

ON DELTA FOR PARAMETERIZED CURVE SINGULARITIES

GERT-MARTIN GREUEL AND GERHARD PFISTER

ABSTRACT. We consider families of parameterizations of reduced curve singularities over a Noetherian base scheme and prove that the delta invariant is semicontinuous. In our setting, each curve singularity in the family is the image of a parameterization and not the fiber of a morphism. The problem came up in connection with the right-left classification of parameterizations of curve singularities defined over a field of positive characteristic. We prove a bound for right-left determinacy of a parameterization in terms of delta and the semicontinuity theorem provides a simultaneous bound for the determinacy in a family. The fact that the base space can be an arbitrary Noetherian scheme causes some difficulties but is (not only) of interest for computational purposes.

Mathematics Subject Classification – MSC2020. 13B22, 13B40, 14B05, 14B07.

Keywords. Curve singularity, parameterization, delta invariant, completed tensor product, semicontinuity, determinacy.

INTRODUCTION

Let R be the local ring of a reduced curve singularity and \overline{R} its normalization, i.e., the integral closure in the total quotient ring of R . If \mathbb{k} is a field and $\mathbb{k} \rightarrow R$ a ring map, then $\delta_{\mathbb{k}}(R) := \dim_{\mathbb{k}} \overline{R}/R$ is called the delta invariant of R (w.r.t. \mathbb{k}), the most important numerical invariant of a reduced curve singularity. In the present paper we consider families of parameterizations of reduced curve singularities, defined over an arbitrary field, and prove that the delta invariant behaves upper semicontinuous.

A parameterization of a reduced algebroid curve singularity is given (in the irreducible case) by a non-zero morphism

$$\varphi : \mathbb{k}[[x_1, \dots, x_n]] \rightarrow \mathbb{k}[[t]], \quad \varphi(x_i) = \varphi^i(t), \quad i = 1, \dots, n,$$

where \mathbb{k} is an arbitrary field. The image $\mathbb{k}[[\varphi^1(t), \dots, \varphi^n(t)]] \subset \mathbb{k}[[t]]$ is a reduced 1-dimensional complete local ring, which we call a reduced curve singularity. A family of parameterization over a Noetherian ring A is a morphism of A -algebras

$$\varphi_A : A[[x_1, \dots, x_n]] \rightarrow A[[t]], \quad \varphi_A(x_i) = \varphi_A^i(t), \quad i = 1, \dots, n.$$

For a prime ideal $\mathfrak{p} \subset A$ with $k(\mathfrak{p}) = \text{Quot}(A/\mathfrak{p})$ the residue field of \mathfrak{p} , let

$$\varphi_{\mathfrak{p}} : k(\mathfrak{p})[[x_1, \dots, x_n]] \rightarrow k(\mathfrak{p})[[t]], \quad \varphi_{\mathfrak{p}}(x_i) = \varphi_{\mathfrak{p}}^i(t), \quad i = 1, \dots, n,$$

be the induced map. If $\varphi_{\mathfrak{p}} \neq 0$, then the image of the parameterization $\varphi_{\mathfrak{p}}$ is $k(\mathfrak{p})[[\varphi_{\mathfrak{p}}^1(t), \dots, \varphi_{\mathfrak{p}}^n(t)]] \cong R_{\mathfrak{p}} := k(\mathfrak{p})[[x_1, \dots, x_n]]/\text{Ker}(\varphi_{\mathfrak{p}})$, a reduced curve singularity, and we prove in Theorem 18 (also for not necessarily irreducible curve singularities) that the delta invariant is semicontinuous, i.e.,

$$\delta_{k(\mathfrak{p})}(R_{\mathfrak{p}}) \geq \delta_{k(\mathfrak{q})}(R_{\mathfrak{q}})$$

for \mathfrak{q} in some open neighbourhood of \mathfrak{p} in $\text{Spec } A$.

For the proof we use some of the semicontinuity results from [GP20], together with the fact that the parameterization $\varphi_{\mathfrak{q}}$ is already determined (up to right-left equivalence) by its terms of order $\leq 4\delta_{k(\mathfrak{q})}(R_{\mathfrak{q}}) - 2$, which we prove in Corollary 12. The semicontinuity of delta implies that this bound holds also for all parameterizations (of a given family) in a neighbourhood of \mathfrak{p} .

The delta invariant of a reduced curve singularity has been and still is a continuous subject of research (e.g. [Hi65], [Te78], [Ca80], [Ca05], [CL06], [Ng16], [CLMN19], [IL20], to name a few). In all these papers the curve singularities appear as fibers X_s of a flat morphism $X \rightarrow S$, and in this situation it is well known that $\delta(X_s)$ is semicontinuous, assuming that the fibers are reduced (cf. [Te78] for $S = \mathbb{C}$ and [CL06] or [GLS07] for S normal). The assumption, that the fibers are reduced is very restrictive, and it implies e.g. for $S = \mathbb{C}$ that X is a Cohen-Macaulay surface singularity.

In our situation the curve singularities $R_{\mathfrak{q}}$ are for each \mathfrak{q} the image of a parameterization and thus reduced as a subring of a reduced ring, but they are in general not the fiber of some morphism (see Remark 17 and Example 22). We do not make any assumption on flatness and the base space of the family can be an arbitrary Noetherian scheme (e.g. \mathbb{Z}). This fact and that we allow non closed points has interesting computational consequences, see Remark 24. At the end of the paper we state an analogous result in the analytic case, for which the proofs are much easier, see Proposition 25 and Remark 26.

1. PARAMETERIZED CURVE SINGULARITIES

Throughout the paper we assume all rings to be associative, commutative and with 1, that ring maps map 1 to 1 and that maps of local rings respect maximal ideals. \mathbb{k} denotes an arbitrary field and A and R denote Noetherian rings.

Let R be reduced. Then $\text{Quot}(R)$, the *total quotient ring* of R , is a direct product of the fields $\text{Quot}(R/P_i)$, where P_1, \dots, P_r are the minimal primes of R ([Mat86, p.183 proof of Th. 23.8]). \overline{R} denotes the *integral closure* of R in $\text{Quot}(R)$, it is the direct product of the rings $\overline{R/P_1}, \dots, \overline{R/P_r}$, and the natural inclusion $R \hookrightarrow \overline{R}$ is called the *normalization* of R . The annihilator

of the R -module \overline{R}/R , $\mathcal{C} := \text{Ann}_R(\overline{R}/R) \subset R$, is called the *conductor ideal* of $R \subset \overline{R}$.

Remark 1. The non-normal locus $\text{Supp}_R(\overline{R}/R)$ may not be closed in $\text{Spec } R$. However, if \overline{R}/R is (module-) finite over R , then $\text{Supp}_R(\overline{R}/R)$ coincides with $V(\mathcal{C}) = \{\mathfrak{p} \in \text{Spec } R \mid \mathfrak{p} \supset \mathcal{C}\}$ and is therefore closed in $\text{Spec } R$.

The following characterization of the normalization is useful.

Proposition 2. *A morphism of reduced Noetherian rings $\nu : R \rightarrow \tilde{R}$ is the normalization map $\Leftrightarrow \tilde{R}$ is normal and ν is integral and birational.¹*

Proof. The direction \Rightarrow follows from [Stack, Lemma 28.52.5(2) and Lemma 28.52.7] and the converse from [Stack, Lemma 28.52.5(3), Lemma 28.52.7, and Lemma 28.5.8]. \square

Definition 3. Let (R, \mathfrak{m}) be a reduced local ring with normalization \overline{R} , \mathbb{k} a field, and $\mathbb{k} \rightarrow R$ a ring map.

(i) The delta invariant of R (w.r.t. \mathbb{k}) is defined as

$$\delta_{\mathbb{k}}(R) := \dim_{\mathbb{k}} \overline{R}/R.$$

(ii) The (multiplicity of the) conductor of R (w.r.t. \mathbb{k}) is defined as

$$c_{\mathbb{k}}(R) := \dim_{\mathbb{k}} \overline{R}/\mathcal{C}.$$

Remark 4. If (R, \mathfrak{m}) is a one-dimensional reduced Nagata ring and \hat{R} its \mathfrak{m} -adic completion, then $\hat{\overline{R}}$ is the normalization of \hat{R} and $\delta_{\mathbb{k}}(R) = \delta_{\mathbb{k}}(\hat{R})$. This is a consequence of Lemma 32.40.2 in [Stack].

We want to study the behavior of $\delta_{\mathbb{k}}(R)$ in a family of parameterizations. We start with a parameterizations of a reduced and irreducible curve singularity, the *uni-branch case*. Let \mathbb{k} be a field and x_1, \dots, x_n, t independent variables. Consider the power series rings

$$P := \mathbb{k}[[x]] := \mathbb{k}[[x_1, \dots, x_n]], \text{ resp. } \tilde{R} := \mathbb{k}[[t]],$$

with maximal ideals $\langle x \rangle := \langle x_1, \dots, x_n \rangle P$ resp. $\tilde{\mathfrak{m}} := t\tilde{R}$.

Definition 5. Let

$$\varphi : P \rightarrow \tilde{R}, \quad \varphi(x_i) =: \varphi^i(t) \in t\mathbb{k}[[t]], \quad i = 1, \dots, n,$$

be a morphism of local \mathbb{k} -algebras such that $\varphi(x_i) \neq 0$ for at least one i . We set

$$R := P/\text{Ker}(\varphi),$$

which is isomorphic to $\varphi(P) = \mathbb{k}[[\varphi(x_1), \dots, \varphi(x_n)]] \subset \mathbb{k}[[t]] = \tilde{R}$.

