On the Primary Decomposition of Some Determinantal Hyperedge Ideal

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Abstract

In this paper we describe the method which we applied to successfully compute the primary decomposition of a certain ideal coming from applications in combinatorial algebra and algebraic statistics regarding conditional independence statements with hidden variables. While our method is based on the algorithm for primary decomposition by Gianni, Trager and Zacharias, we were not able to decompose the ideal using the standard form of that algorithm, nor by any other method known to us.

Keywords: primary decomposition, determinantal ideal

1. Introduction

The aim of this paper is to describe the method which we applied to compute the primary decomposition of a certain ideal. The importance of this ideal comes from applications in combinatorial algebra and algebraic statistics, in particular regarding conditional independence statements, see [M], [MR], [EH] or [EHHM].

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Conditional independence (CI) is an important tool in statistical modelling. There has been a lot of recent activity on ideals associated to conditional independence statements, when all random variables are observed. However, in applications such as causal reasoning, some variables are hidden and it is important to know what constraints on the observed variables are caused by hidden variables.

From algebraic point of view, knowing the primary decomposition of CI ideals leads to important results in causal inference. We have studied a particular ideal of interest in causal inference whose primary decomposition has led to interesting theoretical and computational results. It is shown in this paper that this ideal is equidimensiomal and has two prime components. In particular, one of the prime components is not a determinantal ideal which is a surprising result. In addition, both components are important in the sense of causal inference, as none of them contains a variable, see [CMR, Example 4.2].

We tried the algorithms for primary decomposition which are available in SINGULAR, that is, the one of Gianni, Trager and Zacharias [GTZ] as well as the one of Shimoyama and Yokoyama [SY], but none of them succeeded to decompose the ideal within one month.

The outline of this paper is quite simple: The input ideal and some of its combinatorial properties are given in Section 2. This ideal can be decomposed using the computational method presented in Section 3. This method combines four improvements to the well-known algorithm for primary decomposition by Gianni, Trager and Zacharias [GTZ] which are described in detail in separate subsections. Finally, in Section 4, we specify the result of the primary decomposition and reveal some of the combinatorial properties of the computed prime components. However, the combinatorial structure of the second prime component is not yet fully understood.

2. Input ideal

The ideal I of which we would like to compute a primary decomposition is a determinantal hyperedge ideal. Over the polynomial ring $R := \mathbb{Q}[x_1, \ldots, x_{12}, y_1, \ldots, y_{12}, z_1, \ldots, z_{12}]$, consider the 3×12-matrix

$$M := \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12} \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} & y_{11} & y_{12} \\ z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & z_7 & z_8 & z_9 & z_{10} & z_{11} & z_{12} \end{pmatrix}.$$

Following the notation for determinantal hyperedge ideals in [CMR], set

$$\begin{aligned} R_1 &:= \{1,2,3\} =: N, & C_1 &:= \{1,4,7,10\}, \\ R_2 &:= \{4,5,6\}, & C_2 &:= \{2,5,8,11\}, \\ R_3 &:= \{7,8,9\}, & C_3 &:= \{3,6,9,12\}, \\ R_4 &:= \{10,11,12\}, & & \end{aligned}$$

and denote the minor of M with row indices A and column indices B by $[A|B]_M$. Furthermore, for any set X and any non-negative integer s, we denote by $\binom{X}{s}$ the set of all subsets of X which contain exactly s elements.

With this notation, we define the ideal $I \subseteq R$ as

$$I := \left\langle [N|B]_M \;\middle|\; B = R_i,\ i \in \{1,2,3,4\} \text{ or } B \in \binom{C_j}{3},\ j \in \{1,2,3\} \right\rangle,$$

