangle$,
2) $J_2 = \langle b_{3,-1}, b_{11}, a_{11}, b_{2,-1}\rangle$,
3) $J_3 = \langle a_{-13}, b_{11}, a_{11}, a_{-12}\rangle$,
4) $J_4 = \langle b_{11}^2 - b_{3,-1}a_{-13}, -b_{01}b_{11} + b_{2,-1}a_{-13}, b_{2,-1}b_{11} - b_{01}b_{3,-1}, -a_{-12}b_{11} + a_{10}a_{-13}, a_{10}b_{11} - a_{-12}b_{3,-1}, -a_{-12}b_{2,-1} + a_{10}b_{01}, a_{11} - b_{11}, \rangle$
5) $J_5 = \langle a_{-12}^4 b_{3,-1}^3 - b_{2,-1}^4 a_{-13}^3, -a_{-12}b_{01}b_{3,-1} + a_{10}b_{2,-1}a_{-13}, a_{10}a_{-12}^3 b_{3,-1}^2 - b_{2,-1}^3 b_{01} a_{-13}^2, a_{10}^2 a_{-12}^2 b_{3,-1} - b_{2,-1}^2 b_{01}^2 a_{-13}, a_{10}^3 a_{-12} - b_{2,-1}b_{01}^3, -b_{01}^4 b_{3,-1} + a_{10}^4 a_{-13}, a_{11} - b_{11}\rangle$,
6) $J_6 = \langle a_{11} - b_{11}, b_{01}, a_{10}\rangle$.

Proof. With the algorithm of Table 3.1 and a simple modification of the MATHEMATICAcode of Figures 6.1 and 6.2 of [23] we have computed the first nine focus quantities of (10). The first five form the ideal $\mathcal{B}_5 = \langle g_{11}, \dots, g_{55}\rangle$ [1]. Then, computing with `minAssGTZ` [7] procedure of SINGULAR[12] we obtain the ideals $J_1, \dots, J_6$.

Now we have to show that all systems corresponding to $\mathbf{V}(J_1), \dots, \mathbf{V}(J_6)$ are integrable.

In Case 1 the system is the quadratic system with 3 invariant lines and, as it is well-known, is integrable (see, e.g. [4, 23]).

In Case 2 we are not able to find an invariant curve of the system rather then $y = 0$, so we cannot construct a Darboux integral, but we can show that the corresponding system

$$\dot{x} = x - a_{10}x^2 - a_{-12}y^2 - a_{-13}y^3 = P(x,y), \quad \dot{y} = y(-1 + b_{01}y) = Q(x,y) \tag{11}$$

is integrable. We look for a first integral in the form

$$\Psi(x,y) = \sum_{k=1}^{\infty} f_k(x) y^k. \tag{12}$$

Then the function Ψ should satisfy the partial differential equation $\frac{\partial \Psi}{\partial x}P + \frac{\partial \Psi}{\partial y}Q = 0$. Substituting in this equation the series (12) and equating in the obtained expression the coefficient of the same power of y we find $f_1 = x/(1 - a_{10}x)$ and for each $k = 2, 3, \dots$ we obtain the linear ordinary differential equation

$$(x - a_{10}x^2)f_k'(x) - kf_k(x) + (k-1)b_{01}f_{k-1}(x) - a_{-12}f_{k-2}'(x) - a_{-13}f_{k-3}'(x) = 0, \tag{13}$$

[1] The polynomials are too long, so we do not present them here, however they are available at `www.math.rwth-aachen.de/ Viktor.Levandovskyy/filez/cyclicity/fqcubic.tst`. One can also obtain the polynomials generating $\mathcal{B}_5$ applying to polynomials $g_{11}^{(F)}, \dots g_{55}^{(F)}$ in Section 4 the homomorphism (23).

where it is assumed that $f_{-1} = f_0 = 0$. Let p_s denote a polynomial of degree at most s. Using induction on k we wish to show that

$$f_m = \frac{p_m(x)}{(1 - a_{10}x)^m}. \tag{14}$$

Assuming that for $m < k$ (14) holds we find that a solution to (13) is

$$f_k(x) = \frac{p_k}{(1 - a_{10}x)^k},$$

as required. Thus, the system (11) admits a formal integral of the form (8), which yields also the existence of an analytic integral (8).

Case 3 is dual to Case 2 under the involution $a_{ij} \leftrightarrow b_{ji}$.

In Case 4 the system can be written as

(15)

$$\begin{aligned}\dot{x} =& -(-b_{01}b_{2,-1}x + a_{10}b_{01}b_{2,-1}x^2 + a_{-13}b_{2,-1}^2x^2y + a_{10}b_{01}^2y^2 + a_{-13}b_{01}b_{2,-1}y^3)/(b_{01}b_{2,-1}),\\ \dot{y} =& (b_{01}^2b_{2,-1}x^2 + a_{-13}b_{2,-1}^2x^3 - b_{01}^2y + b_{01}^3y^2 + a_{-13}b_{01}b_{2,-1}xy^2)/b_{01}^2.\end{aligned}$$

To avoid cumbersome expressions involving radicals without loss of generality we assume that $b_{01} = b_{2,-1} = 1$. Then the system has the invariant curves

$f_1 = 1 + ia_{-13}x^2 - iy(-i + a_{10} + a_{-13}y) - x(-i + a_{10} + 2a_{-13}y)$, $f_2 = 1 - ia_{-13}x^2 + iy(i + a_{10} + a_{-13}y) - x(i + a_{10} + 2a_{-13}y)$, which allow us to construct an analytical integrating factor of the Darboux type $\mu = (f_1f_2)^{-1}$. Therefore the system has a center at the origin.

