Constructive General Neron Desingularization for one dimensional local rings

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Abstract

An algorithmic proof of General Neron Desingularization is given here for one dimensional local rings and it is implemented in Singular. Also a theorem recalling Greenberg' strong approximation theorem is presented for one dimensional local rings.

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1. Introduction

A ring morphism $u : A \to A'$ has *regular fibers* if for all prime ideals $P \in \operatorname{Spec} A$ the ring A'/PA' is a regular ring, i.e. its localizations are regular local rings. It has *geometrically regular fibers* if for all prime ideals $P \in \operatorname{Spec} A$ and all finite field extensions K of the fraction field of A/P the ring $K \otimes_{A/P} A'/PA'$ is regular. If for all $P \in \operatorname{Spec} A$ the fraction field of A/P has characteristic 0 then the regular fibers of u are geometrically regular fibers. A flat morphism u is *regular* if its fibers are geometrically regular. If u is regular of finite type then u is called *smooth*. A localization of a smooth algebra is called *essentially smooth*.

In Artin approximation theory (introduced in [2]) an important result (see [16]) is the following theorem, generalizing the Neron Desingularization [9], [2].

Theorem 1 (General Neron Desingularization, Popescu [11], [12], [13], André [1], Swan [20], Spivakovski [19]). *Let $u : A \to A'$ be a regular morphism of Noetherian rings and B a finite type A-algebra. Then any A-morphism $v : B \to A'$ factors through a smooth A-algebra C, that is v is a composite A-morphism $B \to C \to A'$.*

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The purpose of this paper is to give an algorithmic proof of the above theorem when A, A' are one dimensional local rings, that is Theorem 2. When A, A' are domains such algorithm is given in [10], the case when A, A' are discrete valuation rings proved by Néron [9] is given in a different way in [15] with applications in arcs frame. The present algorithm was implemented by the authors in the Computer Algebra system SINGULAR [3] and will be as soon as possible found in a development version at
`https://github.com/Singular/Sources/blob/spielwiese/Singular/LIB/.`

The proof of Theorem 2 splits essentially in three steps. We will give here the idea in case A and A' are domains. In step 1 we reduce the problem to the case when $H_{B/A} \cap A \neq 0$, $H_{B/A}$ being the ideal defining the nonsmooth locus of B over A. Let $0 \neq d \in H_{B/A} \cap A$. This means geometrically that $\operatorname{Spec} B_d \to \operatorname{Spec} A_d$ is smooth. In the second step we construct a smooth A-algebra D, $A \subset D \subset A'$ and an A-morphism $v' : B \to D/d^3D$ such that $v \equiv v'$ modulo d^3A'. If A' is the completion $\hat{A}$ of A we can use $D = A$. The third step resolves the singularity. If $B = A[Y]/I$, $Y = (Y_1, \ldots, Y_n)$ then we can find $f = (f_1, \ldots, f_r)$, $r \leq n$ a system of polynomials from I (given a tuple $(b_1, \ldots b_s)$ we denote usually by the corresponding unindexed letter b the vector $(b_1, \ldots b_s)$), and an $r \times r$-minor M of the Jacobian matrix $(\partial f_i/\partial Y_j)$ such that $d \equiv MN$ modulo I for some $N \in (f) : I)$, where (f) denotes the ideal generated by the system f. Then $v'(MN) = ds$ for some $s \in 1 + dD$. Assume that $M = \det(\partial f_i/\partial Y_j)_{1 \leq i,j \leq r}$. Let H be the matrix obtained by adding to $(\partial f/\partial Y)$ the boarder block $(0|\mathrm{Id}_{n-r})$ and let G' be the adjoined matrix of H and $G = NG'$. Consider in $D[Y, T]$, $T = (T_1, \ldots, T_n)$ the ideal $J = ((f, s(Y - y') - dG(y')T) : d^2)$, where $y' \in D^n$ is lifting $v'(Y)$. Then C is a suitable localization of the $B \otimes_A D$-algebra $D[Y, T]/(I, J)$ and v extends to C by $v(T) = t = (1/d^2)H(y')(v(Y) - y')$.

Consider the following example. Let $A = \mathbf{Q}[x]_{(x)}$, $A' = \mathbf{C}[[x]]$, $B = A[Y_1, Y_2]/(Y_1^2 + Y_2^2)$, $a \in \mathbf{C}$ a transcendental element over $\mathbf{Q}$, $\bar{u} \in \mathbf{C}[[x]] \setminus \mathbf{C}[x]_{(x)}$ and $u = a + x^6\bar{u}$. Let v be given by $v(Y_1) = xu$, $v(Y_2) = xiu$, where $i = \sqrt{-1}$. In step 1 we change B by $B_1 = A[Y_1, Y_2, Y_3]/I$, $I = (Y_1^2 + Y_2^2, x - 2Y_1Y_3)$ and extend v by $v(Y_3) = 1/(2u)$. We have $4Y_1^2Y_3^2 \in H_{B/A}$ which implies $d = x^2 \in H_{B/A} \cap A$. We define $D = A[a, a^{-1}, i]$ and $v'(Y) = y' = (xa, xia, 1/(2a))$. This is step two.

To understand step 3 we simplify the example taking $B = A[Y_1, Y_2]/(Y_1Y_2 - x^2)$, $u = 1 + x^6\bar{u}$ and v given by $v(Y_1) = xu$, $v(Y_2) = x/u$. Then $d = x \in H_{B/A}$, $D = A$, $y_1' = x = y_2'$. We obtain $H = \begin{pmatrix} Y_2 & Y_1 \\ 0 & 1 \end{pmatrix}$, $G = G' = \begin{pmatrix} 1 & -Y_1 \\ 0 & Y_2 \end{pmatrix}$, $N = 1$ and $J = ((Y_1Y_2 - x^2, Y_1 - x - xT_1 + x^2T_2, Y_2 - x - x^2T_2) : x^2)$. We have $J = (xT_1T_2 - x^2T_2^2 + T_1, Y_1 - x - xT_1 + x^2T_2, Y_2 - x - x^2T_2)$ and we obtain that $C \cong (A[T_1, T_2]/(xT_1T_2 - x^2T_2^2 + T_1))_{1+xT_2} \cong (A[T_2])_{1+xT_2}$ is a smooth A-algebra.