(1) We call $\varphi : P \rightarrow \tilde{R}$ a parameterization of the uni-branch curve singularity $\varphi(P)$ (or of R).

¹A morphism $\varphi : A \rightarrow B$ between Noetherian rings is *birational* if φ induces a bijection between the minimal primes of B and A and if for every minimal prime \mathfrak{p} of B the induced morphism of local rings $A_{\varphi^{-1}(\mathfrak{p})} \rightarrow B_{\mathfrak{p}}$ is an isomorphism.

- (2) φ is called primitive if for any morphism $\tilde{\varphi} : P \rightarrow \mathbb{k}[[t]]$, satisfying $\varphi^i(t) = \tilde{\varphi}^i(\tau(t))$ for some power series τ and $i = 1, \dots, n$, the order of τ is 1.

Lemma 6. Let $\varphi : P \rightarrow \tilde{R}$ be a parameterization of the uni-branch curve singularity $\varphi(P)$ as in Definition 5. Then φ is a finite morphism and $R \cong \varphi(P)$ is a Noetherian, reduced and irreducible complete local ring of dimension 1 with maximal ideal $\langle x \rangle R \cong \langle \varphi(x) \rangle \mathbb{k}[[t]]$. Moreover, the following are equivalent:

- (i) φ is a primitive parameterization,
- (ii) $\delta_\varphi := \dim_{\mathbb{k}} \tilde{R}/\varphi(P) < \infty$,
- (iii) $\varphi(P) \hookrightarrow \tilde{R}$ is the normalization of $\varphi(P)$.

Proof. Since $\varphi^i \neq 0$ for some i , the inclusion $\varphi(P) \subset \mathbb{k}[[t]]$ is quasi-finite and hence finite by the Weierstrass Finite Theorem [GLS07, Theorem 1.10]. By [GP08, Corollary 3.3.3] $\dim \varphi(P) = \dim \mathbb{k}[[t]] = 1$. Moreover, $\text{Spec } \mathbb{k}[[t]] \rightarrow \text{Spec } \varphi(P)$ is surjective ([Stack, Lemma 10.35.17]) and thus $\varphi(P)$ and hence R is the local ring of an irreducible, reduced curve singularity.

It follows that the normalization \overline{R} (integral closure in $\text{Quot}(R)$) of R is of the form $\mathbb{k}[[\tau]]$ for some variable τ and $\dim_{\mathbb{k}} \overline{R}/R < \infty$ ([Stack, 32.40.2] together with Cohen's structure theorem). Since $\mathbb{k}[[t]]$ is integrally closed $k[[\tau]] \hookrightarrow \mathbb{k}[[t]]$, $\tau \mapsto \tau(t)$, and we have inclusions

$$\varphi(P) = \mathbb{k}[[\varphi^1(t), \dots, \varphi^n(t)]] \cong R \subset \overline{R} \cong k[[\tau]] \hookrightarrow \mathbb{k}[[t]] = \tilde{R}.$$

If φ is primitive, then $\text{ord}(\tau(t)) = 1$, $\overline{R} \cong \tilde{R}$, and \tilde{R} is the normalization of $\varphi(P)$.

If φ is not primitive there exists a morphism $\tilde{\varphi} : P \rightarrow \mathbb{k}[[t]]$, satisfying $\varphi^i(t) = \tilde{\varphi}^i(\tau(t))$, $i = 1, \dots, n$, for some power series τ with order of $\tau \geq 2$. We have $\mathbb{k}[[\varphi^1(t), \dots, \varphi^n(t)]] \subset \mathbb{k}[[\tau(t)]] \subset \mathbb{k}[[t]]$ and $\dim_{\mathbb{k}} \mathbb{k}[[t]]/\mathbb{k}[[\tau(t)]] = \infty$. Hence $\dim_{\mathbb{k}} \tilde{R}/\varphi(P) = \infty$ and \tilde{R} is not the normalization of R . \square

Note that the parameterizations $\varphi_1 = (t^2, t^3)$ and $\varphi_2 = (t^4, t^6)$ define the same curve singularity $R = \mathbb{k}[[x, y]]/\langle x^3 - y^2 \rangle$, but with different embeddings $\varphi_1(P)$ and $\varphi_2(P)$ in $\mathbb{k}[[t]]$. The first parameterization is primitive, the second not.

We consider now the *multi-branch case*, i.e., the parameterization of a curve singularity with several branches. Let t_1, \dots, t_r be independent variables, set

$$\tilde{R} := \mathbb{k}[[t_1]] \oplus \dots \oplus \mathbb{k}[[t_r]],$$

and let $\pi_j : \tilde{R} \rightarrow \tilde{R}_j := \mathbb{k}[[t_j]]$ denote the projection. \tilde{R} is a complete, semilocal, principal ideal ring with Jacobson radical

$$\tilde{\mathfrak{m}} := \tilde{\mathfrak{m}}_1 \oplus \dots \oplus \tilde{\mathfrak{m}}_r, \quad \tilde{\mathfrak{m}}_j := t_j \mathbb{k}[[t_j]],$$

and $\tilde{\mathfrak{m}}_j$ the maximal ideal of the local ring \tilde{R}_j .

Definition 7. Let

$$\begin{aligned}\varphi : P = \mathbb{k}[[x_1, \dots, x_n]] &\rightarrow \tilde{R}, \\ \varphi(x_i) &=: (\varphi_1^i(t_1), \dots, \varphi_r^i(t_r)) \in t_1\mathbb{k}[[t_1]] \oplus \dots \oplus t_r\mathbb{k}[[t_r]],\end{aligned}$$

be a morphism of \mathbb{k} -algebras. For $j = 1, \dots, r$ let

$$\varphi_j := \pi_j \circ \varphi : P \rightarrow \tilde{R}_j = \mathbb{k}[[t_j]]$$

be the composition. We set

$$\begin{aligned}R &:= P/\text{Ker}(\varphi) \cong \varphi(P), \\ R_j &:= P/\text{Ker}(\varphi_j) \cong \varphi_j(P).\end{aligned}$$

- (1) We call $\varphi = (\varphi_1, \dots, \varphi_r)$ a parameterization of the curve singularity $\varphi(P)$ (or of R) with r branches if φ_j is a parameterization of $\varphi_j(P)$ as in Definition (5) and if $\varphi_j(P) \neq \varphi_{j'}(P)$ for $j \neq j'$. The morphism of local rings φ_j , $j = 1, \dots, r$, is called a parameterization of the branch $\varphi_j(P)$, (or of R_j).
- (2) The parameterization φ is called primitive if φ_j is primitive for each j .

The rings

$$\begin{aligned}\varphi(P) &= \mathbb{k}[[\varphi(x_1), \dots, \varphi(x_n)]] \subset \mathbb{k}[[t_1]] \oplus \dots \oplus \mathbb{k}[[t_r]] \text{ and} \\ \varphi_j(P) &= \mathbb{k}[[\varphi_j^1(t_j), \dots, \varphi_j^n(t_j)]] \subset \mathbb{k}[[t_j]]\end{aligned}$$

are Noetherian, reduced, complete local rings with maximal ideals $\mathfrak{m} := \varphi(\langle x \rangle)$ and $\mathfrak{m}_j := \varphi_j(\langle x \rangle)$ respectively. It follows from Lemma 6 that $\varphi_j(P)$ is the local ring of a reduced, irreducible curve singularity. Hence $\varphi(P)$ is the local ring of a reduced curve singularity with r branches.

Defining the conductor ideal of φ as

$$\mathcal{C}_\varphi := \text{Ann}_{\varphi(P)}(\tilde{R}/\varphi(P)),$$

we get the following result for multi-branch curve singularities.

Lemma 8. Any parameterization $\varphi : P \rightarrow \tilde{R}$ of $\varphi(P)$ as in Definition 7 is a finite morphism and $R \cong \varphi(P)$ is a Noetherian, reduced and complete local ring of dimension 1. Moreover, the following are equivalent:

- (i) φ is a primitive parameterization,
- (ii) $\delta_\varphi := \dim_{\mathbb{k}} \tilde{R}/\varphi(P) < \infty$ (delta-invariant of φ),
- (iii) $c_\varphi := \dim_{\mathbb{k}} \tilde{R}/\mathcal{C}_\varphi < \infty$ (conductor of φ),
- (iv) $\varphi(P) \hookrightarrow \tilde{R}$ is birational,
- (v) $\varphi(P) \hookrightarrow \tilde{R}$ is the normalization of $\varphi(P)$.

If any of these conditions is fulfilled, then $\delta_{\mathbb{k}}(R) = \delta_\varphi$ and $c_{\mathbb{k}}(R) = c_\varphi$. Moreover, $\delta_{\mathbb{k}}(R) \leq c_{\mathbb{k}}(R) \leq 2\delta_{\mathbb{k}}(R)$ and $c_{\mathbb{k}}(R) = 2\delta_{\mathbb{k}}(R)$ iff R is Gorenstein.

Proof. \tilde{R}_j is a finite R_j -module by Lemma 6 and since $R_1 \oplus \dots \oplus R_n$ is finite over R , $\tilde{R} = \tilde{R}_1 \oplus \dots \oplus \tilde{R}_n$ is finite over R .