Systems corresponding to Case 5 are time-reversible (by Theorem 6 of [15]) and those corresponding to Case 6 are Hamiltonian with the Hamiltonian $H = -(b_{2,-1}x^3)/3 - (b_{3,-1}x^4)/4 + xy - 1/2b_{11}x^2y^2 - (a_{-12}y^3)/3 - (a_{-13}y^4)/4$. □

4. The cyclicity of system (4)

To resolve the cyclicity problem for system (4) we use the following specific structure of the focus quantities which we briefly describe now. Fix a family (5), hence the index set $S = \{(p_1, q_1), \ldots, (p_\ell, q_\ell)\} \subset (\{-1\} \cup \mathbb{N}_0) \times \mathbb{N}_0$. For $\nu = (\nu_1, \ldots, \nu_{2\ell}) \in \mathbb{N}_0^{2\ell}$ let L be the map from $\mathbb{N}_0^{2\ell}$ to $\mathbb{Z}^2$ defined by

$$L(\nu) = \begin{pmatrix} L^1(\nu) \\ L^2(\nu) \end{pmatrix} = \begin{pmatrix} p_1 \\ q_1 \end{pmatrix}\nu_1 + \cdots + \begin{pmatrix} p_\ell \\ q_\ell \end{pmatrix}\nu_\ell + \begin{pmatrix} q_\ell \\ p_\ell \end{pmatrix}\nu_{\ell+1} + \cdots + \begin{pmatrix} q_1 \\ p_1 \end{pmatrix}\nu_{2\ell}. \tag{16}$$

Define the monoid $\mathcal{M} \subset \mathbb{N}_0^{2\ell}$ by

$$\mathcal{M} = \left\{ \nu \in \mathbb{N}_0^{2\ell} : \text{there exists } \ k \in \mathbb{N} \text{ such that } L(\nu) = \begin{pmatrix} k \\ k \end{pmatrix} \right\}. \tag{17}$$

For $\nu = (\nu_1, \ldots, \nu_{2\ell}) \in \mathbb{N}_0^{2\ell}$ let $[\nu]$ denote the monomial in $\mathbb{C}[a, b]$ given by

$$[\nu] = a_{p_1q_1}^{\nu_1} \cdots a_{p_\ell q_\ell}^{\nu_\ell} b_{q_\ell p_\ell}^{\nu_{\ell+1}} \cdots b_{q_1p_1}^{\nu_{2\ell}}, \tag{18}$$

and $\hat{\nu}$ denote the involution of ν, $\hat{\nu} = (\nu_{2\ell}, \nu_{2\ell-1}, \ldots, \nu_1)$. Let $\{\mu_1, \ldots, \mu_s\}$ be a Hilbert basis of the monoid $\mathcal{M}$. We denote by $\mathbb{C}[\mathcal{M}]$ the polynomial subalgebra $\mathbb{C}[[\mu_1], \ldots, [\mu_s]]$.

It is shown in [23] (similar result has been obtained also in [6, 17]) that the focus quantities of system (7) have the form

$$g_{kk} = \sum_{\nu : L(\nu)=\binom{k}{k}} g_{(\nu)}([\nu]-[\hat{\nu}]), \tag{19}$$

with $ig_{(\nu)} \in \mathbb{Q},\ k = 1, 2, \ldots$. In particular, (19) implies that $g_{kk} \in \mathbb{C}[\mathcal{M}]$ for all $k \in \mathbb{N}$.

The following statement is a simple generalization of Proposition 1 of [20].

Proposition 3. *Let $I = \langle g_1, \ldots, g_t\rangle$ be an ideal in $\mathbb{C}[x_1, \ldots, x_n]$ such that the primary decomposition of I is given as*

$$I = P_1 \cap \cdots \cap P_k \cap Q_1 \cap \cdots \cap Q_m,$$

where $P_s = \sqrt{P_s}$ for $s = 1, \ldots, k$, and $Q_j \neq \sqrt{Q_j}$ for $j = 1, \ldots, m$.

Let $Q = Q_1 \cap \cdots \cap Q_m$ and g be a polynomial vanishing on $\mathbf{V}(I)$. Let $x^* = (x_1^*, \ldots, x_n^*)$ be an arbitrary point of $\mathbf{V}(I) \setminus \mathbf{V}(Q)$. Then in a small neighborhood of x^*

$$g = g_1 f_1 + \cdots + g_t f_t,$$

where $f_1, \ldots, f_t$ are power series convergent at x^*.

Proof. From the condition of the proposition we have that

$$\sqrt{I} = P_1 \cap \cdots \cap P_k \cap \sqrt{Q_1} \cap \cdots \cap \sqrt{Q_m}.$$

Let $g \in \sqrt{I}$. Then $g \in P_1 \cap \cdots \cap P_k$ and $g \in \sqrt{Q}$. For any polynomial $q \in Q$ we have $qg \in P_1 \cap \cdots \cap P_k$ and $qg \in Q$, hence $qg \in I$; in particular there exist $f_1, \ldots, f_t \in \mathbb{C}[x_1, \ldots, x_n]$ such that

$$qg = f_1 g_1 + \cdots + f_t g_t. \tag{20}$$

Since $x^* = (x_1^*, \ldots, x_n^*) \in \mathbf{V}(I) \setminus \mathbf{V}(Q)$, there exists $q \in Q$ such that $q(x^*) \neq 0$. Since such q is invertible in the local ring at x^*, we can write:

$$g = \frac{f_1}{q} g_1 + \cdots + \frac{f_t}{q} g_t. \tag{21}$$

Clearly, for any $l = 1, \ldots, t$ $\frac{f_l}{q}$ can be expressed as a power series in a neighborhood of x^*. □