When A' is the completion of a Noetherian local ring A of dimension one we show that we may have a linear Artin function as it happens in the Greenberg's case (see [5] and [10, Theorem 18]). More precisely, the Artin function is given by $c \to (\rho + 1)(e + 1) + c$, where e, ρ depend on A and the polynomial system of equations defining B (see Theorem 14).

2. Theorem 1 in one dimensional local rings

Let $u : A \to A'$ be a flat morphism of Noetherian local rings of dimension 1. Suppose that the maximal ideal $\mathfrak{m}$ of A generates the maximal ideal of A'. Moreover suppose that u is a regular morphism, k is infinite and there exist canonical inclusions $k = A/\mathfrak{m} \subset A$, $k' = A'/\mathfrak{m}A' \subset A'$ such that $u(k) \subset k'$.

Let $B = A[Y]/I$, $Y = (Y_1, \dots, Y_n)$. If $f = (f_1, \dots, f_r)$, $r \leq n$ is a system of polynomials from I then we can define the ideal Δ_f generated by all $r \times r$-minors of the Jacobian matrix $(\partial f_i / \partial Y_j)$. After Elkik [4] let $H_{B/A}$ be the radical of the ideal $\sum_f ((f) : I)\Delta_f B$, where the sum is taken over all systems of polynomials f from I with $r \leq n$. Then B_P, $P \in \operatorname{Spec} B$ is essentially smooth over A if and only if $P \not\supset H_{B/A}$ by the Jacobian criterion for smoothness. Thus $H_{B/A}$ measures the non smooth locus of B over A. B is *standard smooth* over A if there exists f in I as above such that $B = ((f) : I)\Delta_f B$.

The aim of this paper is to give an easy algorithmic proof of the following theorem.

Theorem 2. *Any A-morphism $v : B \to A'$ factors through a standard smooth A-algebra B'.*

We consider in the algorithmic part the following assumption, which we will keep in the whole paper

$(*)$ A is essentially of finite type over a field k, let us say $A \cong (k[x]/J)_{(x)}$ for some variables $x = (x_1, \dots x_m)$, and the completion of A' is $k'[[x]]/(J)$.

When v is defined by polynomials y from $k'[x]$ then our problem is easy. Let L be the field obtained by adjoining to k all coefficients of y. Then $R = (L[x]/(J))_{(x)}$ is a subring of A' containing $\operatorname{Im} v$ which is essentially smooth over A. Then we may take B' as a standard smooth A-algebra such that R is a localization of B'. Thus we will not suppose in this paper that y is polynomial and therefore L is not necessarily a finite type field extension of k.

In the proof we need to know that $v(H_{B/A})$ is not contained in any minimal prime ideal of A'. In theory, we may reduce to this case as it follows. Let $\mathfrak{p} \in \operatorname{Min} A'$. Since u is regular, it induces a regular map $u_{\mathfrak{p}} : A_{(\mathfrak{p} \cap A)} \to A'_{\mathfrak{p}}$ of local Artinian rings (in particular $k(\mathfrak{p}) \otimes_{A_{\mathfrak{p} \cap A}} u_p$, $k(\mathfrak{p}) = A_{(\mathfrak{p} \cap A)}/(\mathfrak{p} \cap A)A_{(\mathfrak{p} \cap A)}$ is a separable field extension). Note that $A_{(\mathfrak{p} \cap A)} \supset k(\mathfrak{p})$ because of $(*)$ and $A'_{\mathfrak{p}}$ is a filtered inductive limit of its subrings of the form $E_{F_{\mathfrak{p}}} = A_{(\mathfrak{p} \cap A)} \otimes_{k(\mathfrak{p})} F_{\mathfrak{p}}$ for all finite type field subextension $F_{\mathfrak{p}}/k(\mathfrak{p})$ of $(A'_{\mathfrak{p}}/\mathfrak{p}A'_{\mathfrak{p}})/k(\mathfrak{p})$. We may change B by a finite type B-algebra $\tilde{B}$ of A' such that $\tilde{B}_{\mathfrak{p} \cap \tilde{B}} \cong E_{F_{\mathfrak{p}}}$. It follows that $v(H_{B/A}) \not\subset \mathfrak{p}A'$ for all $\mathfrak{p} \in \operatorname{Min} A'$. Next we will assume from the beginning that

$(**)$ $v(H_{B/A}) \not\subset \mathfrak{p}A'$ for all $\mathfrak{p} \in \operatorname{Min} A'$.

Unfortunately, the computer cannot decide this since we are not able to give the whole information concerning the coefficients of y. But we are able to decide if $v(H_{B/A})$ is not contained in $\mathfrak{m}^N A' + \mathfrak{p}$ for $N >> 0$ and one $\mathfrak{p} \in \operatorname{Min} A'$. In the following we suppose that $v(H_{B/A}) \not\subset \mathfrak{m}^N A' + \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Min} A'$ and a certain $N >> 0$. Choose $\gamma \in H_{B/A}$ such that $v(\gamma)$ is not in $\cup_{\mathfrak{p} \in \operatorname{Min} A'} \mathfrak{p} + \mathfrak{m}^N A'$.

The idea of the proof of Theorem 2 is to find $f = (f_1, \dots, f_r)$ in I and a $d \in v(((f) : I)\Delta_f)A' \cap A$ which is not in $\cup_{\mathfrak{p} \in \operatorname{Min} A'} \mathfrak{p}$. The assumption $(**)$ gives just that there exists $f_{\mathfrak{p}}$ for any $\mathfrak{p} \in \operatorname{Min} A'$ such that $\Delta_{f_{\mathfrak{p}}}((f_{\mathfrak{p}}) : I) \not\subset \mathfrak{p}A'$ for all $\mathfrak{p} \in \operatorname{Min} A'$. The main problem is to reduce to the case when $f_{\mathfrak{p}}$ does not depend of $\mathfrak{p}$. Actually, this follows if I_γ / I_γ^2 is free over B_γ. In the next three lemmas containing some results of Elkik [4] (in the form used in [11], [20], [14]), we see that this is true if we reduce to the case when Ω_{B_γ}/A is free over B_γ. In general, the last module is projective but not free as the following example shows.