With $I_j := \text{Ker}(\varphi_j)$ we have $\varphi_j(P) \cong P/I_j$, $j = 1, \dots, r$. Consider, for $r = 2$, the inclusions

$$\varphi(P) \cong P/I_1 \cap I_2 \hookrightarrow P/I_1 \oplus P/I_2 \hookrightarrow \tilde{R}_1 \oplus \tilde{R}_2 = \tilde{R}$$

and the exact sequence

$$0 \rightarrow P/I_1 \cap I_2 \rightarrow P/I_1 \oplus P/I_2 \rightarrow P/(I_1 + I_2) \rightarrow 0.$$

It follows $\delta_\varphi = \delta_{\varphi_1} + \delta_{\varphi_2} + \dim_{\mathbb{k}} P/(I_1 + I_2)$, with $\dim_{\mathbb{k}} P/(I_1 + I_2) < \infty$ since $P/(I_1 + I_2)$ is concentrated on \mathfrak{m} (by definition of a parameterization). By induction on $r \geq 2$ we see that $\delta_\varphi < \infty$ iff $\delta_{\varphi_j} < \infty$ for all $j = 1, \dots, r$.

The equivalence of (i), (ii), (v) is now a consequence of Lemma 6. The equivalence of (iv) and (v) follows from Proposition 2.

The equivalence of (ii) and (iii) can be seen as follows: The exact sequence

$$0 \rightarrow \varphi(P)/\mathcal{C}_\varphi \rightarrow \tilde{R}/\mathcal{C}_\varphi \rightarrow \tilde{R}/\varphi(P) \rightarrow 0$$

implies $c_\varphi = \delta_\varphi + \dim_{\mathbb{k}} \varphi(P)/\mathcal{C}_\varphi$ and hence $\delta_\varphi \leq c_\varphi$. We claim $\delta_\varphi < \infty$ implies $c_\varphi < \infty$. In fact, \mathcal{C}_φ is an \tilde{R} -ideal in $\varphi(P)$ and since \tilde{R} is a principal ideal ring, we get $\mathcal{C}_\varphi = t^{c_\varphi} \tilde{R}$ if $\mathcal{C}_\varphi \neq 0$. We have $\varphi(x_i) \neq 0$ for some i and hence $\varphi(x_i)^k (\tilde{R}/\varphi(P)) = 0$ for some k ($\delta_\varphi < \infty$) by Nakayama's lemma since $\tilde{R}/\varphi(P)$ is finite over $\varphi(P)$. It follows that $\mathcal{C}_\varphi \neq 0$ and hence $c_\varphi < \infty$.

The relations between $\delta_{\mathbb{k}}(R)$ and $c_{\mathbb{k}}(R)$ are well known. In fact, from the exact sequence $0 \rightarrow R/\mathcal{C} \rightarrow \tilde{R}/\mathcal{C} \rightarrow \tilde{R}/R \rightarrow 0$ we get $c_{\mathbb{k}}(R) = \delta_{\mathbb{k}}(R) + \dim_{\mathbb{k}} R/\mathcal{C}$. By [HK71, Korollar 3.7] we have $2 \dim_{\mathbb{k}} R/\mathcal{C} \leq c_{\mathbb{k}}(R)$, with equality iff R is Gorenstein. Hence $\dim_{\mathbb{k}} R/\mathcal{C} \leq \delta_{\mathbb{k}}(R)$ with equality iff R is Gorenstein. This implies the claim. \square

Remark 9. 1. By Lemma 8 δ_φ and c_φ depend only on R (and not on the embedding φ) iff φ is primitive.

2. If φ is a uni-brach parameterization, we can define the *semigroup (of values) of φ* by setting $\Gamma_\varphi := \{v(g) \mid g \in \varphi(P)\} \subset \mathbb{N}$ with $v(g)$ the order of $g \in \mathbb{k}[[t]]$. It is easy to see that the set $\mathbb{N} \setminus \Gamma_\varphi$ is finite iff the greatest common divisor of the integers in Γ_φ (equivalently, of any set of semigroup generators of Γ_φ) is 1. The cardinality $\#(\mathbb{N} \setminus \Gamma_\varphi)$ ("number of gaps") is called the *delta invariant* of Γ_φ and denoted by $\delta(\Gamma_\varphi)$. The smallest integer c in Γ_φ such that $c + \mathbb{N} \subset \Gamma_\varphi$ is called the *conductor* of Γ_φ and denoted by $c(\Gamma_\varphi)$ if it exists, otherwise we set $c(\Gamma_\varphi) := \infty$.

We have $\delta_\varphi = \delta(\Gamma_\varphi)$ and $c_\varphi = c(\Gamma_\varphi)$ for any parameterization φ and (by Lemma 8) $\delta_{\mathbb{k}}(R) = \delta(\Gamma_\varphi)$ and $c_{\mathbb{k}}(R) = c(\Gamma_\varphi)$ iff φ is primitive.

At the end of this section we show that a primitive parameterization $\varphi = (\varphi_1, \dots, \varphi_r)$ is isomorphic to a polynomial parameterization of degree $< 2c_\varphi$, i.e. the $\varphi_j \in \mathbb{k}[[t_j]]$ are determined by their terms of degree $< 2c_\varphi$.

More precisely, with $\text{Aut}_{\mathbb{k}}(P)$ resp. $\text{Aut}_{\mathbb{k}}(\tilde{R})$ the groups of \mathbb{k} -algebra automorphisms of P resp. \tilde{R} , we make the following definition.

Definition 10. (1) Let $\varphi, \psi : P \rightarrow \tilde{R}$ be two parameterizations. We say that φ and ψ are right-left or \mathcal{A} -equivalent ($\varphi \sim_{\mathcal{A}} \psi$) if there exist $\sigma \in \text{Aut}_{\mathbb{k}}(P)$ and $\tau \in \text{Aut}_{\mathbb{k}}(\tilde{R})$ such that $\tau \circ \varphi \circ \sigma^{-1} = \psi$. If $\tau = \text{id}$, we say that φ and ψ are left or \mathcal{L} -equivalent ($\varphi \sim_{\mathcal{L}} \psi$).
(2) φ is called $k - \mathcal{A}$ -determined (resp. $k - \mathcal{L}$ -determined) if $\varphi \equiv \psi \pmod{\tilde{\mathfrak{m}}^{k+1}}$ implies $\varphi \sim_{\mathcal{A}} \psi$ (resp. $\varphi \sim_{\mathcal{L}} \psi$).

Proposition 11. Let $\varphi : P \rightarrow \tilde{R}$ be a parameterization with finite conductor c_{φ} . Then φ is $(2c_{\varphi} - 1) - \mathcal{L}$ -determined and hence $(2c_{\varphi} - 1) - \mathcal{A}$ -determined.

Proof. Since \tilde{R} is a principal ideal ring, $\mathcal{C} = f\tilde{R}$, with $f = (t_1^{c_1}, \dots, t_r^{c_r})$, $c_i \geq 1$, and $c_{\varphi} = c_1 + \dots + c_r$. Set $c := \max\{c_1, \dots, c_r\}$ and we are going to show that φ is $(2c - 1) - \mathcal{L}$ -determined.

Since $c \geq c_i$ for all i we get the $\tilde{\mathfrak{m}}^c \subset \mathcal{C} \subset \tilde{\mathfrak{m}}$ with $\tilde{\mathfrak{m}} = (t_1, \dots, t_r)\tilde{R}$ the Jacobson radical of \tilde{R} . Moreover, each $h \in \mathcal{C} \subset \varphi(P)$ satisfies $h = H(\varphi(x_1), \dots, \varphi(x_n))$ for a suitable $H \in \langle x \rangle \mathbb{k}[[x_1, \dots, x_n]]$.

Assume now that $\varphi \equiv \psi \pmod{\tilde{\mathfrak{m}}^{2c}}$, i.e. $\varphi(x_i) = \psi(x_i) + h_i$ with $h_i \in \tilde{\mathfrak{m}}^{2c} \subset f\tilde{\mathfrak{m}}^c$, i.e. $h_i = fg_i$ with $g_i \in \mathfrak{m}^c$. We obtain that $f = F(\varphi(x_1), \dots, \varphi(x_n))$ and $g_i = G_i(\varphi(x_1), \dots, \varphi(x_n))$ for suitable $F, G_i \in \langle x \rangle \mathbb{k}[[x_1, \dots, x_n]]$. Hence $h_i = H_i((\varphi(x_1), \dots, \varphi(x_n)))$ with $H_i = FG_i \in \langle x \rangle^2 \mathbb{k}[[x_1, \dots, x_n]]$. We obtain that

$$\varphi(x_i) - H_i(\varphi(x_1), \dots, \varphi(x_n)) = \psi(x_i).$$

Now let $\sigma : P \rightarrow P$ be the automorphism defined by

$$\sigma(x_i) = x_i - H_i(x_1, \dots, x_n)$$

then we obtain $\phi \circ \sigma = \psi$ and this implies $\varphi \sim_{\mathcal{L}} \psi$. \square

Corollary 12. Let $\varphi : P \rightarrow \tilde{R}$ be a parameterization. If $\delta_{\varphi} < \infty$ then φ is $(4\delta_{\varphi} - 2) - \mathcal{L}$ -determined.