Following Algorithm 5.1 of [23, p.235] to find the generators of $\mathbb{C}[\mathcal{M}]$ we compute the reduced Gröbner basis of

$$\mathcal{J} = \langle 1 - w\alpha^5, a_{10} - t_1,\ b_{01} - \alpha t_1, a_{-12} - t_2, \alpha^3 b_{2,-1} - t_2, a_{11} - t_3, b_{11} - t_3, a_{-13} - t_4, \alpha^4 b_{3,-1} - t_4\rangle$$

with respect to the lexicographic order with $w \succ \alpha \succ \{t_i\} \succ \{a_{ij}, b_{ji}\}$, and then take from the output list the binomials which do not depend on w, α, t_j. Then the monomials of the binomials together with the "symmetric" binomials $a_{10}b_{01}$, $a_{-12}b_{2,-1}$ and $a_{-13}b_{3,-1}$ generate $\mathbb{C}[\mathcal{M}]$. The polynomials of the output list that do not depend on $w, \alpha, t_1, t_2, t_3, t_4$

are exactly the binomials given in 5) of Theorem 2. Denote the monomials of these binomials by $\mu_1, \ldots, \mu_{14}$, that is,

$$\mu_1 = a_{11},\ \mu_2 = b_{11},\ \mu_3 = (a_{-13}^3 b_{2,-1}^4),\ \mu_4 = a_{-12}^4 b_{3,-1}^3,\ \mu_5 = a_{-13}^2 b_{01} b_{2,-1}^3,$$
$$\mu_6 = a_{10} a_{-12}^3 b_{3,-1}^2,\ \mu_7 = a_{10} a_{-13} b_{2,-1},\ \mu_8 = a_{-12} b_{01} b_{3,-1},\ \mu_9 = a_{-13} b_{01}^2 b_{2,-1}^2,$$
$$\mu_{10} = a_{10}^2 a_{-12}^2 b_{3,-1},\ \mu_{11} = a_{10}^3 a_{-12},\ \mu_{12} = b_{01}^3 b_{2,-1},\ \mu_{13} = a_{10}^4 a_{-13},\ \mu_{14} = b_{01}^4 b_{3,-1}.$$

Thus, the monomials $\mu_1, \ldots, \mu_{14}$ together with the monomials

$$\mu_{15} = a_{-12} b_{2,-1}, \mu_{16} = a_{10} b_{01},\ \mu_{17} = a_{13} b_{3,-1}$$

generate the subalgebra $\mathbb{C}[\mathcal{M}]$ for system (10), that is, for this system $\mathbb{C}[\mathcal{M}] = \mathbb{C}[\mu_1, \ldots, \mu_{17}]$.

Theorem 4. *The center at the origin of system (4), where* $|a_{11}|+|a_{-13}| \neq 0$*, has cyclicity at most 6.*

Proof. By Theorem 1 an upper bound for cyclicity of system (4) with fixed coefficients (a^*) is equal to the number m such that $\mathcal{B} = \mathcal{B}_m$ in $\mathcal{G}_{(a^*,\bar{a}^*)}$, where $\mathcal{G}_{(a^*,\bar{a}^*)}$ is the ring of germs of complex analytic functions at $(a^*, \bar{a}^*)$. Thus in order to prove the theorem it is sufficient to show that for any (a^*) such that $|a_{11}| + |a_{-13}| \neq 0$, and $k > 6$

$$g_{kk} = g_{11} f_1 + g_{22} f_2 + g_{33} f_3 + g_{44} f_4 + g_{55} f_5 + g_{66} f_6 \tag{22}$$

in $\mathcal{G}_{(a^*,\bar{a}^*)}$.

Let $h = (h_1, \ldots, h_{17})$ and denote by J the ideal in $\mathbb{C}[a, b, h]$ defined by $h_j - \mu_j(a, b)$, that is,

$$J = \langle h_j - \mu_j(a, b)\ :\ j = 1, \ldots, 17\rangle.$$

We also define the polynomial mapping

$$F : \mathbb{C}^8 \to \mathbb{C}^{17} : (a, b) \mapsto (h_1, \ldots, h_{17}) = (\mu_1(a, b), \ldots, \mu_{17}(a, b)).$$

F induces the $\mathbb{C}$-algebra homomorphism

$$F_* : \mathbb{C}[h] \to \mathbb{C}[a, b] : \sum c^{(\alpha)} h_1^{\alpha_1} \cdots h_{17}^{\alpha_{17}} \mapsto \sum c^{(\alpha)} \mu_1^{\alpha_1}(a, b) \cdots \mu_{17}^{\alpha_{17}}(a, b). \tag{23}$$

That is, instead of variables (a, b) we introduce new variables $h_1, \ldots, h_{17}$. Computing the normal forms of g_{ii} according to Proposition 7 of §7.3 of [5] we find that the expressions of the preimages of g_{ii} in $\mathbb{C}[h]$ up to a constant factor which are given in the Appendix. Here each focus quantity is reduced in $\mathbb{C}[h]$ modulo a Gröbner basis of the previous ones.

Denote by W the image of $\mathbb{C}^8$ under F and by $\overline{W}$ its Zariski closure. Let $\widetilde{G}_6$ be the ideal $\langle g_{11}^{(F)}, \ldots, g_{66}^{(F)}\rangle$ in $\mathbb{C}[\overline{W}]$ and let R be the kernel of F_*. Then $R = \ker F_* = J \cap \mathbb{C}[h]$ and, by the Elimination Theorem (see e.g. Theorem 2 of §3.1 of [5])), $R = \langle g \in J_G : g \in \mathbb{C}[h]\rangle$. The unique reduced Gröbner Basis of the ideal contains 105 binomials $r_1, \ldots, r_{105}$, that is, $R = \langle r_1, \ldots, r_{105}\rangle$. We do not present these polynomials in the paper, however the interested reader can easily obtain them computing the reduced Gröbner basis of J with help of any available computer algebra system. Note that by Theorem 1 of §3.3 of [5] $\overline{W} = \mathbf{V}(R)$.

Denote by V the variety $\mathbf{V}(\mathcal{B})$ and by V_h the image of V under F, $V_h = F(V)$. Similarly as in [16] we check that $V_h = \mathbf{V}(\langle g_{kk}^{(F)} : k \in \mathbb{N}\rangle) = V_6$, where $V_6 = \mathbf{V}(\langle \widetilde{G}_6, R\rangle)$.