Example 3. Let k be a subfield of $\mathbf{R}$, $A = (k[x_1, x_2, x_3]/(x_1^2 - x_2 x_3, x_3^2 - x_1 x_2))_{(x_1, x_2, x_3)}$, and $\alpha = x_1 Y_1^2 + x_2 Y_2^2 + x_3 Y_3^2 - x_1 - x_2 - x_3 \in A[Y]$, $Y = (Y_1, Y_2, Y_3)$. Set $f_1 = x_2 \alpha = x_3^2 Y_1^2 + x_2^2 Y_2^2 + x_1^2 Y_3^2 - x_3^2 - x_2^2 - x_1^2$, $f_2 = x_1 \alpha$, $f_3 = x_3 \alpha$ and $I = (f)$. Then $x_2 I \subset (f_1)$, $x_1 I \subset (f_2)$, $x_3 I \subset (f_3)$. Also note that $x_2^2 Y_2 \in \Delta_{f_1}$, $x_1^2 Y_1 \in \Delta_{f_2}$, $x_3^2 Y_3 \in \Delta_{f_3}$.

Let $B = A[Y]/I$, $A' = k'[[x_1, x_2, x_3]]/(x_1^2 - x_2 x_3, x_3^2 - x_1 x_2)$, where $k \subset k'$ is a field extension. Let u_1, u_2 be two algebraically independent elements of $k'[[x_1, x_2, x_3]]$ over $k[x_1, x_2, x_3]$. Set $y_1 =$

$x_3u_1 - 1$, $y_2 = x_3u_2 - 1$ and we may find y_3 such that $y_3^2 = 1 - x_1x_3u_1^2 + 2x_1u_1 - x_2x_3u_2^2 + 2x_2u_2$. Clearly, $\alpha(y) = 0$, that is $\alpha(y_i) = 0$ for all $1 \le i \le n$, and so we get an A-morphism $v : B \to A'$ given by $Y \to y$.

Note that $H_{B/A} = (x_1, x_2, x_3)$. Take $\gamma = x_1 + x_2 + x_3$ and $\beta = f_1 + f_2 + f_3$. Then $I_\gamma = (\beta)_\gamma = (\alpha)_\gamma$ and we claim that $\Omega_{B_\gamma/A}$ is projective but not free. Indeed, $\Omega_{B/A} = BdY_1 \oplus BdY_2 \oplus BdY_3/(x_1Y_1dY_1 + x_2Y_2dY_2 + x_3Y_3dY_3)$ and $\Omega_{B_\gamma/A}$ is projective because its Fitting ideal is $(x_1Y_1, x_2Y_2, x_3Y_3)B_\gamma \supset (x_1Y_1^2 + x_2Y_2^2 + x_3Y_3^2)B_\gamma = \gamma B_\gamma = B_\gamma$ (see e.g. [7, Proposition 1.3.8]).

Now suppose that $\Omega_{B_\gamma/A}$ is free over B_γ. Then $\lambda = x_1Y_1 + x_2Y_2 + x_3Y_3$ can be included in a basis of $B_\gamma dY_1 \oplus B_\gamma dY_2 \oplus B_\gamma dY_3$. More precisely, there exists a 3×3 invertible matrix (a_{ij}), $a_{ij} = a_{ij}(x, Y)$ over B_γ with $a_{1j} = x_jY_j$ for $j \in [3]$, let us say $a_{ij} = b_{ij}(x, Y)/c(x)$, $j = 2, 3$ with $b_{ij} \in k[x, Y]$, $c \in k[x]$. We may choose some positive real numbers $x'_1.x'_2, x'_3$ such that $x'^2_1 = x'_2x'_3$, $x'^2_3 = x'_1x'_2$, $c(x') \neq 0$ and the matrix $(a_{ij}(x', Y))$ invertible in $B' = \mathbf{R}[Y]/(x'_1Y_1^2 + x'_2Y_2^2 + x'_3Y_3^2 - x'_1 - x'_2 - x'_3)$. It follows that $P' = (B'dY_1 \oplus B'dY_2 \oplus B'dY_3)/ < x'_1Y_1dY_1 + x'_2Y_2dY_2 + x'_3Y_3dY_3 >$ is free over B'. Changing Y_i by $\sqrt{x'_i/(x'_1 + x'_2 + x'_3)}Y_i$ we see that over $B'' = \mathbf{R}[Y]/(Y_1^2 + Y_2^2 + Y_3^2 - 1)$ the module $P'' = (B''dY_1 \oplus B''dY_2 \oplus B''dY_3)/ < Y_1dY_1 + Y_2dY_2 + Y_3dY_3 >$ is free, that is the tangent bundle over the real sphere is trivial which contradicts for example [8, page 114].

On the other hand, note that $\Omega_{B_\gamma[Y_4]/A} =$

$$B_\gamma[Y_4]dY_1 \oplus B_\gamma[Y_4]dY_2 \oplus B_\gamma[Y_4]dY_3 \oplus B_\gamma[Y_4]dY_4/ < x_1Y_1dY_1 + x_2Y_2dY_2 + x_3Y_3dY_3 >$$

is free because in $\sum_{i=1}^4 B_\gamma[Y_4]dY_i$ we have the basis

$$\{x_1Y_1dY_1 + x_2Y_2dY_2 + x_3Y_3dY_3, dY_1 - Y_1dY_4, dY_2 - Y_2dY_4, dY_3 - Y_3dY_4\}.$$

Let B_1 be the symmetric algebra $S_B(I/I^2)$ of I/I^2 over B. Then $(B_1)_\gamma$ is the symmetric algebra $S_{B_\gamma}((I/I^2)_\gamma)$ of $(I/I^2)_\gamma$ over B_γ. But $(I/I^2)_\gamma$ is free generated by α and so $(B_1)_\gamma \cong B_\gamma[Y_4]$. Consequently, $\Omega_{(B_1)_\gamma/A}$ is free from above.

Lemma 4. *([11, Lemma 3.4]) Let B_1 be the symmetric algebra $S_B(I/I^2)$ of I/I^2 over*[1] *B. Then $H_{B/A}B_1 \subset H_{B_1/A}$ and $(\Omega_{B_1/A})_\gamma$ is free over $(B_1)_\gamma$.*

Lemma 5. *([20, Proposition 4.6]) Suppose that $(\Omega_{B/A})_\gamma$ is free over B_γ. Let $I' = (I, Y') \subset A[Y, Y']$, $Y' = (Y'_1, \ldots, Y'_n)$. Then $(I'/I'^2)_\gamma$ is free over B_γ.*

Lemma 6. *([14, Corollary 5.10]) Suppose that $(I/I^2)_\gamma$ is free over B_γ. Then a power of γ is in $((g) : I)\Delta_g$ for some $g = (g_1, \ldots g_r)$, $r \le n$ in I.*

Step 1. Reduction to the case when $\Omega_{B_\gamma/A}$ is free over B_γ .