Proof. Since $2\delta_{\varphi} \geq c_{\varphi}$ by Lemma 8 we obtain the result. \square

Remark 13. 1. \mathcal{A} -equivalence of φ and ψ implies that the \mathbb{k} -algebras $\varphi(P)$ and $\psi(P)$ are isomorphic (this is sometimes called *contact equivalence*). Of course, the multiplicity of the conductor and the delta invariant depend only on the contact equivalence class of a parameterization. Hironaka showed in [Hi65, Theorem B] that a reduced algebraic curve singularity R , defined over an algebraically closed field \mathbb{k} , is $3\delta_{\mathbb{k}}(R) + 1$ contact determined, which can be improved for plane curve singularities over the complex numbers to $2\delta_{\mathbb{C}}(R) - r + 2$ (cf. e.g. [GLS07, Corollary I.2.24]).

2. The proof of Proposition 11 shows a bit more than claimed. First, φ $2c - \mathcal{L}$ -determined with $c = \max\{c_1, \dots, c_r\} \leq c_{\varphi}$. Second, the automorphism $\sigma : P \rightarrow P$ is the identity on linear terms, hence φ and ψ are in the same orbit of the corresponding unipotent subgroup of $\text{Aut}_{\mathbb{k}}(P)$.

3. For an integer $c > 1$ the parameterization $\varphi = (t^c, t^{c+1}, \dots, t^{2c-1})$ has $c_\varphi = c$ and $\delta_\varphi = c - 1$. By Proposition 11 φ is $(2c - 1) - \mathcal{L}$ -determined, but it is obviously not $2c - k$ -determined for $k > 1$, showing that the bound is sharp in this example (contrary to the too optimistic bound c that can be found in the literature, e.g in [Ca05, Proposition 2.1]). Parameterizations of plane curve singularities are $(c_\varphi + 1) - \mathcal{A}$ -determined in any characteristic as shown in [Ng16].

4. The advantage of the bound $4\delta_\varphi - 2$ in Corollary 12 compared to $2c_\varphi - 1$ in Proposition 11 is that the delta invariant is semicontinuous in a family (Theorem 18) while the multiplicity of the conductor is not semicontinuous (Example 21). This is important for the construction of versal deformations or for the classification of parameterizations but also for computational purposes (Remark 24).

2. SEMICONTINUITY OF DELTA FOR FAMILIES OF PARAMETERIZATIONS

We analyze the semicontinuity of delta in a family of parameterizations of reduced curve singularities, which we define now.

Definition 14. Let A be a Noetherian ring.

(1) Consider a morphism of A -algebras

$$\varphi_A : P_A := A[[x]] = A[[x_1, \dots, x_n]] \rightarrow \tilde{R}_A := A[[t_1]] \oplus \dots \oplus A[[t_r]]$$

and denote by

$$\varphi_{A,j} : P_A \rightarrow \tilde{R}_{A,j} := A[[t_j]]$$

the composition of φ_A with the projection $\tilde{R}_A \rightarrow A[[t_j]]$.

We assume that $\varphi_{A,j}(x_i) \in t_j A[[t_j]]$ for $i = 1, \dots, n$, $j = 1, \dots, r$ and that $\varphi_{A,j}(P) \neq \varphi_{A,j'}(P)$ for $j \neq j'$.

(2) The A -algebras

$$\varphi_A(P_A) = A[[\varphi_A(x_1), \dots, \varphi_A(x_n)]] \subset A[[t_1]] \oplus \dots \oplus A[[t_r]] \text{ and}$$

$$\varphi_{A,j}(P_A) = A[[\varphi_{A,j}^1(t_j), \dots, \varphi_{A,j}^n(t_j)]] \subset A[[t_j]]$$

are Noetherian and we set

$$R_A := P_A / \text{Ker}(\varphi_A) \cong \varphi_A(P_A),$$

$$R_{A,j} := P_A / \text{Ker}(\varphi_{A,j}) \cong \varphi_{A,j}(P_A).$$

(3) For a prime ideal \mathfrak{p} of A let $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = \text{Quot}(A/\mathfrak{p})$ be the residue field of \mathfrak{p} . We define

$$P_{\mathfrak{p}} := k(\mathfrak{p})[[x_1, \dots, x_n]],$$

$$\tilde{R}_{\mathfrak{p}} := k(\mathfrak{p})[[t_1]] \oplus \dots \oplus k(\mathfrak{p})[[t_r]],$$

$$\tilde{R}_{\mathfrak{p},j} := k(\mathfrak{p})[[t_j]].$$

Then φ_A induces morphisms

$$\begin{aligned}\varphi_{\mathfrak{p}} : P_{\mathfrak{p}} &\rightarrow \tilde{R}_{\mathfrak{p}}, \\ \varphi_{\mathfrak{p},j} : P_{\mathfrak{p}} &\rightarrow \tilde{R}_{\mathfrak{p},j},\end{aligned}$$

with $\varphi_{\mathfrak{p},j}$ the composition of $\varphi_{\mathfrak{p}}$ with the projection to $\tilde{R}_{\mathfrak{p},j}$. We set

$$\begin{aligned}R_{\mathfrak{p}} &:= P_{\mathfrak{p}}/\text{Ker}(\varphi_{\mathfrak{p}}) \cong \varphi_{\mathfrak{p}}(P_{\mathfrak{p}}), \\ R_{\mathfrak{p},j} &:= P_{\mathfrak{p}}/\text{Ker}(\varphi_{\mathfrak{p},j}) \cong \varphi_{\mathfrak{p},j}(P_{\mathfrak{p}}).\end{aligned}$$

and call $\varphi_A : P_A \rightarrow \tilde{R}_A$ or $\{\varphi_{\mathfrak{p}} : P_{\mathfrak{p}} \rightarrow \tilde{R}_{\mathfrak{p}}\}_{\mathfrak{p} \in \text{Spec } A}$ a family of parameterizations over A .

Remark 15. Fix $\mathfrak{p} \in \text{Spec } A$. Then $\varphi_{\mathfrak{p}}$ is a parameterization of the reduced curve singularity $\varphi_{\mathfrak{p}}(P_{\mathfrak{p}})$ with r branches $\varphi_{\mathfrak{p},1}(P_{\mathfrak{p}}), \dots, \varphi_{\mathfrak{p},r}(P_{\mathfrak{p}})$ in the sense of Definition 7 if and only if the following condition holds (cf. Lemma 8):

- (*) • for each j there is at least one i s.t. $\varphi_{\mathfrak{p},j}(x_i) \neq 0$,
- for $j \neq j'$ we have $\varphi_{\mathfrak{p},j}(P_{\mathfrak{p}}) \neq \varphi_{\mathfrak{p},j'}(P_{\mathfrak{p}})$.

If condition (*) holds for \mathfrak{p} then it holds for \mathfrak{q} in some open neighbourhood U of \mathfrak{p} in $\text{Spec } A$. In particular, the number of branches of $\varphi_{\mathfrak{q}}(P_{\mathfrak{q}})$ is constant for $\mathfrak{q} \in U$.

Let us recall from [GP20] the completed tensor product and the completed fiber, the main tool for our approach.

Let M be any P_A -module and $\mathfrak{p} \in \text{Spec } A$. Considering M as an A -module, we set

$$M(\mathfrak{p}) := M \otimes_A k(\mathfrak{p})$$

and call it the *fiber of M over \mathfrak{p}* . The *completed fiber* of M over \mathfrak{p} is defined as

$$\hat{M}(\mathfrak{p}) := M \hat{\otimes}_A k(\mathfrak{p}) \cong M(\mathfrak{p})^\wedge,$$

where N^\wedge denotes the $\langle x \rangle$ -adic completion of a P_A -module N and $\hat{\otimes}$ denotes the *completed tensor product*, see [GP20, Definitions 2 and 11, and Proposition 3]. In general, if B is any A -algebra, then

$$A[[x]] \hat{\otimes}_A B = (A[[x]] \otimes_A B)^\wedge = B[[x]]$$

([GP20, Proposition 3]). If M is a finitely presented $A[[x]]$ -module, then $M \hat{\otimes}_A B$ is a finitely presented $B[[x]]$ -module ([GP20, Corollary 6]).

Similarly we have $A[[t]] \hat{\otimes}_A B = (A[[t]] \otimes_A B)^\wedge = B[[t]]$, where here \wedge denotes the $\langle t \rangle$ -adic completion, and $\hat{\otimes}_A$ commutes with direct sums.

Lemma 16. With the notations from above and from Definition 14, let \tilde{R}_A be finite over P_A . For $\mathfrak{p} \in \text{Spec } A$ we have

- (1) $P_{\mathfrak{p}} = P_A \hat{\otimes}_A k(\mathfrak{p})$,
- (2) $\tilde{R}_{\mathfrak{p}} = \tilde{R}_A \hat{\otimes}_A k(\mathfrak{p})$,

- (3) $\varphi_{\mathfrak{p}} = \varphi_A \hat{\otimes}_A k(\mathfrak{p}) : P_{\mathfrak{p}} \rightarrow \tilde{R}_{\mathfrak{p}}$,
factoring as $\varphi_{\mathfrak{p}} : P_{\mathfrak{p}} \twoheadrightarrow R_{\mathfrak{p}} \cong \varphi_{\mathfrak{p}}(P_{\mathfrak{p}}) \hookrightarrow \tilde{R}_{\mathfrak{p}}$,
- (4) $(\tilde{R}_A/R_A) \hat{\otimes}_A k(\mathfrak{p}) = \tilde{R}_{\mathfrak{p}}/R_{\mathfrak{p}}$.