Let $K = \langle g_{11}^{(F)}, \dots, g_{55}^{(F)}, g_{66}^{(F)}, R\rangle \subset \mathbb{C}[h]$. For every $k \in \mathbb{N}$ $g_{kk}^{(F)}$ vanishes on V_h, the subvariety of $\tilde{G}_6$ in $\overline{W}$. That means $g_{kk}^{(F)}$ as a polynomial of $\mathbb{C}[h]$ vanishes on the variety of $K \subset \mathbb{C}[h]$. Computing the primary decomposition of $K \subset \mathbb{C}[h]$ using `primdecGTZ` [7, 9] we find that

$$K = P_1 \cap \cdots \cap P_5 \cap Q_1 \cap Q_2 \cap Q_3, \tag{24}$$

where $P_1, \dots, P_5$ are prime ideals, and Q_i are not prime, but primary ideals, such that $\sqrt{Q_1} = \langle \{h_i \mid 1 \le i \le 10, 13 \le i \le 17\}, h_{11}h_{12} - h_{15}h_{16}^3\rangle$ $\sqrt{Q_2} = \langle h_1, \dots, h_{17}\rangle$, $\sqrt{Q_3} = \langle \{h_i \mid 3 \le i \le 16\}, h_1 - h_2, 13975h_2^4 - 11562h_2^2h_{17} + 5971h_{17}^2\rangle$.

Thus, K has the structure as in Proposition 3. The minimal associate primes of $\cap_{i=1}^3 F(Q_i)$ are $T_1 = \langle a_{-13}, b_{11}, a_{11}, b_{01}, a_{-12}\rangle$, $T_2 = \langle b_{3,-1}, b_{11}, a_{11}, b_{2,-1}, a_{10}\rangle$, $T_3 = \langle a_{-13}, b_{3,-1}, b_{11}, a_{11}\rangle$. Obviously, the intersection of $\mathbf{V}(Q_1) \cup \mathbf{V}(Q_2) \cup \mathbf{V}(Q_3)$ with the parameter space $E(a)$ is the set $a_{11} = a_{-13} = 0$.

Let now $(a^*, \bar{a}^*)$ be a point from $E(a,b)$ corresponding to a system (5). If $|a_{11}^*| + |a_{-13}^*| \ne 0$ then $F((a^*, \bar{a}^*) \notin \mathbf{V}(Q_1) \cup \mathbf{V}(Q_2) \cup \mathbf{V}(Q_3)$. Therefore, by the proposition, there exist rational functions $f_{j,k}, s_{j,k}$ such that for $h \in \overline{W}$ in a neighborhood of $F((a^*, \bar{a}^*)$ with $|a_{11}^*| + |a_{-13}^*| \ne 0$

$$g_{kk}^{(F)}(h) = g_{11}^{(F)}(h)f_{1,k}(h) + \cdots + g_{55}^{(F)}(h)f_{5,k}(h) + g_{66}^{(F)}(h)f_{6,k}(h) + \sum_{j=1}^{105} r_j(h)s_{j,k}(h). \tag{25}$$

Applying F_* to (25) we see that (22) holds in a neighborhood of $(a^*, \bar{a}^*)$.

Thus, G_6 is a basis of $\mathcal{B}$ in $\mathcal{G}_{(a^*,\bar{a}^*)}$ at any $(a^*, \bar{a}^*)$ with $|a_{11}^*| + |a_{-13}^*| \ne 0$. Therefore, by Theorem 1 the cyclicity of the center at the origin of the corresponding system (4) is at most 6. □

For (10) $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_5)$, however it appears that $\mathcal{B} = \mathcal{B}_8$. Computations show that for $K = \langle g_{11}^{(F)}, \dots, g_{55}^{(F)}, g_{66}^{(F)}, g_{77}^{(F)}, R\rangle$ and $K = \langle g_{11}^{(F)}, \dots, g_{55}^{(F)}, g_{66}^{(F)}, g_{77}^{(F)}, g_{88}^{(F)}, R\rangle$ the ideal K has a simpler structure with the embedded component Q_3 being removed, $K = P_1 \cap \cdots \cap P_5 \cap Q_1 \cap Q_2$. Also in these cases the computations are much faster. However if we deal with $\mathcal{B}_7$ by Theorem 1 we have to conclude that the upper bound for the cyclicity is 7, that is, the conclusion is weaker, than given by Theorem 4.

We believe it would be possible to obtain five as an upper bound for the cyclicity of the system if we were able to find a primary decomposition of the ideal

$$K = \langle g_{11}^{(F)}, \dots, g_{55}^{(F)}, R\rangle. \tag{26}$$

However we were unable to complete the calculations with Singularon our computational facilities due to the very high memory consumption of the calculation process. We have tried to apply modular calculations to this ideal. The result of calculation is 9 components five of which are the radical ones and four are not. However in this case the lifting (reconstruction) to the rational numbers gives an ideal which does not coincide with K, that is the modular calculations do not give true primary ideals of K.

5. The cyclicity of components of the center variety

In this section we study bifurcations of limit cycles from each component of the center variety.

Let $I = \langle f_1, \ldots, f_m \rangle \subset k[x_1, \ldots, x_n]$ be an ideal and $\mathbf{V}(I)$ be its variety, decomposition of $V = \mathbf{V}(I)$ is given and let p be a point from V. The tangent space to V at p is defined as $T_p = p + \{v | J_p(I)v = 0\}$, where $J(I)$ is the Jacobian of the polynomials $f_1, \ldots, f_m$ and J_p indicates that it is evaluated at p. It follows that $\dim T_p = n - rank(J_p(I))$. It is said that p is a smooth point of V if $\dim T_p = \dim V_p$. Let C be a component of V of codimension k and assume that $p \in C$, $rank(J_p(I)) = s$. Then $k \geq s$ and p is a smooth point C if and only if $k = s$; in this case $rank(J_q(I)) = k$ at any smooth point of C.