Let B_1 be given by Lemma 4. The inclusion $B \subset B_1$ has a retraction w which maps I/I^2 to zero. For the reduction we change B, v by B_1, vw.

Step 2. Reduction to the case when $(I/I^2)_\gamma$ is free over B_γ.

Since $\Omega_{B_\gamma/A}$ is free over B_γ we see using Lemma 5 that changing I with $(I, Y') \subset A[Y, Y']$ we may suppose that $(I/I^2)_\gamma$ is free over B_γ.

Step 3. Reduction to the case when a power of γ is in $((f) : I)\Delta_f$ for some $f = (f_1, \ldots f_r)$, $r \le n$ in I.

We reduced to the case when $(I/I^2)_\gamma$ is free over B_γ. Then it is enough to use Lemma 6.

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[1] Let M b e a finitely represented B-module and $B^m \xrightarrow{(a_{ij})} B^n \to M \to 0$ a presentation then $S_B(M) = B[T_1, \ldots, T_n]/J$ with $J = (\{\sum_{i=1}^n a_{ij}T_i\}_{j=1,\ldots,m})$.

Step 4. The Jacobian matrix $(\partial f/\partial Y)$ can be completed with $(n-r)$ rows from k^n obtaining a square n matrix H with $v(\det H) \notin \cup_{\mathfrak{p}\in \mathrm{Min}\, A'}\mathfrak{p}$.

We may suppose that $r < n$, otherwise there exist nothing to show. Note that the rows of $(\partial f/\partial Y)$ are mapped by v in r linear independent vectors from $(A'/\mathfrak{p})^n$ for each $\mathfrak{p} \in \mathrm{Min}\, A'$. Fix a $\mathfrak{p}$ and consider the set $\Lambda_{\mathfrak{p}}$ of all $(n-r)$ linear independent vectors from k^n which define a basis in $Q(A'/\mathfrak{p})^n$ together with the rows of $v(\partial f/\partial Y)$. Clearly $L_{\mathfrak{p}}$ is a nonempty open Zariski set of $k^{n(n-r)}$. Since $k^{n(n-r)}$ is irreducible we get $\cap_{\mathfrak{p}\in \mathrm{Min}\, A'} L_{\mathfrak{p}} \neq \emptyset$. Choosing $(n-r)$ rows from $\cap_{\mathfrak{p}\in \mathrm{Min}\, A'} L_{\mathfrak{p}}$ we may complete $(\partial f/\partial Y)$ to the wanted matrix H.

Step 5. Reduction to the case when $((\det H)((f):I)) \cap A$ is $\mathfrak{m}$ primary.

By Step 3 there exists a polynomial $R' \in ((f):I)$ such that $v(R')$ is not in $\cup_{p\in \mathrm{Min}\, A'} p$. Set $P' = R' \det H$. Then $v(P')$ generates in A' an ideal of height 1 which must be $\mathfrak{m}A'$ primary. Then $(v(P')) \cap A$ is $\mathfrak{m}$ primary too and we may choose $d' \in (v(P')) \cap A$ such that $d'A$ is $\mathfrak{m}$ primary, let us say $d' = v(P')z$ for some $z \in A'$. Set $B_1 = B[Z]/(f_{r+1})$, where $f_{r+1} = -d' + P'Z$ and let $v_1 : B_1 \to A'$ be the map of B-algebras given by $Z \to z$. It follows that $d' \in ((f, f_{r+1}) : (I, f_{r+1}))$ and $d' \in \Delta_f$, $d' \in \Delta_{f_{r+1}}$. Then $d = d'^2 \equiv P$ modulo (I, f_{r+1}) for $P = P'^2 Z^2 \in H_{B_1/A}$. For the reduction change B by B_1 and H by $\begin{pmatrix} H & 0 \\ * & P' \end{pmatrix}$. Note that $d \in ((\det H)((f):I)) \cap A$. The determinant of the new H is the determinant of the old H multiplied with P'. Thus P is the determinant of the new H multiplied with $R = R'Z^2$

Remark 7. In Example 3 the module $\Omega_{B_\gamma/A}$ is not free and so we may apply Step 1. In fact we do not need to apply the Steps 1, 2, 3 because $(I/I^2)_\gamma$ is already free.

3. Proof of the case when $((\det H)((f):I)) \cap A$ is $\mathfrak{m}$ primary.

Let $(0) = \cap_{\mathfrak{p}\in \mathrm{Ass}\, A} Q_{\mathfrak{p}}$ be a reduced primary decomposition [2] of (0) in A, where $Q_{\mathfrak{p}}$ is a primary ideal with $\sqrt{Q_{\mathfrak{p}}} = \mathfrak{p}$. Let d be defined as in the end of Section 2. Define e by $(0 :_A d^e) = (0 :_A d^{e+1})$. This equality happens for example taking e such that $\mathfrak{p}^e \subset Q_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Ass}\, A$. Set $\bar{A} = A/(d^{2e+1})$, $\bar{A}' = A'/d^{2e+1}A'$, $\bar{u} = \bar{A} \otimes_A u$, $\bar{B} = B/d^{2e+1}B$, $\bar{v} = \bar{A} \otimes_A v$. By base change $\bar{u}$ is a regular morphism of Artinian local rings.

Step 6. There exists a smooth A-algebra and an A-morphism $\omega : D \to A'$ such that $y = v(Y) \in \mathrm{Im}\, \omega + d^{2e+1}A'$.

We extend the proof of [15, Theorem 10] in our case. But now A' is not the completion of A, that is the coefficients of y in x are not necessarily from k. Fortunately, as in the proof of [15, Theorem 10] we need only a finite number of this coefficients, namely those of monomials x which are not in $d^{2e+1}A'$. This is the reason to ask for the existence of such D, ω.

By [6, 19,7.1.5] for every field extension L/k there exists a flat complete Noetherian local $\bar{A}$-algebra $\tilde{A}$, unique up to an isomorphism, such that $\mathfrak{m}\tilde{A}$ is the maximal ideal of $\tilde{A}$ and $\tilde{A}/\mathfrak{m}\tilde{A} \cong L$. It follows that $\tilde{A}$ is Artinian. On the other hand, we may consider the localization A_L of $L \otimes_k \bar{A}$ in $\mathfrak{m}(L \otimes_k \bar{A})$ which is Artinian and so complete. By uniqueness we see that $A_L \cong \tilde{A}$. It follows that $\bar{A}' \cong A_{k'}$. Note that A_L is essentially smooth over A by base change and $\bar{A}'$ is a filtered union of sub-$\bar{A}$-algebras A_L with L/k finite type field sub extensions of k'/k.