Proof. (1), (2) and (3) follow from [GP20, Proposition 3]. If we tensor the exact sequence

$$P_A \xrightarrow{\varphi_A} \tilde{R}_A \rightarrow \tilde{R}_A/\varphi_A(P_A) \cong \tilde{R}_A/R_A \rightarrow 0$$

with $\hat{\otimes}_A k(\mathfrak{p})$ we get

$$(\tilde{R}_A/R_A) \hat{\otimes}_A k(\mathfrak{p}) = (\tilde{R}_A \hat{\otimes}_A k(\mathfrak{p}) / \text{Im}(\varphi_A \hat{\otimes}_A k(\mathfrak{p}))) = \tilde{R}_{\mathfrak{p}}/R_{\mathfrak{p}},$$

since $\hat{\otimes}_A$ is right exact for finitely generated P_A -modules ([GP20, Corollary 4]). \square

Remark 17. In general $\varphi_{\mathfrak{p}}$ is not the restriction of φ_A to the fiber over \mathfrak{p} for an arbitrary prime $\mathfrak{p} \in \text{Spec } A$, i.e., $\varphi_{\mathfrak{p}} \neq \varphi_A \otimes_A k(\mathfrak{p})$ (cf. [GP20, Remark 13]), except for maximal ideals. Namely, if \mathfrak{p} is a maximal ideal of A , then $k(\mathfrak{p}) = A/\mathfrak{p}$ and it follows $P_{\mathfrak{p}} = k(\mathfrak{p})[[x]] = P_A \otimes_A k(\mathfrak{p})$ and $\tilde{R}_{\mathfrak{p}} = \tilde{R}_A \otimes_A k(\mathfrak{p})$ and hence $\varphi_{\mathfrak{p}} = \varphi_A \otimes_A k(\mathfrak{p})$.

However $R_{\mathfrak{p}} \neq R_A \otimes_A k(\mathfrak{p})$ and $R_{\mathfrak{p}} \neq R_A \hat{\otimes}_A k(\mathfrak{p})$ in general, even if \mathfrak{p} is maximal (cf. Example 22). The relation between $R_{\mathfrak{p}}$ and $R_A \hat{\otimes}_A k(\mathfrak{p})$ for arbitrary $\mathfrak{p} \in \text{Spec } A$ is as follows. Tensoring

$$0 \rightarrow \text{Ker}(\varphi_A) \rightarrow P_A \rightarrow R_A \rightarrow 0$$

with $\hat{\otimes}_A k(\mathfrak{p})$ we have by the right-exactness of $\hat{\otimes}_A$ ([GP20, Corollary 4])

$$R_A \hat{\otimes}_A k(\mathfrak{p}) = P_{\mathfrak{p}}/\text{Ker}(\varphi_A)P_{\mathfrak{p}}$$

with $\text{Ker}(\varphi_A)P_{\mathfrak{p}}$ the image of $\text{Ker}(\varphi_A) \hat{\otimes}_A k(\mathfrak{p})$ in $P_{\mathfrak{p}}$. Moreover, tensoring

$$\varphi_A : P_A \twoheadrightarrow R_A \cong \varphi_A(P_A) \subset \tilde{R}_A$$

with $\hat{\otimes}_A k(\mathfrak{p})$ we get

$$\varphi_{\mathfrak{p}} : P_{\mathfrak{p}} \twoheadrightarrow R_A \hat{\otimes}_A k(\mathfrak{p}) \rightarrow \tilde{R}_{\mathfrak{p}}.$$

Since $R_{\mathfrak{p}} = P_{\mathfrak{p}}/\text{Ker}(\varphi_{\mathfrak{p}}) \cong \varphi_{\mathfrak{p}}(P_{\mathfrak{p}})$ we have an induced maps

$$\mathbb{R}_A \hat{\otimes}_A k(\mathfrak{p}) \twoheadrightarrow R_{\mathfrak{p}} \hookrightarrow \tilde{R}_{\mathfrak{p}}.$$

In contrast to $R_{\mathfrak{p}}$ the ring $R_A \hat{\otimes}_A k(\mathfrak{p})$ may not be reduced and $\mathbb{R}_A \hat{\otimes}_A k(\mathfrak{p}) \rightarrow R_{\mathfrak{p}}$ is in general not injective as Example 22 shows.

The main result of this paper is the following theorem, confirming for a family of parameterizations of reduced curve singularities the (*upper*) *semi-continuity* of the function

$$\mathfrak{p} \mapsto \delta_{k(\mathfrak{p})}(R_{\mathfrak{p}}) = \dim_{k(\mathfrak{p})}(\overline{R}_{\mathfrak{p}}/R_{\mathfrak{p}})$$

on $\text{Spec } A$. This means that for $\mathfrak{p} \in A$, there exists an open neighbourhood U of \mathfrak{p} in $\text{Spec } A$ such that $\delta_{k(\mathfrak{q})}(R_{\mathfrak{q}}) \leq \delta_{k(\mathfrak{p})}(R_{\mathfrak{p}})$ for each $\mathfrak{q} \in U$.

Theorem 18. *Let $\varphi_A : P_A \rightarrow \tilde{R}_A$ be a family of parameterizations as in Definition 14 and fix $\mathfrak{p} \in \text{Spec } A$. Assume that $\varphi_{\mathfrak{p}} : P_{\mathfrak{p}} \rightarrow \tilde{R}_{\mathfrak{p}}$ is a parameterization of the reduced curve singularity $R_{\mathfrak{p}} \cong \varphi_{\mathfrak{p}}(P_{\mathfrak{p}})$ (satisfying condition (*) of Remark 15) with $\dim_{k(\mathfrak{p})}(\tilde{R}_{\mathfrak{p}}/R_{\mathfrak{p}}) < \infty$. Then there exists an open neighbourhood $U \subset \text{Spec } A$ of \mathfrak{p} such that*

- (1) $R_{\mathfrak{q}} \hookrightarrow \tilde{R}_{\mathfrak{q}}$ is the normalization of $R_{\mathfrak{q}}$ for each $\mathfrak{q} \in U$ and
- (2) $\mathfrak{q} \mapsto \delta_{k(\mathfrak{q})}(R_{\mathfrak{q}})$ is upper semicontinuous on U .

Proof. By Lemma 8 we have $\dim_{k(\mathfrak{p})}(\tilde{R}_{\mathfrak{p}}/R_{\mathfrak{p}}) = \delta_{k(\mathfrak{p})}(R_{\mathfrak{p}})$ and with $N := 4\delta_{k(\mathfrak{p})}(R_{\mathfrak{p}}) - 2$ we get from Corollary 12 that $\varphi_{\mathfrak{p}}$ is $N - \mathcal{L}$ -determined. We define

$$\bar{\varphi}_A : P_A \rightarrow \tilde{R}_A, \quad x_i \mapsto \bar{\varphi}_A(x_i)$$

with $\bar{\varphi}_A(x_i) = \varphi_A(x_i) \bmod t_1^{N+1}A[[t_1]] \oplus \dots \oplus t_r^{N+1}A[[t_r]]$, identified with its terms up to order N in $\tilde{R}_A = A[[t_1]] \oplus \dots \oplus A[[t_r]]$.

For $\mathfrak{q} \in \text{Spec } A$ we have $\bar{\varphi}_A \hat{\otimes}_A k(\mathfrak{q}) = \bar{\varphi}_{\mathfrak{q}} : P_{\mathfrak{q}} \rightarrow \tilde{R}_{\mathfrak{q}}$, with $\bar{\varphi}_{\mathfrak{q}}(x_i) = \varphi_{\mathfrak{q}}(x_i) \bmod \tilde{\mathfrak{m}}^{N+1}$ and $\bar{\varphi}_{\mathfrak{p}} \sim_{\mathcal{L}} \varphi_{\mathfrak{p}}$ by Corollary 12. $\bar{\varphi}_A$ is a polynomial map and Proposition 19 below implies $\delta_{\bar{\varphi}_{\mathfrak{q}}} = \dim_{k(\mathfrak{q})}(\tilde{R}_{\mathfrak{q}}/\bar{\varphi}_{\mathfrak{q}}(P_{\mathfrak{q}})) \leq \delta_{\bar{\varphi}_{\mathfrak{p}}}$ for \mathfrak{q} in some neighbourhood U of \mathfrak{p} . Hence $\bar{\varphi}_{\mathfrak{q}}$ is $N - \mathcal{L}$ -determined for all $\mathfrak{q} \in U$. Since the delta invariant is invariant under \mathcal{L} -equivalence, we get

$$\delta_{k(\mathfrak{p})}(R_{\mathfrak{p}}) = \delta_{\varphi_{\mathfrak{p}}} = \delta_{\bar{\varphi}_{\mathfrak{p}}} \geq \delta_{\bar{\varphi}_{\mathfrak{q}}} = \delta_{\varphi_{\mathfrak{q}}} = \delta_{k(\mathfrak{q})}(R_{\mathfrak{q}})$$

for $\mathfrak{q} \in U$. Together with Lemma 8 this proves the theorem. \square

Proposition 19. *With the notations and assumptions of Theorem 18, assume moreover that $\varphi_A : P_A \rightarrow \tilde{R}_A$ is defined by algebraic power series, i.e., $\varphi_{A,j}(x_i) \in A\langle t_j \rangle$ (e.g. $\in A[t_j]$) for $j = 1, \dots, r$, $i = 1, \dots, n$.*

Then there exists an étale map $A \rightarrow B$, with $\text{Spec } B$ an étale neighbourhood of \mathfrak{p} , such that for the induced map

$$\varphi_B^h : P_B^h := B\langle x_1, \dots, x_n \rangle \rightarrow \tilde{R}_B^h := B\langle t_1 \rangle \oplus \dots \oplus B\langle t_r \rangle$$

\tilde{R}_B^h is module-finite over P_B^h . Moreover, there exists an open neighbourhood $U \subset \text{Spec } A$ of \mathfrak{p} such that for each $\mathfrak{q} \in U$ we have $\delta_{k(\mathfrak{q})}(R_{\mathfrak{q}}) \leq \delta_{k(\mathfrak{p})}(R_{\mathfrak{p}})$.