Denote by $g_{kk}^{\mathbb{R}}$ the polynomials obtained after the substituting b_{qp} with $\bar{a}_{pq}$ into g_{kk}. Then the center variety of the real system (5) with $\lambda = 0$ is the the variety $V^{\mathbb{R}}$ in $E(\bar{a})$ of the ideal $\mathcal{B}^{\mathbb{R}} = \langle g_{11}^{\mathbb{R}}, g_{22}^{\mathbb{R}}, \ldots \rangle$.

The following statement is slightly reformulated Theorem 2.1 of [3].

Theorem 5 (C. Christopher)**.** *Assume that for system* (5) *with* $\lambda = 0$ $p \in V^{\mathbb{R}}$ *and* $rank\, J_p(\mathcal{B}_k^{\mathbb{R}}) = k$*. Then* p *lies on a component of* $V^{\mathbb{R}}$ *of codimension at least* k *and there are bifurcations of* (5) *which produce* k *limit cycles locally from the center corresponding to the parameter value* p*.*

If furthermore, p *lies on a component* C *of* $V^{\mathbb{R}}$ *of codimension* k *then* p *is a smooth point of the center variety, and the cyclicity of* p *and any generic point of* C *is exactly* k*.*

Thus, the cyclicity of generic point of a component of the center variety can be easily determined if we know the dimension of the center variety. The dimension of complex variety can be easily computed using algorithms of computational algebra, since it is equal to the degree of the affine Hilbert polynomial of any ideal defining the variety. However determining dimensions of real varieties is more difficult problem.

We go from the complex system to the real one by setting

(27)
$$a_{10} = A_{10} + IB_{10},\ b_{01} = A_{10} - IB_{10},\ a_{-12} = A_{-12} + IB_{-12},\ b_{2,-1} = A_{-12} - IB_{-12},$$
$$a_{11} = A_{11} + IB_{11},\ b_{11} = A_{11} - IB_{11},\ a_{-13} = A_{-13} + IB_{-13},\ b_{3,-1} = A_{-13} - IB_{-13},$$

where I stands for the complex root of unity.

The following statement describes the center variety of (4).

Theorem 6. *The center variety in* $\mathbb{R}^8$ *of the real system* (4) *with* $\lambda = 0$ *consists of the following four irreducible components:*
1) $B_{11} = A_{11} = A_{-13} = B_{-13} = 0,$
2) $B_{11} = -A_{-13}B_{10} + A_{11}B_{-12} - A_{10}B_{-13} = -A_{10}^2 + A_{-12}^2 - B_{10}^2 + B_{-12}^2 = A_{11}B_{10} - A_{-13}B_{-12} + A_{-12}B_{-13} = -A_{10}A_{11} + A_{-12}A_{-13} + B_{-12}B_{-13} = A_{11}A_{-12} - A_{10}A_{-13} + B_{10}B_{-13} = A_{11}^2 - A_{-13}^2 - B_{-13}^2 = -A_{11}A_{-13}B_{10} + A_{-13}^2B_{-12} - A_{10}A_{11}B_{-13} + B_{-12}B_{-13}^2 = 0,$
3) $-4A_{-12}^3A_{-13}^3B_{-12} + 4A_{-12}A_{-13}^3B_{-12}^3 + 3A_{-12}^4A_{-13}^2B_{-13} - 18A_{-12}^2A_{-13}^2B_{-12}^2B_{-13} + 3A_{-13}^2B_{-12}^4B_{-13} + 12A_{-12}^3A_{-13}B_{-12}B_{-13}^2 - 12A_{-12}A_{-13}B_{-12}^3B_{-13}^2 - A_{-12}^4B_{-13}^3 + 6A_{-12}^2B_{-12}^2B_{-13}^3 - B_{-12}^4B_{-13}^3 = -3A_{10}^2A_{-12}B_{10} + A_{-12}B_{10}^3 - A_{10}^3B_{-12} + 3A_{10}B_{10}^2B_{-12} = -4A_{10}^3A_{-13}B_{10} + 4A_{10}A_{-13}B_{10}^3 - A_{10}^4B_{-13} + 6A_{10}^2B_{10}^2B_{-13} - B_{10}^4B_{-13} = -B_{11} = 0$
4) $A_{10} = B_{10} = B_{11} = 0,$
of dimensions 4, 4, 5, 5, respectively.

Proof. Performing the substitution (27) we obtain from 1), 4) and 6) of Theorem 2 the conditions 1), 2) and 4), respectively. The ideals 2) and 3) of Theorem 2 yield a subvariety of 1) of the present theorem. By Theorem 3.3 of [22] the variety $\mathbf{V}(J_5)$ is the same as the variety of $\langle a^4_{-12}b^3_{3,-1} - b^4_{2,-1}a^3_{-13}, a^3_{10}a_{-12} - b_{2,-1}b^3_{01}, -b^4_{01}b_{3,-1} + a^4_{10}a_{-13}, a_{11} - b_{11}\rangle$, which, after the substitution (27), gives the component 3).

The dimension of components 1) and 4) is obvious. To find dimensions of the other components we look for their rational parametrization.

We guess that 2) can be parametrized as

(28)
$$\begin{gathered}B_{11} = f_0,\ A_{11} = f_1(r_1),\ A_{10} = f_2(r_1, r_2, t_1, t_2)/g_2(t_2),\ B_{10} = f_3(r_1, r_2, t_1, t_2)/g_2(t_2),\\ A_{-12} = f_4(r_1, r_2, t_1, t_2)/(g_1(t_1)g_2(t_2)),\ B_{-12} = f_5(r_1, r_2, t_1, t_2)/(g_1(t_1)g_2(t_2)),\\ A_{-13} = f_6(r_1, r_2, t_1, t_2)/g_1(t_1), B_{-13} = f_7(r_1, r_2, t_1, t_2)/g_1(t_1),\end{gathered}$$

where $f_0 = 0, f_1 = r_1, f_2 = r_2(1-t_2^2), f_3 = 2r_2t_2), f_4 = r_2(-1-t_1-t_2+t_1t_2)(-1+t_1+t_2+t_1t_2), f_5 = -2r_2(t_1 + t_2)(-1 + t_1t_2), f_6 = r_1(1 - t_1^2), f_7 = 2r_1t_1, g_1(t_1) = 1 + t_1^2, g_2(t_2) = 1 + t_2^2$.