[2] The primary decomposition of an ideal in a polynomial ring and its localizations by a maximal ideal can be computed using the SINGULAR library primdec.lib. This is a very difficult task in the computational algebra usually needing a lot of Gröbner basis computations with respect to the lexicographical ordering.

Choose L/k a finite type field extension such that A_L contains the residue class $\bar{y} \in \bar{A}'^n$ induced by y. In fact $\bar{y}$ is a vector of polynomials in the generators of $\mathfrak{m}$ with the coefficients c_ν in k' and we may take $L = k((c_\nu)_\nu)$. Then $\bar{v}$ factors through A_L. Assume that $k[(c_\nu)_\nu] \cong k[(U_\nu)_\nu]/\bar{J}$ for some new variables U and a prime ideal $\bar{J} \subset k[U]$. We have $H_{L/k} \neq 0$ because L/k is separable. Then we may assume that there exist $w = (w_1, \ldots, w_p)$ in $\bar{J}^p$ such that $\rho = \det(\partial w_i/\partial U_\nu)_{i,\nu \in [p]} \neq 0$ and a nonzero polynomial $\tau \in ((w) : \bar{J}) \setminus \bar{J}$ (we set $[p] = \{1, \ldots, p\}$). Actually, we may reduce to the case when $p = 1$, but this means a complication for our algorithm. Thus L is a fraction ring of the smooth k-algebra $(k[U]/(w))_{\rho\tau}$. Note that w, ρ, τ can be considered in $A[U]$ because $k \subset A$ and $c_\nu \in A'$ because $k' \subset A'$.

Then $\bar{v}$ factors through a smooth $\bar{A}$-algebra $C \cong (\bar{A}[U]/(w))_{\rho\tau\gamma}$ for some polynomial γ which is not in $\mathfrak{m}(\bar{A}[U]/(w))_{\rho\tau}$.

Lemma 8. *There exists a smooth A-algebra D such that $\bar{v}$ factors through $\bar{D} = \bar{A} \otimes_A D$.*

Proof. By our assumptions $u(k) \subset k'$. Set $D = (A[U]/(w))_{\rho\tau\gamma}$ and $\omega : D \to A'$ be the map given by $U_\nu \to c_\nu$. We have $C \cong \bar{A} \otimes_A D$. Certainly, $\bar{v}$ factors through $\bar{\omega} = \bar{A} \otimes_A \omega$ but in general v does not factor through ω. □

It is worth recalling the following two remarks from [10].

Remark 9. If $A' = \hat{A}$ then $\bar{A} \cong \bar{A}'$ and we may take $D = A$.

Remark 10. If $k \subset A$ but $L \not\subset A'$ then $D = (A[U,Z]/(w - d^{2e+1}Z))_{\rho\tau\gamma}$, $Z = (Z_1, \ldots, Z_p)$, $U = (U_1, \ldots, U_q)$ is a smooth A-algebra and $\bar{D} \cong C[Z]$. Note that $\bar{v}$ factors through a map $C \to \bar{A}'$ given let us say by $U \to \lambda + d^{2e+1}A'^q$ for some λ in A'^q. Thus $w(\lambda) = d^{2e+1}z$ for some z in A'^p and we get a map $\omega : D \to A'$, $(U, Z) \to (\lambda, z)$. As above $\bar{v}$ factors through $\bar{\omega} = \bar{A} \otimes_A \omega$ but in general v does not factor through ω. If also $k \not\subset A$ then the construction of D goes using a lifting of w, τ, γ from $k[U]$ to $A[U]$. In both cases we may use D as it follows.

Let $\delta : B \otimes_A D \cong D[Y]/ID[Y] \to A'$ be the A-morphism given by $b \otimes \lambda \to v(b)\omega(\lambda)$.

Step 7. δ factors through a special finite type $B \otimes_A D$-algebra E.

Note that the map $\bar{B} \to \bar{D}$ is given by $Y \to y' + d^{2e+1}D$ for some $y' \in D^n$. Thus $I(y') \equiv 0$ modulo $d^{2e+1}D$. Since $\bar{v}$ factors through $\bar{\omega}$ we see that $\bar{\omega}(y' + d^{2e+1}D) = \bar{y}$. Set $\tilde{y} = \omega(y')$. We get $y - \tilde{y} = v(Y) - \tilde{y} \in d^{2e+1}A'^n$, let us say $y - \tilde{y} = d^{e+1}\varepsilon$ for $\varepsilon \in d^e A'^n$.

Recall that $P = R \det H$ for $R \in ((f) : I)$ (see Step 5). We have $d \equiv P$ modulo I and so $P(y') \equiv d$ modulo d^{2e+1} in D because $I(y') \equiv 0$ modulo $d^{2e+1}D$. Thus $P(y') = ds$ for a certain $s \in D$ with $s \equiv 1$ modulo d. Let G' be the adjoint matrix of H and $G = RG'$. We have $GH = HG = P\mathrm{Id}_n$ and so

$$ds\mathrm{Id}_n = P(y')\mathrm{Id}_n = G(y')H(y').$$

But H is the matrix $(\partial f_i/\partial Y_j)_{i \in [r], j \in [n]}$ completed with some $(n - r)$ rows from k^n. Especially we obtain

$$(\partial f/\partial Y)G = (P\mathrm{Id}_r|0). \tag{1}$$

Then $t := H(y')\varepsilon \in d^e A'^n$ satisfies

$$G(y')t = P(y')\varepsilon = ds\varepsilon$$

and so

$$s(y - \tilde{y}) = d^e \omega(G(y'))t.$$

Let

$$h = s(Y - y') - d^e G(y')T, \tag{2}$$

where $T = (T_1, \ldots, T_n)$ are new variables. The kernel of the map $\varphi : D[Y, T] \to A'$ given by $Y \to y$, $T \to t$ contains h. Since

$$s(Y - y') \equiv d^e G(y')T \text{ modulo } h$$

and

$$f(Y) - f(y') \equiv \sum_j \frac{\partial f}{\partial Y_j}(y')(Y_j - y'_j)$$

modulo higher order terms in $Y_j - y'_j$, by Taylor's formula we see that for $p = \max_i \deg f_i$ we have