Note that in general, \tilde{R}_A is finite over $P_A \Leftrightarrow \tilde{R}_A/R_A$ is finite over $R_A \Leftrightarrow \tilde{R}_A/\mathcal{C}_A$ is finite over R_A , with $\mathcal{C}_A := \text{Ann}_{R_A}(\tilde{R}_A/R_A)$.

Proof. Let A^h be the Henselization of the local ring $A_{\mathfrak{p}}$ and let

$$\varphi_{A^h}^h : P_{A^h}^h := A^h\langle x_1, \dots, x_n \rangle \rightarrow \tilde{R}_{A^h}^h := A^h\langle t_1 \rangle \oplus \dots \oplus A^h\langle t_r \rangle$$

be the canonical extension of φ_A . A^h is a local ring with maximal ideal $\mathfrak{m} = \mathfrak{p}A^h$, $A^h/\mathfrak{m} = k(\mathfrak{p})$, $P_{A^h}^h$ is a local Noetherian Henselian ring with maximal ideal $\langle \mathfrak{m}, x \rangle$ and residue field $k(\mathfrak{p})$, and $\tilde{R}_{A^h}^h$ is a Henselian semilocal ring with radical $\langle \mathfrak{m}, x \rangle \tilde{R}_{A^h}^h = \langle \mathfrak{m}, t_1 \rangle \oplus \dots \oplus \langle \mathfrak{m}, t_r \rangle$. Both are A^h -algebras of finite type and then $\tilde{R}_{A^h}^h$ is also a finite type $P_{A^h}^h$ -algebra (here finite type means finite

type in the Henselian sense²). Since $\varphi_{\mathfrak{p}}$ is a parameterization, $\varphi_{\mathfrak{p},j} \neq 0$ for all j and hence $\dim_{k(\mathfrak{p})}(\tilde{R}_{\mathfrak{p}}/\varphi_{\mathfrak{p}}(\langle x \rangle)\tilde{R}_{\mathfrak{p}}) = \dim_{k(\mathfrak{p})}(\tilde{R}_{A^h}^h/\varphi_{A^h}(\langle \mathfrak{m}, x \rangle)\tilde{R}_{A^h}^h) < \infty$, saying that $\tilde{R}_{A^h}^h$ is a quasifinite and hence a finite $P_{A^h}^h$ -module (by [KPP78, Proposition 1.5]).

By definition of the Henselisation A^h , there exists an étale ring map $A_{\mathfrak{p}} \rightarrow D$ inducing $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \cong D/\mathfrak{p}D$ such that $\varphi_{A^h}^h(x_i) \in B\langle t_1 \rangle \oplus \dots \oplus B\langle t_r \rangle$ for all i . By [GP20, Proposition 28 (9)] there is an étale map $A \rightarrow B$ with $D = B_{\mathfrak{p}}$. By Lemma 33 of [GP20] there exists $\mathfrak{b} \in \text{Spec } B$ such that $\mathfrak{b} \cap A = \mathfrak{p}$. Let $\varphi_B^h : P_B^h := B\langle x_1, \dots, x_n \rangle \rightarrow \tilde{R}_B^h := B\langle t_1 \rangle \oplus \dots \oplus B\langle t_r \rangle$ be the map induced by φ_A and set $R_B^h := P_B^h/\text{Ker}(\varphi_B^h)$. It follows that \tilde{R}_B^h and hence \tilde{R}_B^h/R_B^h is a finite P_B^h -module. Thus \tilde{R}_B^h/R_B^h has a finite presentation over P_B^h with presentation matrix $T : (P_B^h)^p \rightarrow (P_B^h)^q$.

Let $P_B^\wedge := B[[x_1, \dots, x_n]]$ be the $\langle x \rangle$ -adic completion of P_B^h . We get an induced map $\varphi_B^\wedge : P_B^\wedge \rightarrow \tilde{R}^\wedge := B[[t_1]] \oplus \dots \oplus B[[t_r]]$ and set $R_B^\wedge := P_B^\wedge/\text{Ker}(\varphi_B^\wedge)$. Then $\tilde{R}^\wedge/R_B^\wedge$ (the $\langle x \rangle$ -adic completion of \tilde{R}_B^h/R_B^h) has the algebraic presentation $T : (P_B^\wedge)^p \rightarrow (P_B^\wedge)^q$. We can now apply [GP20, Theorem 42] and get that $\dim_{k(\mathfrak{c})}(\tilde{R}_B^\wedge/R_B^\wedge) \hat{\otimes}_B k(\mathfrak{c})$ is semicontinuous for \mathfrak{c} in some open neighbourhood $\tilde{U} \subset \text{Spec } B$ of \mathfrak{b} .

Since $\pi : \text{Spec } B \rightarrow \text{Spec } A$ is étale it is open, $U := \pi(\tilde{U})$ is an open neighbourhood of \mathfrak{p} in $\text{Spec } A$, and for any $\mathfrak{q} \in U$ there exists a $\mathfrak{c} \in \tilde{U}$ with $\mathfrak{c} \cap A = \mathfrak{q}$. From Lemma [GP20, Lemma 39] we obtain

$$\begin{aligned} \dim_{k(\mathfrak{c})}(\tilde{R}_B^\wedge/R_B^\wedge) \hat{\otimes}_B k(\mathfrak{c}) &= \dim_{k(\mathfrak{c})}(\tilde{R}_B^h/R_B^h) \otimes_B^h k(\mathfrak{c}) \\ &= \dim_{k(\mathfrak{q})}(\tilde{R}_A/R_A) \hat{\otimes}_A k(\mathfrak{q}). \end{aligned}$$

We have $(\tilde{R}_A/R_A) \hat{\otimes}_A k(\mathfrak{q}) = \tilde{R}_{\mathfrak{q}}/R_{\mathfrak{q}}$ by Lemma 16 and $\dim_{k(\mathfrak{q})} \tilde{R}_{\mathfrak{q}}/R_{\mathfrak{q}} = \delta_{k(\mathfrak{q})}(R_{\mathfrak{q}})$ by Lemma 8. This implies the semicontinuity of $\delta_{k(\mathfrak{q})}(R_{\mathfrak{q}})$ on U . \square

Remark 20. In contrast to delta we cannot expect any semicontinuity for the conductor. In Example 21 (2) $c_{\mathbb{k}}(R_{\mathfrak{p}_{\lambda}})$ is not upper semicontinuous. In general $c_{\mathbb{k}}$ is also not lower semicontinuous (e.g. for plane curves $c_{\mathbb{k}} = 2\delta_{\mathbb{k}}$ and $\delta_{\mathbb{k}}$ is not lower semicontinuous.)

Two important examples are $A = \mathbb{k}[s]$ and $A = \mathbb{Z}$, treated in the examples below.

Example 21. We give two concrete examples with $A = \mathbb{k}[s]$. In both examples delta is of course upper semicontinuous, but in the second example the conductor is not.

(1) Consider the family of parameterizations over A ,

$$\begin{aligned} \varphi_A : P_A &= A[[x, y, z, u]] \rightarrow \tilde{R}_A = A[[t]], \\ x &\mapsto t^5, y \mapsto t^6, z \mapsto st^4 + t^8, u \mapsto t^9, \end{aligned}$$

²Let A be a Henselian ring. An A -algebra R is an A -algebra of finite type in the Henselian sense if $R \cong A\langle y_1, \dots, y_s \rangle$ for suitable $y_1, \dots, y_s \in R$.

shortly $\varphi_A(s, t) = (t^5, t^6, st^4 + t^8, t^9)$, with $\varphi_A(P_A) = \mathbb{k}[s][[t^5, t^6, st^4 + t^8, t^9]] \subset \mathbb{k}[s][[t]]$. For the maximal ideals (closed points in $\text{Spec } A$) $\mathfrak{p}_\lambda = \langle s - \lambda \rangle$, $\lambda \in \mathbb{k}$, we have thus (in the notations of Definition 14) the family

$$\varphi_{\mathfrak{p}_\lambda} : P_{\mathfrak{p}_\lambda} = \mathbb{k}[[x, y, z, u]] \rightarrow \tilde{R}_{\mathfrak{p}_\lambda} = \mathbb{k}[[t]], \quad \varphi_{\mathfrak{p}_\lambda}(t) = (t^5, t^6, \lambda t^4 + t^8, t^9)$$

of reduced curve singularities $R_{\mathfrak{p}_\lambda} = \mathbb{k}[[x, y, z, u]]/\text{Ker}(\varphi_{\mathfrak{p}_\lambda}) \cong \mathbb{k}[[t^5, t^6, \lambda t^4 + t^8, t^9]] \subset \mathbb{k}[[t]]$. The parameterization is primitive and $\tilde{R}_{\mathfrak{p}_\lambda}$ is the normalization of $R_{\mathfrak{p}_\lambda}$ (Lemma 8). We compute easily $\delta_{\mathbb{k}}(R_{\mathfrak{p}_\lambda}) = 5$ if $\lambda = 0$ and $\delta_{\mathbb{k}}(R_{\mathfrak{p}_\lambda}) = 4$ if $\lambda \neq 0$. Moreover, $c_{\mathbb{k}}(R_{\mathfrak{p}_\lambda}) = 8$ for all λ .