that is,

$$\begin{aligned}
I = \langle & x_1y_4z_7 - x_1y_7z_4 - x_4y_1z_7 + x_4y_7z_1 + x_7y_1z_4 - x_7y_4z_1,\\
& x_1y_4z_{10} - x_1y_{10}z_4 - x_4y_1z_{10} + x_4y_{10}z_1 + x_{10}y_1z_4 - x_{10}y_4z_1,\\
& x_1y_7z_{10} - x_1y_{10}z_7 - x_7y_1z_{10} + x_7y_{10}z_1 + x_{10}y_1z_7 - x_{10}y_7z_1,\\
& x_4y_7z_{10} - x_4y_{10}z_7 - x_7y_4z_{10} + x_7y_{10}z_4 + x_{10}y_4z_7 - x_{10}y_7z_4,\\
& x_2y_5z_8 - x_2y_8z_5 - x_5y_2z_8 + x_5y_8z_2 + x_8y_2z_5 - x_8y_5z_2,\\
& x_2y_5z_{11} - x_2y_{11}z_5 - x_5y_2z_{11} + x_5y_{11}z_2 + x_{11}y_2z_5 - x_{11}y_5z_2,\\
& x_2y_8z_{11} - x_2y_{11}z_8 - x_8y_2z_{11} + x_8y_{11}z_2 + x_{11}y_2z_8 - x_{11}y_8z_2,\\
& x_5y_8z_{11} - x_5y_{11}z_8 - x_8y_5z_{11} + x_8y_{11}z_5 + x_{11}y_5z_8 - x_{11}y_8z_5,\\
& x_3y_6z_9 - x_3y_9z_6 - x_6y_3z_9 + x_6y_9z_3 + x_9y_3z_6 - x_9y_6z_3,\\
& x_3y_6z_{12} - x_3y_{12}z_6 - x_6y_3z_{12} + x_6y_{12}z_3 + x_{12}y_3z_6 - x_{12}y_6z_3,\\
& x_3y_9z_{12} - x_3y_{12}z_9 - x_9y_3z_{12} + x_9y_{12}z_3 + x_{12}y_3z_9 - x_{12}y_9z_3,\\
& x_6y_9z_{12} - x_6y_{12}z_9 - x_9y_6z_{12} + x_9y_{12}z_6 + x_{12}y_6z_9 - x_{12}y_9z_6,\\
& x_1y_2z_3 - x_1y_3z_2 - x_2y_1z_3 + x_2y_3z_1 + x_3y_1z_2 - x_3y_2z_1,\\
& x_4y_5z_6 - x_4y_6z_5 - x_5y_4z_6 + x_5y_6z_4 + x_6y_4z_5 - x_6y_5z_4,\\
& x_7y_8z_9 - x_7y_9z_8 - x_8y_7z_9 + x_8y_9z_7 + x_9y_7z_8 - x_9y_8z_7,\\
& x_{10}y_{11}z_{12} - x_{10}y_{12}z_{11} - x_{11}y_{10}z_{12} + x_{11}y_{12}z_{10} + x_{12}y_{10}z_{11} - x_{12}y_{11}z_{10}\rangle\\
\subseteq\ & \mathbb{Q}[x_1,\ldots,x_{12},\ y_1,\ldots,y_{12},\ z_1,\ldots,z_{12}]\,.
\end{aligned}$$

The ideal I exhibits a kind of symmetry in the sense that it stays invariant under certain permutations of the variables in the polynomial ring R. In fact, the following operations on the matrix M in the above construction lead to the same ideal I:

- permuting the rows
- permuting the sets of columns which correspond to the index sets R_1, R_2, R_3 and R_4
- permuting the sets of columns which correspond to the index sets C_1, C_2 and C_3

Let S_n be the symmetric group of order n as usual. Then the above observations imply that the symmetry group $\mathcal{S} \leq S_{36}$ of I has a subgroup $\mathcal{S}'$ isomorphic to $S_3 \times S_4 \times S_3$. We believe that indeed $\mathcal{S} \cong \mathcal{S}'$. However, this is more difficult to prove.

3. Computational method

Our computational method is based on the GTZ algorithm for primary decomposition by Gianni, Trager and Zacharias [GTZ] as presented in [GP, Chapter 4]. Let us recall the fact that the GTZ algorithm computes some maximal dimensional associated primary ideals $Q_1, \ldots, Q_s$ of I using maximal independent sets to reduce the problem to the zero-dimensional case, as well as a polynomial $h^m \notin I$ such that $I : \langle h^m \rangle = I : \langle h^\infty \rangle = Q_1 \cap \ldots \cap Q_s$ and $I = (I : \langle h^m \rangle) \cap \langle I, h^m \rangle$.