$$s^p f(Y) - s^p f(y') \equiv \sum_j s^{p-1} d^e \frac{\partial f}{\partial Y_j}(y') G_j(y') T_j + d^{2e} Q \tag{3}$$

modulo h where $Q \in T^2 D[T]^r$. We have $f(y') = d^{e+1} b$ for some $b \in d^e D^r$. Then

$$g_i = s^p b_i + s^p T_i + d^{e-1} Q_i, \qquad i \in [r] \tag{4}$$

is in the kernel of φ. Indeed, we have $s^p f_i = d^{e+1} g_i$ modulo h because of (1) and $P(y') = ds$. Thus $d^{e+1}\varphi(g) = d^{e+1} g(t) \in (h(y,t), f(y)) = (0)$. Since $Q \in T^2 D[T]^r$ and $t \in d^e A'^n$ we get $g(t) \in d^e A'^r$ and so $g(t) \in (0 :_{A'} d^{e+1}) \cap d^e A' = 0$, because u is flat and $(0 :_{A'} d^e) = (0 :_{A'} d^{e+1})$. Set $E = D[Y, T]/(I, g, h)$ and let $\psi : E \to A'$ be the map induced by φ. Clearly, v factors through ψ because v is the composed map $B \to B \otimes_A D \cong D[Y]/I \to E \xrightarrow{\psi} A'$.

Step 8. There exist $s', s'' \in E$ such that $E_{ss's''}$ is smooth over A and ψ factors through $E_{ss's''}$.

Note that the $r \times r$-minor s' of $(\partial g/\partial T)$ given by the first r-variables T is from $s^{rp} + (T) \subset 1 + (d, T)$ because $Q \in (T)^2$. Then $V = (D[Y, T]/(h, g))_{ss'}$ is smooth over D. We claim that $I \subset (h, g)D[Y, T]_{ss's''}$ for some other $s'' \in 1 + (d, T)D[Y, T]$. Indeed, we have $PI \subset (h, g)D[Y, T]_s$ and so $P(y' + s^{-1} d^e G(y')T)I \subset (h, g)D[Y, T]_s$. Since $P(y' + s^{-1} d^e G(y')T) \in P(y') + d^e(T)$ we get $P(y' + s^{-1} d^e G(y')T) = ds''$ for some $s'' \in 1 + (d, T)D[Y, T]$. It follows that $s'' I \subset ((h, g) : d)D[Y, T]_{ss'}$. On the other hand, $I \equiv I(y')$ modulo $(d^e, h)D[Y, T]$ and $I(y') \subset d^{2e+1} D$. Thus $s'' I \subset (0 :_V d) \cap d^e V = 0$ because $(0 :_A d) \cap d^e A = 0$ and the maps $A \to D$, $D \to V$ are flat, which shows our claim. It follows that $I \subset (h, g)D[Y, T]_{ss's''}$. Thus $E_{ss's''} \cong V_{s''}$ is a B-algebra which is also standard smooth over D and A.

As $\omega(s) \equiv 1$ modulo d and $\psi(s'), \psi(s'') \equiv 1$ modulo (d, t), $d, t \in \mathfrak{m}A'$ we see that $\omega(s), \psi(s'), \psi(s'')$ are invertible because A' is local and ψ (thus v) factors through the standard smooth A-algebra $E_{ss's''}$.

Remark 11. When A' is the completion of A then the algorithmic proof is much easier (one reason is given by Remark 9) and it is somehow similar to the proof of Theorem 14. Certainly, in this case the next algorithm could be substantially easier.

Remark 12. If we want to study the case when $\dim A = 2$ then we need first to treat the case when $\dim A = 0$, $\dim A' = 1$ and after that the case when A, A' are one dimensional but not local rings. We expect that an algorithmic proof in the last case is a hard goal. However, if we are lucky to get such proof it is doubtful that the corresponding algorithm will really work.

4. The algorithm

We obtain the following algorithm:

Algorithm 1 Neron Desingularization

Input: $N \in \mathbb{Z}_{>0}$ a bound

$A := k[x]_{(x)}/J, J = (h_1, \dots, h_g) \subseteq k[x], x = (x_1, \dots, x_t), k$ an infinite field

$k' := Q(k[U]/\overline{J}), \overline{J} = (a_1, \dots, a_r) \subseteq k[U], U = (U_1, \dots, U_w)$ separable over k

$B := A[Y]/I, I = (g_1, \dots, g_l) \subseteq k[x, Y], Y = (Y_1, \dots, Y_n)$

$v : B \to A' \subseteq K[[x]]/JK[[x]]$ an A–morphism, given by $y'_1, \dots, y'_n \in k[U, x]$, approximations $\bmod (x)^N$ of $v(Y_i), K \supseteq k'$ a field.

Output: A Neron desingularization of $v : B \to A'$ or the message "the algorithm fails since the bound N is too small"

1: Compute $P_1, \dots, P_s$ the minimal associated primes of A.

2: Compute $w = (a_{i_1}, \dots, a_{i_p})$ and a p–minor ρ of $\left(\frac{\partial a_{i_v}}{\partial U_j}\right)$ such that $\rho \notin \overline{J}$.
Compute $\tau \in (w) : \overline{J}$ such that $k[U]_{\rho\tau}/(a_1, \dots, a_r) = k[U]_{\rho\tau}/(w)$.
$D := A[U]_{\rho\tau}/(w)$.

3: Compute $H_{B/A}$ and $H_{B/A} \cap A$.

4: If $\dim(A/H_{B/A} \cap A) = 1$
$B := S_B(I/I^2)$, v trivially extended, write $B = A[Y]/I$, $n := \#Y$
$Y := Y, Z, I := (I, Z), B := A[Y]/I,\ Z = (Z_1 \dots Z_n)$, v trivially extended

5: Compute $f = (f_1, \dots, f_r)$ in I such that $v(((f) : I)\Delta_f) \not\subseteq P_i + (x)^N$ for all i

6: Complete $\left(\frac{\partial f_i}{\partial Y_j}\right)$ by random vectors from k^n to obtain a square matrix H with $v(\det(H)) \notin P_i + (x)^N$ for all i

7: Compute $R \in (f) : I$ such that $v(R) \notin P_i + (x)^N$ for all i.