Every closed point of $\text{Spec } A$ is in the closure of the generic point $\eta = \langle 0 \rangle$. Theorem 18 implies that $\delta_{\mathbb{k}(t)}(R_\eta) \leq \delta_{\mathbb{k}}(R_{\mathfrak{p}_\lambda})$, $R_\eta = k(s)[[t^5, t^6, st^4 + t^8, t^9]]$, for any $\lambda \in \mathbb{k}$.

(2) Let $\varphi_A(s, t) = (t^5, t^6, st^7 + t^8, t^9)$. Then for $\mathfrak{p} = \langle s - \lambda \rangle$, $\lambda \in \mathbb{k}$, we get $R_{\mathfrak{p}_\lambda} \cong \mathbb{k}[[t^5, t^6, \lambda t^7 + t^8, t^9]]$ and we compute $\delta_{\mathbb{k}}(R_{\mathfrak{p}_\lambda}) = 5$ for all $\lambda \in \mathbb{k}$. On the other hand we get $c_{\mathbb{k}}(R_{\mathfrak{p}_\lambda}) = 8$ if $\lambda = 0$ and $c_{\mathbb{k}}(R_{\mathfrak{p}_\lambda}) = 9$ if $\lambda \neq 0$.

Example 22. We illustrate the difference between $R_{\mathfrak{p}}$ and $R_A \hat{\otimes}_A k(\mathfrak{p})$ from Remark 17. Consider the previous example $\varphi_A(s, t) = (t^5, t^6, st^4 + t^8, t^9)$, and use SINGULAR [DGPS] for the following computations.

We eliminate t from $\langle x - t^5, y - t^6, z - st^4 + t^8, u - t^9 \rangle$ and get $R_A = \mathbb{k}[s][[x, y, z, u]]/I$ with $I := \text{Ker}(\varphi_A) = \langle yz - xu - x^2s, yu - x^3, zu - xy^2 - xzs + us^2, u^2 - x^2z + xus, y^3 - x^2z + xus, xz^2 - y^2u - zus - xy^2s, z^3 - yu^2 - 2y^2zs - xyus - x^2ys^2 - y^2s^3 \rangle$.

For $\mathfrak{p}_\lambda = \langle s - \lambda \rangle$, $\lambda \in \mathbb{k}$, we have $R_\lambda := R_A \hat{\otimes}_A k(\mathfrak{p}_\lambda) = \mathbb{k}[[x, y, z, u]]/(I|_{s=\lambda})$. If $\lambda \neq 0$ R_λ is reduced and coincides with the reduced curve $R_{\mathfrak{p}_\lambda}$ considered in Example 21. But for $\lambda = 0$ the ring R_0 has an embedded component (it can be computed as $\langle u, z^3, yz, y^3, xz^2, xy^2, x^2z, x^3 \rangle$), while the reduction of R_0 coincides with $R_{\mathfrak{p}_0}$.

Example 23. The case $A = \mathbb{Z}$ is of special interest. Consider a parameterization of an irreducible curve singularity with integer coefficients, $t \mapsto (\varphi^1(t), \dots, \varphi^n(t))$ with $\varphi^i(t) \in \mathbb{Z}[[t]]$. For $\mathfrak{p} \subset \mathbb{Z}$ a prime ideal we write $\varphi_p^i \in \mathbb{F}_p[[t]]$ if $\mathfrak{p} = \langle p \rangle$ with p a prime number, and $\varphi_0^i \in \mathbb{Q}[[t]]$ if $\mathfrak{p} = \langle 0 \rangle$.

Theorem 18 implies: If for some fixed prime p the parameterization φ_p is primitive, then $\dim_{\mathbb{F}_p} \mathbb{F}_p[[t]]/\mathbb{F}_p[[\varphi_p^1(t), \dots, \varphi_p^n(t)]] = \delta_{\mathbb{F}_p}(R_p)$ is finite and $\delta_{\mathbb{F}_p}(R_p) \geq \delta_{\mathbb{Q}}(R_0)$ as well as $\delta_{\mathbb{F}_p}(R_p) \geq \delta_{\mathbb{F}_q}(R_q)$ for all except finitely many primes $q \in \mathbb{Z}$. In particular, if there exists a prime number p with $\delta_{\mathbb{F}_p}(R_p) < \infty$ then $\delta_{\mathbb{Q}}(R_0) < \infty$. Conversely, if $\delta_{\mathbb{Q}}(R_0)$ is finite, then $\delta_{\mathbb{Q}}(R_0) \geq \delta_{\mathbb{F}_q}(R_q)$ (and hence “=” for all except finitely many prime numbers $q \in \mathbb{Z}$).

The same result holds for algebraic power series, i.e., if we replace $[\dots]$ by $\langle \dots \rangle$.

Remark 24 (Computational consequences). Let A be an integral domain and $\eta = \langle 0 \rangle$ the generic point of $\text{Spec } A$. Then any other point of $\text{Spec } A$ is in the closure of η and it follows for a family of parameterizations

$\varphi_A : P_A \rightarrow \tilde{R}_A$ (assumptions as in Theorem 18) that

$$(**) \quad \delta_{k(\eta)}(R_\eta) \leq \delta_{k(\mathfrak{p})}(R_\mathfrak{p})$$

for any $\mathfrak{p} \in \text{Spec } A$.

There exist several algorithms to compute $\delta_{\mathbb{k}}$ for a given concrete parameterization, eg. by computing the normalization as in [GP08] by the Grauert-Remmert algorithm or by the Singh-Swanson algorithm in positive characteristic, or by computing a Puiseux expansion (in char 0) resp. a Hamburger-Noether expansion (in char > 0) as in [Ca80] or by still other algorithms (see also the Manual of SINGULAR [DGPS]).

- *Computing in special points:* In applications it is often required to know the delta invariant (or at least a bound for it) at a generic point η . However the computation over $k(\eta) = \text{Quot}(A)$ are often extremely expansive (e.g. $k(\eta) = \mathbb{k}(t)$ for $A = \mathbb{k}[t]$ or $k(\eta) = \mathbb{Q}$ for $A = \mathbb{Z}$), while the computations at a special point (e.g. $\lambda \in \mathbb{k}$ or p a prime number) are usually much faster and this can be used as a bound for delta at the generic point. Depending on the chosen algorithm, the bound computed at a special point may be used also for early termination of the computation of delta at the generic point.

Our result says also that delta at the generic point coincides with delta at the special points in an open dense subset of $\text{Spec } A$ (a result which can also be obtained by the theory of Gröbner or standard bases) but this open set is usually not known. We like to emphasize, that the estimate $(**)$ is true for *every* special point, a result that cannot be deduced by Gröbner basis theory.

- *Cutting off higher order terms during computations:* By Corollary 12 the parameterization $\varphi_{\mathfrak{p}}$ of a reduced curve singularity $R_{\mathfrak{p}}$ is $(4\delta - 2)$ -determined, $\delta := \delta_{k(\mathfrak{p})}(R_{\mathfrak{p}})$. The semicontinuity implies that for any point \mathfrak{q} in some neighbourhood U of \mathfrak{p} (e.g. for the generic point $\mathfrak{q} = \eta$) in the power series $\varphi_{\mathfrak{q},j}(x_i) \in k(\mathfrak{q})[[t]]$ the terms of degree $> 4\delta - 2$ can be cut off without changing $\delta_{\varphi_{\mathfrak{q}}}$.

It is well known, that intermediate terms of very high degree may occur during a Gröbner or Sagbi basis computation, which cancel at the end. The bound $4\delta - 2$ can be used to cut off higher order terms that might appear during the computation of any \mathcal{L} (or \mathcal{A} or contact)-invariant of $\varphi_{\mathfrak{q}}$.

The analytic case: An analogous semicontinuity result for complex analytic families (with a much easier proof) may be of independent interest. A morphism of analytic \mathbb{C} -algebras $\varphi : \mathbb{C}\{x_1, \dots, x_n\} \rightarrow \mathbb{C}\{t_1\} \oplus \dots \oplus \mathbb{C}\{t_r\}$ is called a parameterization of the analytic curve singularity $Im(\varphi)$ with r branches if it satisfies the conditions as in Definition 7, with formal power series replaced by convergent ones. The notion of a primitive parameterization carries over to analytic \mathbb{C} -algebras and Lemma 8 holds also in the

analytic case (in particular, φ is primitive $\Leftrightarrow \varphi$ is the normalization map $\Leftrightarrow \dim_{\mathbb{C}} \mathbb{C}\{t_1\} \oplus \dots \oplus \mathbb{C}\{t_r\}/Im(\varphi) < \infty$).