We eliminate from the ideal

$$\begin{gathered}\langle 1 - wg_1g_2, A_{11} - r_1, g_2A_{10} - f_2,\ g_2B_{10} - f_3, g_1g_2A_{-12} - f_4,\ g_1g_2B_{-12} - f_5,\\ g_1A_{-13} - f_6,\ g_1B_{-13} - f_7 \subset \mathbb{R}[w, r_1, r_2, t_1, t_2, A_{10}, B_{10}, A_{-12}, B_{-12}, A_{11}, A_{-13}, B_{-13}]\end{gathered}$$

variables w, r_1, r_2, t_1, t_2 and obtain the ideal which coincides with the ideal defined by the equations of 2). By Theorem 2 of [5, §3.3] it means that (28) give a rational parametrization of the third component and the dimension of the component is less or equal to four. Computing the Jacobian of the functions $f_0, \dots, f_7$ at the randomly chosen point $t_1 = 1, t_2 = 2, r_1 = 3, r_2 = 4$ we see that it is four. Therefore, the dimension of the component is four as well.

Similarly, one can check that a polynomial parametrization of 3) is given by

$$\begin{gathered}(29)\quad A_{11} = u_1, A_{-13} = u_2(u_3^4 - 6u_3^2u_4^2 + u_4^4),\ B_{-13} = 4u_2u_3u_4(-u_3^2 + u_4^2),\\ A_{-12} = u_3u_5(u_3^2 - 3u_4^2),\ B_{-12} = u_5u_4(-3u_3^2 + u_4^2),\ A_{10} = u_3,\ B_{10} = u_4.\end{gathered}$$

Indeed, eliminating from (29) u_1, u_2, u_3, u_4, u_5 we obtain an ideal whose radical is equal to the radical of the ideal defining 3. Similarly as above one can easily check the dimension of the component is equal to five. □

We denote by $J_p^{(k)}$ the rank of the Jacobian of the first k focus quantities evaluated at the point p.

Theorem 7. *The cyclicity of a generic point of components 2) is equal to four, and of a generic point of 3) is equal to 3. Define the polynomials* $F_1 = 3A_{10}^2B_{10}A_{-12} - B_{10}^3A_{-12} + A_{10}^3B_{-12} - 3A_{10}B_{10}^2B_{-12}$ *and* $F_4 = 4A^3_{-12}B_{-12}A^3_{-13} - 4A_{-12}B^3_{-12}A^3_{-13} - 3A^4_{-12}A^2_{-13}B_{-13} + 18A^2_{-12}B^2_{-12}A^2_{-13}B_{-13} - 3B^4_{-12}A^2_{-13}B_{-13} - 12A^3_{-12}B_{-12}A_{-13}B^2_{-13} + 12A_{-12}B^3_{-12}A_{-13}B^2_{-13} + A^4_{-12}B^3_{-13} - 6A^2_{-12}B^2_{-12}B^3_{-13} + B^4_{-12}B^3_{-13}$. *Then the cyclicity of a point p of 1) with $F_1(p) \neq 0$ and of a point p' of 4) with $F_4(p') \neq 0$ is at least 3.*

Proof. We take a random point on the component 2), e.g. p with the coordinates $A_{11} = 3, B_{11} = 0, A_{-13} = 0, B_{-13} = 3, A_{10} = -12/5, B_{10} = 16/5, A_{-12} = -16/5, B_{-12} = -12/5$. Calculations yield that $rank J_p^{(4)} = 4$. Since the dimension of 2) is 4, by Theorem 5 the cyclicity of a generic point of 2) is 4. Similarly, computing $J_p^{(3)}$ at the random point p : $(A_{-13} = 41, B_{-13} = 840, A_{-12} = -568, B_{-12} = 260, A_{10} = 2, B_{10} = 5, A_{11} = 3, B_{11} = 0)$ we see that $J_p^{(3)} = 3$. Therefore by Christopher's theorem the cyclicity of a generic point of the component is 3.

Computing Fitting ideals (see e. g. [11]), we obtain that for the component 1) $rank J_p^{(3)} = rank J_p^{(4)} = 3$ at p with $F_1(p) \neq 0$. Similarly for component 4) we have $rank J_p^{(3)} = rank J_p^{(4)} = 3$ at the point p' with $F_4(p') \neq 0$.

Thus, by Theorem 5 three limit cycles bifurcate from the origin for the system corresponding to p. □

6. The cyclicity of a generic time-reversible cubic system

By Theorem 6 of [15], Theorem 3.3 of [22], and Theorem 3.2 of [21] the Zariski closure of the set of time-reversible systems in the family (3) is the variety of the ideal

$$I = \langle a_{-13}b_{20}^2 - a_{02}^2 b_{3,-1}, a_{11} - b_{11}, a_{02}a_{20} - b_{02}b_{20}, a_{-13}a_{20}^2 - b_{02}^2 b_{3,-1}, a_{-12}^2 b_{20}^3 - a_{02}^3 b_{2,-1}^2,$$
$$a_{-12}^2 a_{20}^3 - b_{02}^3 b_{2,-1}^2, -a_{-13}^3 b_{2,-1}^4 + a_{-12}^4 b_{3,-1}^3, -a_{02}a_{10}^2 + b_{01}^2 b_{20},$$
$$a_{20}b_{01}^2 - a_{10}^2 b_{02}, -a_{10}^3 a_{-12} + b_{01}^3 b_{2,-1}, -a_{10}^4 a_{-13} + b_{01}^4 b_{3,-1}, -a_{01}a_{10} + b_{01}b10,$$
$$a_{02}b10^2 - a_{01}^2 b_{20}, -a_{01}^2 a_{20} + b_{02}b10^2, a_{-12}b10^3 - a_{01}^3 b_{2,-1}, a_{-13}b10^4 - a_{01}^4 b_{3,-1}\rangle$$