8: $P := R \cdot \det(H)$, compute $(P) \cap A$

9: If $\dim(A/(P) \cap A) = 1$
Compute d in $v(P) \cap A$ an active element. $f_{r+1} := PY_{n+1} - d$,
$B := B[Y_{n+1}]/f_{r+1}\ f := f, f_{r+1}$, $Y := Y, Y_{n+1}, y' := y', \frac{d}{v(P)}$, $n := n + 1$, r:=r+1
$H := \begin{pmatrix} H & 0 \\ * & P \end{pmatrix}$ $d := d^2$,$R := RY_n^2$, $P := R \cdot \det(H)$

10: Else
Compute d in $((P) \cap A)$ an active element, $P := d$.

11: Compute e such that $(0 : d^e) = (0 : d^{e+1})$

12: If $(x)^N \not\subseteq (d^{2e+1})$ return "the algorithm fails since the bound is too small"

13: Compute $b := \frac{f(y')}{d^{e+1}}$ in $d^e D^r$

14: Compute G' the adjoint matrix of H , $G := RG'$

15: Compute $s \in D$ such that $P(y') = ds$
$h := s(Y - y') - d^e G(y')T$, $T = (T_1, \dots, T_n)$

16: $p := \max\{\deg(f_i)\}_{i=1,\dots,r}$
write $s^p f(Y) - s^p f(y') = \sum_j s^{p-1} d^e \frac{\partial f}{\partial Y_j}(y') G_j(y') T_j + d^{2e} Q$
define $g_i := s^p b_i + s^p T_j + d^{e-1} Q_i$ and $E := D[Y, T]/(I, g, h)$

17: Compute s' the r–minor of $\left(\frac{\partial g}{\partial T}\right)$ given by the first r variables of T

18: find $s'' \in 1 + (d, T)_{D[Y,T]}$ such that $I \subset (h, g)D[Y, T]_{ss's''}$
(s'' is given by $P(y' + s^{-1} d^e G(y')T) = ds''$)

19: return $E_{ss's''}$

Example 13. We assume

$N=12$,
$A=\mathbf{Q}[x_1,x_2]_{(x_1,x_2)}/(x_1x_2)$, $J=(x_1,x_2)$,
$k'=\mathbf{Q}$,
$B=A[Y_1,Y_2]/(x_2Y_1-x_1Y_2)$, $I=(x_2Y_1,x_1Y_2)$,
$v:B\to\mathbf{Q}[[x_1,x_2]]/(x_1x_2)$, $v(Y_1)=x_1u$, $v(Y_2)=x_2w$,
$u,w\in\mathbf{Q}[[x_1,x_2]]$ algebraically independent,
$y_1'=\mathrm{jet}(x_1u,11)$, $y_2'=\mathrm{jet}(x_2w,11)$.
Now we follow the steps of the algorithm.
1. $P_1=(x_1)$, $P_2=(x_2)$,
2. $D=A$,
3. $H_{B/A}=(x_1,x_2)B$, $H_{B/A}\cap A=(x_1,x_2)$,
4. is not true,
5. $f=x_2Y_1+x_1Y_2$,
6. $H:=\begin{pmatrix} x_2 & x_1\\ -1 & 1\end{pmatrix}$
7. $R=x_1+x_2$,
8. $P=(x_1+x_2)^2$, $(P)\cap A=(x_1+x_2)^2$,
9. is not true,
10. $P=d=(x_1+x_2)^2$,
11. $e=1$,
12. is not true $(x_1,x_2)^{12}\subset(x_1x_2,(x_1,x_2)^6)$,
13. $f(y')=0$, $b=0$,
14. $G':=\begin{pmatrix} 1 & -x_1\\ 1 & x_2\end{pmatrix}$, $G=(x_1+x_2)\begin{pmatrix} 1 & -x_1\\ 1 & x_2\end{pmatrix}$,
15. $s=1$, $h=Y-y'-d(x_1+x_2)\begin{pmatrix} 1 & -x_1\\ 1 & x_2\end{pmatrix}\begin{pmatrix} T_1\\ T_2\end{pmatrix}$,
16. $p=1$, $x_2(Y_1-y_1')+x_1(Y_2-y_2')=d^2T_1 \bmod h$,
$g=T_1$, $E=A[Y,T]/(I,g,h)=A[Y,T]/(g,h)$,
17. $s'=1$,
18. $s''=1$,
19. return $A[Y_1,Y_2,T_2]/(Y_1-y_1'-x_1^4T_2,Y_2-y_2'-x_2^4T_2)$.

Thus the General Neron Desingularization B' of B could be $A[T_2]$. Let $\tau:B\to B'$ be the map given $Y_i\to y_i'+x_i^4T_2$, $i=1,2$. We may find $\rho_i\in(x_i)Q[[x_i]]$ such that $x_1u-y_1'=x_1^{11}\rho_1$ and $x_2w-y_2'=x_2^{11}\rho_2$. Set $t_2=d^4(\rho_1+\rho_2)$ and let $\tau':B'\to A'$ be given by $T_2\to t_2$. Clearly, $v=\tau'\tau$, that is v factors through B'.

5. An extension of Greenberg's theorem on strong approximation

Let $(A,\mathfrak{m})$ be a Noetherian local ring of dimension one, $A'=\hat{A}$ the completion of A, $B=A[Y]/I$, $Y=(Y_1,\ldots,Y_n)$ an A-algebra of finite type and $c\in\mathbf{N}$. If A is Henselian excellent DVR then Greenberg [5] showed that there exists a linear map $\nu:\mathbf{N}\to\mathbf{N}$ such that for each A-morphism $v:B\to A/\mathfrak{m}^{\nu(c)}$ there exists an A-morphism $v':B\to A$ such that $v'\equiv v$ modulo $\mathfrak{m}^c$. More general, if A is a DVR then there exists a linear map $\nu : \mathbf{N} \to \mathbf{N}$ such that for each A-morphism $v : B \to A/\mathfrak{m}^{\nu(c)}$ there exists an A-morphism $v' : B \to A'$ such that $v' \equiv v$ modulo $\mathfrak{m}^c A'$.

The aim of this section is to give a result of Greenberg's type for one dimensional rings when the Jacobian locus is not *too small.* Let e be as in the beginning of Section 3 and $\rho \in \mathbf{N}$. We will show that the map ν given by $c \to (e+1)(\rho+1)+c$ works in the special case below. Suppose that there exists an A-morphism $v : B \to A/\mathfrak{m}^{(e+1)(\rho+1)+c}$ such that $(v(((f):I)\Delta_f) \supset \mathfrak{m}^\rho/\mathfrak{m}^{(e+1)(\rho+1)+c}$ for some $f = (f_1, \ldots f_r)$, $r \leq n$ in I.