An implementation of this algorithm can be found in the Singular library `primdec.lib` [DLPS]. However, neither this implementation nor the one of the algorithm of Shimoyama and Yokoyama [SY] in the same library succeeded to compute a primary decomposition of the input ideal I defined in the previous section within one month.

We propose four improvements for applying this algorithm to our particular example:

1. carefully choosing the maximal independent sets, based on data from intermediate steps;
2. if an ideal P is expected to be a prime ideal, applying a special algorithm to check the primality of P;
3. choosing a monomial order for saturation which is compatible with the monomial orders used in the previous steps;
4. making use of the symmetry of the input ideal for the saturation step.

These four improvements will be explained in detail in the following subsections. The last improvement cannot easily be applied in general if the input ideal does not admit any useful symmetry. On the other hand, we expect that especially the first and the third improvement are also beneficial for other classes of ideals.

3.1. Choosing maximal independent sets

In the GTZ algorithm, the reduction to the zero-dimensional case requires a maximal independent set, see [GP, Algorithm 4.3.2]. However, the input ideal might admit several maximal independent sets, any one of which can be used in the subsequent steps.

For the convenience of the reader, we recall the definition of maximal independent sets:

Definition 3.1 ([GP, Definition 3.5.3]). Let $I \subset K[x_1, \ldots, x_n]$ be an ideal. Then a subset

$$u \subset x = \{x_1, \ldots, x_n\}$$

is called an independent set (with respect to I) if $I \cap K[u] = 0$. An independent set $u \subset x$ (with respect to I) is called maximal if $\dim(K[x]/I) = |u|$.

In our case, the SINGULAR command `indepSet(., 0)`, when applied to a Gröbner basis of I w.r.t. the degree reverse lexicographic order with $x_1 > \ldots > x_{12} > y_1 > \ldots > y_{12} > z_1 > \ldots > z_{12}$, yields $17,223$ different maximal independent sets. The ideal I may even have more maximal independent sets, but it is computationally hard to list all of them.

Depending on the choice made in the algorithm, the running time needed for the computation of the next primary component, especially for the saturation step, may vary enormously. Thus we propose to proceed as follows: Let $K[X]$ with $X = \{X_1, \ldots, X_n\}$ be the polynomial ring of the input ideal. Randomly choose a maximal independent set $u \subseteq X$ of I and compute a minimal Gröbner basis $G_u \subseteq K[X]$ of the zero-dimensional ideal $IK(u)[X \setminus u] \subseteq K(u)[X \setminus u]$, see [GP, Proposition 4.3.1 (1)], where $K(u)$ is the field of fractions of $K[u]$. Further compute

$$d_u := \dim_{K(u)[X\setminus u]}\big(K(u)[X \setminus u] \;/\; IK(u)[X \setminus u]\big)$$

as well as the degrees and the number of terms of the leading coefficients of the elements in G_u, considered as polynomials in $K[u]$. Repeat this for several maximal independent sets u and sort them, in this order,

1. by increasing vector space dimension d_u,
2. by increasing (maximal) degree and finally
3. by the (maximal) number of terms of the leading coefficients of the elements in G_u.

This defines a partial order on the maximal independent sets. Choose one of the first sets w.r.t. this order and proceed with the GTZ algorithm as usual.

It is reasonable to expect, and in fact our experiments confirm, that on average the above choice leads to improved running times for the saturation step compared to other, for example random, choices.

There is a tradeoff between the number of maximal independent sets for which the above mentioned data is computed and the overall speedup which can be achieved by this method. In our example, we used about 700 randomly chosen maximal independent sets. Finally we subsequently picked

$$u_1 := \{x_1, \ldots, x_{12}, y_1, \ldots, y_{12}, z_1, z_2\} \text{ and then}$$
$$u_2 := \{x_1, x_2, x_4, x_5, x_8, x_9, x_{11}, x_{12}, y_1, \ldots, y_{12}, z_1, z_4, z_8, z_9, z_{11}, z_{12}\}$$

to find the two primary components described in Section 4. Especially the computation of the second primary component was far out of reach for many other choices of maximal independent sets.