After the substitution

(30)
$$a_{10} = A_{10}+IB_{10}, b_{01} = A_{10}-IB_{10}, a_{01} = A_{01}+IB_{01}, b10 = A_{01}-IB_{01}, a_{-12} = A_{-12}+IB_{-12},$$
$$b_{2,-1} = A_{-12}-IB_{-12}, a_{20} = A_{20}+IB_{20}, b_{02} = A_{20}-IB_{20}, a_{11} = A_{11}+IB_{11}, b_{11} = A_{11}-IB_{11},$$
$$a_{02} = A_{02} + IB_{02}, b_{20} = A_{02} - IB_{02}, a_{-13} = A_{-13} + IB_{-13}, b_{3,-1} = A_{-13} - IB_{-13}$$

we obtain from I the ideal $I_{\mathbb{R}}$ and $\mathbf{V}(I_{\mathbb{R}})$ is the set of all time-reversible systems inside the family (2) with $\lambda = 0$.

Theorem 8. *The cyclicity of a generic time-reversible system in the general family of cubic systems* (2) *is equal to six.*

Proof. We first show that the dimension of $\mathbf{V}(I_{\mathbb{R}})$ is 7. Noting that the necessary condition for time reversibility is vanishing of B_{11} we set in all computations $B_{11} = 0$. Computing in $\mathbb{Q}[t_1, \dots, t_8, A_{10}, \dots, B_{-13}]$ the eighth elimination ideal J_8 of $J = \langle A_{10} - f_1 B_{10} - f_2, A_{01} - f_3, B_{01} - f_4, A_{-12} - f_5, B_{-12} - f_6, A_{20} - f_7, B_{20} - f_8, A_{11} - f_9, A_{02} - f_{10}, B_{02} - f_{11}, A_{-13} - f_{12}, B_{-13} - f_{13}\rangle$, where

$$f_1 = t_1,\ f_2 = t_2,\ f_3 = t_1 t_3,\ f_4 = -t_2 t_3,\ f_5 = t_1(t_1^2 - 3t_2^2)t_5,\ f_6 = t_2(-3t_1^2 + t_2^2)t_5,$$
$$f_7 = (t_1^2 - t_2^2)t_6,\ f_8 = 2t_1 t_2 t_6, f_9 = t8,\ f_{10} = (t_1^2 - t_2^2)t_3^2 t_7,$$
$$f_{11} = (-2t_1 t_2 t_3^2 t_7),\ f_{12} = (t_1^4 - 6t_1^2 t_2^2 + t_2^4)t_4,\ f_{13} = 4t_1 t_2(-t_1^2 + t_2^2)t_4,$$

we find that $J_8 = \sqrt{I}$. By Theorem 1 of [5, §3, Chapter 3] it means that the polynomials $f_1, \dots, f_{13}$ define a polynomial parametrization of $\mathbf{V}(I_{\mathbb{R}})$. Therefore, $\dim(\mathbf{V}(I_{\mathbb{R}})) \le 7$. Computing the Jacobian of the polynomials $f_1, \dots, f_{13}$ at a random point $\{t_i = i,\ 1 \le i \le 7\}$ we see that its rank is equal to 7. Taking into account the restriction $B_{11} = 0$ we conclude that the codimension of $\mathbf{V}(I_{\mathbb{R}})$ is 6.

The calculation of the Jacobian of $g_{11}^{\mathbb{R}}, \dots, g_{66}^{\mathbb{R}}$ at the point corresponding to the parameters $t_i = i$, $1 \le i \le 7$ shows that its rank is 6. Therefore, by Christopher's theorem the cyclicity of the generic point of $\mathbf{V}(I_{\mathbb{R}})$ is 6. □

7. Concluding remarks

We have described an algorithmic approach to obtain an upper bound for the cyclicity of "most of systems" in a polynomial family and have applied it to study the cyclicity of the family (4). For this family we have shown that the cyclicity of a generic system of the component of the center variety defined by the conditions 2), 3) and 4) of Theorem 6 is at most six. The method does not give any bound for the cyclicity of systems corresponding to the first component of Theorem 6. We believe it is possible to improve the bound computing a primary decomposition of the ideal (26), however the calculations are very laborious and cannot be completed with our computational facilities.

We then have studied bifurcations of limit cycle from each component of the center variety using Christopher's theorem (Theorem 5). Using this approach we have shown that for the second and the third components of Theorem 6 the sharp bound for the cyclicity is four and three, respectively. However the approach does not give upper bounds for the cyclicity of systems corresponding to the first and forth components of the theorem.

We have proven also that the cyclicity of a generic time-reversible cubic system is six.

Acknowledgments

We wish to express our gratitude to Wolfram Decker and Eva Zerz for fruitful discussions on the topics of this paper. The second author acknowledges the support by the Slovenian Research Agency, the DAAD, and by the Transnational Access Programme at RISC-Linz of the European Commission Framework 6 Programme for Integrated Infrastructures Initiatives under the project SCIEnce (Contract No. 026133).

Appendix

The first focus quantities each reduced modulo the ideal of the generated by the previous ones and up to a constant factor are as follows:

$g_{11}^{(F)} = h_1 - h_2,$
$g_{22}^{(F)} = 2/3h_7 - 2/3h_8,$
$g_{33}^{(F)} = -1/3h_2h_{11} + 1/3h_2h_{12} - 1/3h_9 + 1/3h_{10},$
$g_{44}^{(F)} = 29/30h_2h_9 - 29/30h_2h_{10} + 1/6h_2h_{13} - 1/6h_2h_{14} + 36/5h_{13}h_{15} - 36/5h_{14}h_{15} + 36/5h_9h_{16} - 36/5h_{10}h_{16} - 11/30h_{11}h_{17} + 11/30h_{12}h_{17} - 7/6h_5 + 7/6h_6,$
$g_{55}^{(F)} = -191/261h_2^2h_{13} + 191/261h_2^2h_{14} + 1688/87h_2h_{13}h_{15} - 1688/87h_2h_{14}h_{15} + 38351/3915h_{13}h_{15}^2 -$