Let $(S, 0)$ be a complex analytic germ and

$$\varphi_S : \mathcal{O}_{S,0}\{x_1, \dots, x_n\} \rightarrow \mathcal{O}_{S,0}\{t_1\} \oplus \dots \oplus \mathcal{O}_{S,0}\{t_r\}$$

an analytic $\mathcal{O}_{S,0}$ -algebra morphism. For S a representative of $(S, 0)$ and $s \in S$ let

$$\begin{aligned} \varphi_s &= \varphi_S \otimes_{\mathcal{O}_{S,s}} \mathbb{C} : \mathbb{C}\{x_1, \dots, x_n\} \rightarrow \mathbb{C}\{t_1\} \oplus \dots \oplus \mathbb{C}\{t_r\} \\ \varphi_{s,j} &: \mathbb{C}\{x_1, \dots, x_n\} \rightarrow \mathbb{C}\{t_j\}, \quad j = 1, \dots, r, \end{aligned}$$

be the induced maps. Then $\mathcal{O}_{C_s,0} := \mathbb{C}\{x_1, \dots, x_n\}/Ker(\varphi_s) \cong Im(\varphi_s) = \mathbb{C}\{\varphi_s(x_1), \dots, \varphi_s(x_n)\} \subset \mathbb{C}\{t_1\} \oplus \dots \oplus \mathbb{C}\{t_r\}$ is the local ring of the germ $(C_s, 0) = V(Ker(\varphi_s)) \subset (\mathbb{C}^n, 0)$.

Proposition 25. *With the above notations assume that φ_0 is a primitive parameterization of the reduced curve singularity with r branches $(C_0, 0)$. Then, for s in a sufficiently small neighbourhood of $0 \in S$, $(C_s, 0)$ is a reduced curve singularity with r branches satisfying*

$$\dim_{\mathbb{C}} \mathbb{C}\{t_1\} \oplus \dots \oplus \mathbb{C}\{t_r\}/\mathbb{C}\{\varphi_s(x_1), \dots, \varphi_s(x_n)\} = \delta_{\mathbb{C}}(\mathcal{O}_{C_s,0}) < \infty$$

and $\delta_{\mathbb{C}}(\mathcal{O}_{C_s,0}) \leq \delta_{\mathbb{C}}(\mathcal{O}_{C_0,0})$.

Proof. That $(C_s, 0)$ is a reduced curve singularity with r branches follows as in the formal case (Remark 15). Since primitive implies (Lemma 8) $\dim_{\mathbb{C}} Coker(\varphi_0) < \infty$ the Weierstrass Finiteness Theorem ([GLS07, Theorem 1.10]) implies that $Coker(\varphi_S)$ is a finite and hence a coherent \mathcal{O}_S -module (by ([GLS07, Theorems 1.66 and 1.67])). The result follows since $Coker(\varphi_S) \otimes_{\mathcal{O}_{S,s}} \mathbb{C} = Coker(\varphi_s)$ and $\dim_{\mathbb{C}} Coker(\varphi_s) = \delta_{\mathbb{C}}(\mathcal{O}_{C_s,0})$ as in Lemma 8. \square

Remark 26. Let \mathbb{k} be a real-valued field with a valuation and A an analytic \mathbb{k} -algebra, i.e. $A \cong \mathbb{k}\{y\}/I$, $y = (y_1, \dots, y_s)$, I an ideal, and $\mathbb{k}\{y\}$ the convergent power series ring over \mathbb{k} (cf. [GLS07]). Then small ε -neighbourhoods in \mathbb{k}^s are defined and the same semicontinuity result for $\delta_{\mathbb{k}}$ as in the complex case (Proposition 25) holds more generally for morphisms of analytic \mathbb{k} -algebras

$$\varphi_A : A\{x_1, \dots, x_n\} \rightarrow A\{t_1\} \oplus \dots \oplus A\{t_r\}.$$

The proof is basically the same as in the complex-analytic case. The Weierstrass Finiteness Theorem holds also in this case ([GLS07, Theorem 1.10]) and instead of the coherence theorem, one can use [GP20, Lemma 1].

Examples of real-valued fields with valuation are:

- Any field \mathbb{k} with the trivial valuation, then $\mathbb{k}\{y\} = \mathbb{k}[[y]]$ is the formal power series ring. Finite fields have only the trivial valuation.
- $\mathbb{k} \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$ with the absolute value as (Archimedean) valuation, then $\mathbb{k}\{y\}$ is the usual convergent power series ring.

- Another example is $\mathbb{k} = \mathbb{Q}$ with the p -adic valuation³, a non-Archimedian valuation on \mathbb{Q} .

- The completion of a real-valued field is again a real-valued field. The completion of the field \mathbb{Q} with respect to the ordinary absolute value is \mathbb{R} , while the completion of \mathbb{Q} with respect to the p -adic valuation is the field of p -adic numbers \mathbb{Q}_p ⁴.

The convergent power series ring over a real-valued field has similar good properties as the formal power series ring. The most important property is the Weierstrass Finiteness Theorem ([GLS07, Theorem 1.10]). Of interest may be that $\mathbb{k}\{y\}$ is excellent⁵ if and only if the completion of \mathbb{k} with respect to the valuation is separable over \mathbb{k} (cf. [Sch82]).

REFERENCES

- [Ca80] Campillo, A.: *Algebroid Curves in Positive Characteristic*. Lecture Notes in Mathematics, Springer (1980).
- [Ca05] Castellanos, J.: *The semigroup of a space curve singularity*. Pac. J. Math. 221, 227-251 (2005).
- [CL06] Chiang-Hsieh, H.-J.; Lipman, J.: *A numerical criterion for simultaneous normalization*, Duke Math. J., 133:2 (2006)
- [CLMN19] Cogolludo-Agustín, J., László, T., Martin-Morales, J., Némethi, A.: *Delta invariant of curves on rational surfaces I. The analytic approach*. arXiv:1911.07539v1 [math.AG] (2019).
- [DGPS] Decker, W.; Greuel, G.-M.; Pfister, G.; Schönemann, H.: SINGULAR 4-1-2 – A computer algebra system for polynomial computations. <http://www.singular.uni-kl.de> (2019).
- [GP08] Greuel, G.-M., Pfister, G.: *A Singular Introduction to Commutative Algebra*, 2nd Edition, Springer, Berlin (2008). With contributions by O. Bachmann, C. Lossen and H. Schönemann.
- [GP20] Greuel, G.-M., Pfister, G.: *Semicontinuity of Singularity Invariants in Families of Formal Power Series*, arXiv:1912.05263v3 (2020). To appear in the Proceedings of the "Némethi60 Conference".
- [GLS07] Greuel, G.-M.; Lossen, C.; Shustin, E.: *Introduction to Singularities and Deformations*. Springer (2007).
- [HK71] Herzog, J.; Kunz, E.: *Der kanonische Modul eines Cohen-Macaulay Rings*. Lecture Notes in Mathematics 238, Springer (1971).
- [Hi65] Hironaka, H.: *On the equivalence of singularities I*. Arithmetical algebraic Geom., Proc. Conf. Purdue Univ. 1963, 153-200 (1965).
- [KPP78] Kurke, H.; Pfister, G.; Popescu, D; Roczen, M.; Mostowski T.: *Die Approximationseigenschaft lokaler Ringe*. Lecture Notes in Mathematics 634, Springer-Verlag Berlin Heidelberg New York (1978).
- [IIL20] Ito, K., Ito, T., Liedtke, C.: *Deformations of rational curves in positive characteristic*. arXiv:1803.04448v5 [math.AG] (2020)
- [Mat86] Matsumura, H.: *Commutative ring theory*. Cambridge Univ. Press (1986).
- [Ng16] Nguyen, H. D.: *Invariants of plane curve singularities and Plücker formulas in positive characteristic*. Ann. Inst. Fourier 66, No. 5, 2047-2066 (2016).

³Write $q \in \mathbb{Q}$ as $q = p^a \cdot \frac{r}{s}$ with r, s coprime to p , then the p -adic valuation is $|q|_p = p^{-a}$.

⁴ $\mathbb{Q}_p = \{\sum_{i=n}^{\infty} a_i p^i | n \in \mathbb{Z}, a_i \in \{0, \dots, p-1\}\}$. $\mathbb{Q} \subset \mathbb{Q}_p$ since $-1 = \sum_0^{\infty} (p-1)p^i$.

⁵For a definition of excellence see [Stack, 15.51].

- [Sch82] Schemmel, K.-P.: *Eine notwendige und hinreichende Bedingung für die Approximationseigenschaft analytischer Potenzreihenringe über einem Körper beliebiger Charakteristik.* Rev. Roum. Math. Pures Appl. 27, 875-884 (1982).
- [Stack] *Stacks project.* <http://stacks.math.columbia.edu>. Version c716e69b, compiled on Sep 21, 2019.
- [Te78] Teissier, B.: *The hunting of invariants in the geometry of discriminants.* In: P. Holm (ed.): Real and Complex Singularities, Oslo 1976, Northholland (1978).

Gert-Martin Greuel and Gerhard Pfister
University of Kaiserslautern
Department of Mathematics
Erwin-Schroedinger Str.
67663 Kaiserslautern
Germany
e-mail: greuel@mathematik.uni-kl.de
e-mail: pfister@mathematik.uni-kl.de