3.2. Checking ideals for primality

While trying to compute a primary decomposition of some given ideal I by whatever means, one may face the situation that one of the computed ideals P is expected to be one of the prime components of I, but it is not known a priori whether or not P is indeed a prime ideal. For example, this may happen if I is defined in a polynomial ring over a field of characteristic zero, but P has been found via computations in positive characteristic. In that situation, Algorithm 1 can be used to check the primality of P. This algorithm may be known to some experts in the field, but we did not find any reference.

Algorithm 1 Primality check

Input: $P \subseteq K[X] = K[X_1, \dots, X_n]$
Output: true if P is a prime ideal, false otherwise
choose a maximal independent set $u \subseteq X$ of P
compute a minimal Gröbner basis $G \subseteq K[X]$ of $PK(u)[X \setminus u]$
if $PK(u)[X \setminus u]$ is not a maximal ideal in $K(u)[X \setminus u]$ **then**
  return false
let $c_1, \dots c_s \in K[u]$ be the leading coefficients of the elements in G
for $i = 1, \dots, s$ **do**
  if $P \neq P : \langle c_i^\infty \rangle$ **then**
    return false
return true

The correctness of this algorithm is based on the following statement:

Proposition 3.2. *Let $P \subseteq K[X] = K[X_1, \dots, X_n]$ be an ideal and let $u \subseteq X$ be a maximal independent set of P. Then P is a prime ideal if and only if $PK(u)[X \setminus u]$ is a maximal ideal in $K(u)[X \setminus u]$ and $P = PK(u)[X \setminus u] \cap K[X]$.*

Proof. The ideal $PK(u)[X \setminus u]$ has dimension zero by [GP, Proposition 4.3.1]. Let $S = K[u] \setminus \{0\}$, then $K(u)[X \setminus u]$ is the localization of $K[X]$ at S. Therefore the result follows from [AM, Proposition 3.11(iv)]. □

Corollary 3.3. *The output of Algorithm 1 is correct.*

Proof. Let $G \subseteq K[X]$ be a minimal Gröbner basis of $PK(u)[X \setminus u]$, let $c_1, \dots c_s \in K[u]$ be the leading coefficients of the elements in G as in the algorithm, and define $h := \mathrm{LCM}(c_1, \dots c_s) \in K[u]$. Then

$$PK(u)[X \setminus u] \cap K[X] = P : \langle h^\infty \rangle = (\dots (P : \langle c_1^\infty \rangle) \dots : \langle c_s^\infty \rangle)$$

by [GP, Proposition 4.3.1 (2)] and the basic properties of ideal quotients. Thus the statement follows from Proposition 3.2. □

In our case, we made use of the above algorithm to confirm the primality of the component P_2, see Section 4. Computationally, this was by far the hardest part of finding a primary decomposition of the input ideal I defined in Section 2.

A variant of Algorithm 1 is the following improvement to the GTZ algorithm: Let $I = \langle f_1, \ldots, f_m \rangle \subseteq K[X]$ be the remaining ideal to decompose, of dimension greater than zero, and let $u \subseteq X$ be a maximal independent set of I. Then as the next step, the algorithm computes a primary decomposition $IK(u)[X \setminus u] = Q_1 \cap \ldots \cap Q_s$ in dimension zero. As the algorithm proceeds, the intersections $Q_i \cap K[X]$ are computed via saturation of Q_i w.r.t. some $d_i \in K[u]$ to obtain primary components of I, see [GP, paragraph below Algorithm 4.3.4].