Theorem 14. *Then there exists an A-morphism $v' : B \to \hat{A}$ such that $v' \equiv v$ modulo $\mathfrak{m}^c$, that is $v'(Y+I) \equiv v(Y+I)$ modulo $\mathfrak{m}^c$.*

Proof. We note that the proof of Theorem 2 can work somehow in this case. Let $y' \in A^n$ be an element inducing $v(Y+I)$. Then $\mathfrak{m}^\rho \subset (((f):I)\Delta_f)(y')) + \mathfrak{m}^{(e+1)(\rho+1)+c} \subset (((f):I)\Delta_f)(y')) + \mathfrak{m}^{(e+1)(\rho+2)+2c} \subset \ldots$ by hypothesis. It follows that $\mathfrak{m}^\rho \subset (((f):I)\Delta_f)(y'))$.

As in Step 4 of Section 1 we may complete the Jacobian matrix $(\partial f/\partial Y)$ with $n-r$ rows from k^n obtaining a square n matrix H with $v(\det H) \notin \cup_{p \in \mathrm{Min}\, A'} p$

Set $d = (\det H)(y')$. Next we follow the proof of Theorem 2 with $D = A$, $s = 1$, $P = L \det H$ and G such that

$$GH = HG = P\mathrm{Id}_n$$

and so

$$d\mathrm{Id}_n = P(y')\mathrm{Id}_n = G(y')H(y').$$

Let

$$h = Y - y' - d^e G(y')T,$$

where $T = (T_1, \ldots, T_n)$ are new variables. We have

$$f(Y) - f(y') \equiv d^e P(y')T + d^{2e} Q$$

modulo h where $Q \in T^2 A[T]^r$. But $f(y') \in \mathfrak{m}^{(e+1)(\rho+1)+c} A^r \subset d^2 \mathfrak{m}^{c+e-1} A^r$ and we get $f(y') = d^2 b$ for some $b \in \mathfrak{m}^c A^r$. Set $g_i = b_i + T_i + d^{e-1} Q_i$, $i \in [r]$ and $E = A[Y,T]/(I,h,g)$. We have an A-morphism $\beta : E \to A/\mathfrak{m}^c$ given by $(Y,T) \to (y',0)$ because $I(y') \equiv 0$ modulo $\mathfrak{m}^c$, $h(y',0) = 0$ and $g(0) = b \in \mathfrak{m}^c A^r$.

As in the proof of Theorem 2 we see that $I \subset (h,g)A[Y,T]_{s's''}$ for some $s'' \in 1 + (d,T)A[Y,T]$. Indeed, we have $PI \subset (h,g)A[Y,T]$ and so $P(y' + d^e G(y')T)I \subset (h,g)A[Y,T]$. Since $P(y' + d^e G(y')T) \in P(y') + d^e(T)$ we get $P(y' + d^e G(y')T) = ds''$ for some $s'' \in 1 + (d,T)A[Y,T]$. It follows that $s''I \subset ((h,g):d)A[Y,T]_{s'}$. On the other hand, $I \equiv I(y')$ modulo $(d^e,h)A[Y,T]$ and $I(y') \subset \mathfrak{m}^{e\rho} \subset d^e A$. Set $V = (A[Y,T]/(h,g))_{s'}$. Then $s''I \subset (0 :_V d) \cap d^e V = 0$ because $(0 :_A d) \cap d^e A = 0$ and the map $A \to V$ is flat, which shows our claim.

Then $E_{s's''} \cong V_{s''}$ is a B-algebra which is also standard smooth over A and β extends to a map $\beta' : E_{s's''} \to A/\mathfrak{m}^c$ as in the proof of Theorem 2. By the Implicit Function Theorem β' can be lifted to a map $w : E_{s's''} \to \hat{A}$ which coincides with β' modulo $\mathfrak{m}^c$. It follows that the composite map v', $B \to E_{s's''} \xrightarrow{w} \hat{A}$ works. $\square$

Corollary 15. *In the assumptions of the above theorem, suppose that $(A, \mathfrak{m})$ is excellent Henselian. Then there exists an A-morphism $v'' : B \to A$ such that $v'' \equiv v$ modulo $\mathfrak{m}^c$, that is $v''(Y+I) \equiv v(Y+I)$ modulo $\mathfrak{m}^c$.*

Proof. An excellent Henselian local ring $(A, \mathfrak{m})$ has the Artin approximation property by [12], that is the solutions in A of a system of polynomial equations f over A are dense in the set of the solutions of f in $\hat{A}$. By Theorem 14 we get an A-morphism $v' : B \to \hat{A}$ such that $v' \equiv v$ modulo $\mathfrak{m}^c$.

Then there exists an A-morphism $v'' : B \to A$ such that $v'' \equiv v' \equiv v$ modulo $\mathfrak{m}^c$ by the Artin approximation property. $\square$

Remark 16. If $\dim A > 1$ then [17, Theorem 1.2] gives an example when there exist no linear map ν as above (see also [18, Theorem 4.3, Remark 4.7] for different cases). Therefore, we believe that the above corollary does not hold when $\dim A > 1$ even it is restricted to some special B, v.

Theorem 17. *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension one, $e \in \mathbf{N}$ as in the beginning of Section 2, $B = A[Y]/I$, $Y = (Y_1, \ldots, Y_n)$ an A-algebra of finite type, $\rho \in \mathbf{N}$ and $f = (f_1, \ldots, f_r)$ a system of polynomials from I. Suppose that A is excellent Henselian and there exists $y' \in A^n$ such that $I(y') \equiv 0$ modulo $\mathfrak{m}^\rho$ and $(((f) : I)\Delta_f))(y') \supset \mathfrak{m}^\rho$. Then the following statements are equivalent:*

(1) there exists $y'' \in A^n$ such that $I(y'') \equiv 0$ modulo $\mathfrak{m}^{(e+2)(\rho+1)}$ and $y'' \equiv y'$ modulo $\mathfrak{m}^\rho$,

(2) there exists $y \in A^n$ such that $I(y) = 0$ and $y \equiv y'$ modulo $\mathfrak{m}^\rho$.

For the proof apply the above corollary and Theorem 14.

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