$38351/3915h_{14}h_{15}^2-187859/37845h_2h_{13}h_{16}+187859/37845h_2h_{14}h_{16}+38351/3915h_9h_{15}h_{16}-38351/3915h_{10}h_{15}h_{16}-$
$826873/2523h_{13}h_{15}h_{16}+826873/2523h_{14}h_{15}h_{16}-826873/2523h_9h_{16}^2+826873/2523h_{10}h_{16}^2-11/3915h_{11}h_{15}h_{17}+$
$11/3915h_{12}h_{15}h_{17}+63602/37845h_{11}h_{16}h_{17}-63602/37845h_{12}h_{16}h_{17}+91/348h_2h_5-91/348h_2h_6-1714/135h_8h_9+$
$1714/135h_8h_{10}+14167/270h_9h_{11}-14167/270h_{10}h_{12}+277/30h_8h_{13}-16983/10h_{12}h_{13}-277/30h_8h_{14}+$
$16983/10h_{11}h_{14}+7/58h_5h_{15}-7/58h_6h_{15}+1435276/37845h_5h_{16}-1435276/37845h_6h_{16}-1144/1305h_9h_{17}+$
$1144/1305h_{10}h_{17}+7/60h_{13}h_{17}-7/60h_{14}h_{17},$
$g_{66}^{(F)}=1/179957124900(314439792836580h_2h_{13}h_{15}^2-314439792836580h_2h_{14}h_{15}^2+167774955434040h_{13}h_{15}^3-$
$167774955434040h_{14}h_{15}^3-459779466666048h_2h_{13}h_{15}h_{16}+459779466666048h_2h_{14}h_{15}h_{16}+167774955434040h_9h_{15}^2h_{16}-$
$167774955434040h_{10}h_{15}^2h_{16}-5564635229863332h_{13}h_{15}^2h_{16}+5564635229863332h_{14}h_{15}^2h_{16}+73798251681168h_2h_{13}h_{16}^2-$
$73798251681168h_2h_{14}h_{16}^2-5564635229863332h_9h_{15}h_{16}^2+5564635229863332h_{10}h_{15}h_{16}^2+3901291066992420h_{13}h_{15}h_{16}^2-$
$3901291066992420h_{14}h_{15}h_{16}^2+3901291066992420h_9h_{16}^3-3901291066992420h_{10}h_{16}^3+144551266134h_{11}h_{15}^2h_{17}-$
$144551266134h_{12}h_{15}^2h_{17}+24525527209260h_{11}h_{15}h_{16}h_{17}-24525527209260h_{12}h_{15}h_{16}h_{17}-157879306893924h_{11}h_{16}^2h_{17}+$
$157879306893924h_{12}h_{16}^2h_{17}-219450333375h_2^2h_5+219450333375h_2^2h_6-7970291720775h_2h_8h_{13}-4709316225452220h_2h_{12}h_1$
$7970291720775h_2h_8h_{14}+4709316225452220h_2h_{12}h_{14}+4386231116505h_2h_5h_{15}-4386231116505h_2h_6h_{15}-$
$217622722151280h_8h_9h_{15}+217622722151280h_8h_{10}h_{15}+10486735145044144h_9h_{11}h_{15}-1048673514504414h_{10}h_{12}h_{15}+$
$150644468531700h_8h_{13}h_{15}-26924686414769718h_{12}h_{13}h_{15}-150644468531700h_8h_{14}h_{15}+28614070529783622h_{11}h_{14}h_{15}-$
$1689384115013904h_{12}h_{14}h_{15}+2023904532510h_5h_{15}^2-2023904532510h_6h_{15}^2+11900099492469h_2h_5h_{16}-$
$11900099492469h_2h_6h_{16}+7909426957536h_8h_9h_{16}-7909426957536h_8h_{10}h_{16}+177784594345584h_9h_{11}h_{16}+$
$1689384115013904h_9h_{12}h_{16}-1867168709359488h_{10}h_{12}h_{16}-4463619138396h_8h_{13}h_{16}+368377476200712h_{12}h_{13}h_{16}+$
$4463619138396h_8h_{14}h_{16}-368377476200712h_{11}h_{14}h_{16}+627432169709646h_5h_{15}h_{16}-627432169709646h_6h_{15}h_{16}-$
$478312580221938h_5h_{16}^2+478312580221938h_6h_{16}^2+3313052243424h_8h_{11}h_{17}+3881784152840h_{11}^2h_{17}-$
$3313052243424h_8h_{12}h_{17}-86033450301634h_{11}h_{12}h_{17}+82151666148794h_{12}^2h_{17}-345194392635h_2h_{13}h_{17}+$
$345194392635h_2h_{14}h_{17}-14279546249892h_9h_{15}h_{17}+14279546249892h_{10}h_{15}h_{17}+3012698412090h_{13}h_{15}h_{17}-$
$3012698412090h_{14}h_{15}h_{17}-5074709871906h_9h_{16}h_{17}+5074709871906h_{10}h_{16}h_{17}-222914772468h_{13}h_{16}h_{17}+$
$222914772468h_{14}h_{16}h_{17}+653031322560h_{11}h_{17}^2-653031322560h_{12}h_{17}^2+2289495926970h_5h_8-2289495926970h_6h_8+$
$220784663929586h_9^2-226815459886126h_9h_{10}+6030795956540h_{10}^2+27773253873122h_5h_{11}-7191789189140h_6h_{11}-$
$266551007225150h_5h_{12}+245969542541168h_6h_{12}+1285321237690193h_9h_{13}-2720790920240h_{10}h_{13}-$
$4745701548305450h_9h_{14}+3463101101535497h_{10}h_{14}-1089026251875h_3h_{16}+1089026251875h_4h_{16}+527287263300h_5h_{17}-$
$527287263300h_6h_{17}).$

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