Now suppose $s = 1$, that is, $IK(u)[X \setminus u]$ is a primary ideal in $K(u)[X \setminus u]$. Let $\{g_1, \ldots, g_r\} \subseteq I \subseteq K[X]$ be a Gröbner basis of $IK(u)[X \setminus u] = Q_1$. Such a Gröbner basis always exists: For example, take a lexicographical Gröbner basis of I with $X \setminus u > u$ and discard the elements not needed for a minimal Gröbner basis of Q_1. The next step in the GTZ algorithm would be the saturation of $I_0 := \langle g_1, \ldots, g_r \rangle_{K[X]} \subseteq I$ w.r.t. $c \in K[u]$, the least commom multiple of the leading coefficients of the g_i considered as elements of $K(u)[X \setminus u]$. But in this case, we have $I_0 : \langle c^\infty \rangle = I : \langle c^\infty \rangle$. Usually, it is computationally easier to saturate I than I_0 because loosely speaking, I is already closer to the final result $I_0 : \langle c^\infty \rangle$ than I_0.

In the case where Q_1 is even a maximal ideal, this trick is computationally equivalent to Algorithm 1.

3.3. Choosing a monomial order for saturation

For ideals of dimension greater than zero, the GTZ algorithm requires a saturation step, see [GP, Chapter 4.3]. As for many applications of Gröbner bases, the running time of this step heavily depends on the choice of the monomial order. Our experiments have shown that it is beneficial to choose a monomial order which is compatible with the maximal independent set that has been used in the preceding steps of the algorithm.

More precisely, let $K[X]$ with $X = \{X_1, \ldots, X_n\}$ be the polynomial ring of the input ideal as in the previous subsection and let $u \subseteq X$ be the chosen maximal independent set. We propose to use an elimination order for $X \setminus u$ on $K[X]$ for the saturation step. For implementational reasons, the lexicographical order with $X \setminus u > u$ is a good choice, see [GP, Algorithm 4.3.2, step 2].

3.4. Making use of symmetry for the saturation step

With notation as in Algorithm 1, if the input ideal P has some kind of symmetry which also occurs among the coefficients c_i, then this can be used to speed up the computation. More precisely, let φ be a K-algebra automorphism of $K[X]$ which leaves P invariant, that is, $\varphi(P) = P$. Furthermore suppose that $\varphi(c_j) = c_i$ for some indices $i, j \in \{1, \dots, s\}$ with $i \neq j$, and that $P = P : \langle c_j^\infty \rangle$ has already been checked in line 7 of Algorithm 1. Because applying φ commutes with saturation, we then have

$$P : \langle c_i^\infty \rangle = \varphi(P) : \langle \varphi(c_j)^\infty \rangle = \varphi(P : \langle c_j^\infty \rangle) = \varphi(P) = P\,.$$

Thus the check can be left out for c_i.

For computing a primary decomposition of the input ideal I defined in Section 2, we made use of this idea in the following way: Let P_1 and P_2 be the two prime components of I as presented in Section 4, see Theorem 4.1. We used Algorithm 1 to check the primality of P_2, so continuing the notation above, let us consider the case $P = P_2$. It turned out that we have indeed $\varphi(c_j) = c_i$ for some indices $i, j \in \{1, \dots, s\}$ with $i \neq j$ where φ is the $\mathbb{Q}$-algebra automorphism of

$$\mathbb{Q}[x_1, \dots, x_{12},\, y_1, \dots, y_{12},\, z_1, \dots, z_{12}]$$

given by

$$\begin{aligned} \varphi(x_i) &:= x_{\sigma(i)}, \\ \varphi(y_i) &:= y_{\sigma(i)}, \\ \varphi(z_i) &:= z_{\sigma(i)}, \end{aligned}$$

with $\sigma = (14)(25)(36) \in S_{12}$. The automorphism φ leaves I and the two prime components P_1 and P_2 invariant, which is easy to see for I and P_1 and can be checked computationally for P_2. Thus computing the saturation $P : \langle c_i^\infty \rangle$ is superfluous if $P = P : \langle c_j^\infty \rangle$ has already been checked. However, we have performed this computation once and it took almost 500 hours which is more than four times as long as the saturation of P w.r.t. c_j, see Section 4. Since in this case, φ is just a permutation of variables, this example underlines once more the importance of choosing a beneficial monomial order for the saturation step as discussed in Subsection 3.3.

4. Main Result

Using the computational method from the previous section, we were able to show the following:

Theorem 4.1. *The ideal* $I \subseteq R = \mathbb{Q}[x_1, \dots, x_{12},\, y_1, \dots, y_{12},\, z_1, \dots, z_{12}]$ *defined in Section 2 is the intersection*

$$I = P_1 \cap P_2$$

of two different prime ideals P_1, P_2 *where all three ideals* I, P_1 *and* P_2 *have Krull dimension* 26. *In particular, this implies that the above intersection is the unique irredundant primary decomposition of* I.

In our case, we first found the prime component P_1, with $I \subsetneqq P_1$, which was relatively easy. We then found the ideal P_2 for which we were able to show that $I = P_1 \cap P_2$ and $I \subsetneqq P_2$. But the hardest part of the computation were the saturations for the primality check of P_2, see line 7 in Algorithm 1. While checking that P_2 is already saturated w.r.t. all the coefficients c_i was easy in some cases, the longest of these computations took 117 hours on an Intel® Core™ i7-6700 CPU with up to 4 GHz. For one coefficient, it takes almost 500 hours, but this computation can be avoided by the method described in Subsection 3.4.

P_1 is the ideal generated by *all* 3×3-minors of the matrix M defined in Section 2. The structure of the prime component P_2 is more involved. Using the SINGULAR command `mstd()`, one can find a generating set G of P_2 which has the following properties[1]:

- G consists of 44 polynomials $p_1, \ldots, p_{44}$. The elements $p_1, \ldots, p_{16}$ are just the generators of I listed in Section 2.
- The coefficients in $p_1, \ldots, p_{44}$ are all 1, -1, 2 or -2 where the vast majority is just 1 or -1.
- Each polynomial in the set $G' := \{p_{17}, \ldots, p_{44}\}$ is multi-homogeneous of multi-degree $(1, 1, \ldots, 1)$ w.r.t.
$$((x_1, y_1, z_1), (x_2, y_2, z_2), \ldots, (x_{12}, y_{12}, z_{12}))\,.$$
In particular, each polynomial in G' is homogeneous of degree 12.
- Each single element in G' is multi-homogeneous w.r.t.
$$(x, y, z) := ((x_i)_{i=1,\ldots,12}, (y_i)_{i=1,\ldots,12}, (z_i)_{i=1,\ldots,12})\,.$$
For example, the polynomial g_{17} is multi-homogeneous of multi-degree $(0, 6, 6)$ w.r.t. (x, y, z).
- The multi-degrees of the 28 elements in G' are in one-to-one correspondence to the 28 possible partitions of twelve items into three categories with no more then six items belonging to the same category.
- For g_{17} which is multi-homogeneous of multi-degree $(0, 6, 6)$ as mentioned above, we get all monomials as follows: From the 3×4-matrix
$$\begin{pmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{pmatrix}$$

[1] The set G can be found in SINGULAR-readable form at the very end of the TEX file of this article which can be downloaded from `https://arxiv.org/abs/1811.09530`.

choose two different columns A, B. Choose two indices from A. Choose two indices from B such that they are not both in the same rows as the two indices chosen from A. Let C, D be the remaining columns. Choose one index from C. Choose one index from D which is in a different row then the one chosen in C. The chosen indices are the y_i's, all the others are the z_i's (or vice versa, of course). Repeating this process for all possible choices, we get exactly the 216 monomials appearing in g_{17}.

- The generators which are multi-homogeneous of multi-degree $(6, 6, 0)$ and $(6, 0, 6)$, respectively, coincide with g_{17} for some permutation of (x, y, z).
- The generator g_{19} is multi-homogeneous of multi-degree $(1, 5, 6)$. It can be mapped to g_{17} via the map $x_i \mapsto y_i$. Some of the 252 terms of g_{19} cancel under this map.

The above properties indicate that the set G has a combinatorial structure which would be interesting to understand completely, including the coefficients. So far, however, we were not able to reveal every detail of this structure.

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