Rank two Cohen-Macaulay modules over singularities of type $\mathbf{x_1^3 + x_2^3 + x_3^3 + x_4^3}$

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Abstract

We describe, by matrix factorizations, all the rank two maximal Cohen–Macaulay modules over singularities of type $x_1^3+x_2^3+x_3^3+x_4^3$.[1]

Acknowledgment

This is a modified version of the paper submitted to Journal of Algebra. We rectified the statements concerning the non–orientable indecomposable maximal Cohen–Macaulay modules of rank two with 4 and 5 generators. We are very grateful to Robin Hartshorne who pointed out the errors in the original version of this paper.

Introduction

Let R be a hypersurface ring, that is $R = S/(f)$ for a regular local ring $(S, \mathfrak{m})$ and $0 \neq f \in \mathfrak{m}$. According to Eisenbud [Ei], any maximal Cohen–Macaulay

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(briefly MCM) module over R has a minimal free resolution of periodicity 2, which is completely given by a matrix factorization $(\varphi, \psi), \varphi, \psi$ being square matrices over S such that $\varphi\psi = \psi\varphi = f\,\mathrm{Id}_n$, for a certain positive integer n. Therefore, in order to describe the MCM R–modules, it is enough to describe their matrix factorizations. In this paper we give the description, by matrix factorizations, of the graded, rank 2, indecomposable, MCM modules over $K[x_1, x_2, x_3, x_4]/(x_1^3 + x_2^3 + x_3^3 + x_4^3)$. Part of this study was done with the help of the Computer Algebra System SINGULAR [GPS].

The MCM modules over the hypersurface $f_3 = x_1^3 + x_2^3 + x_3^3$ were described in [LPP] as 1–parameter families indexed by the points of the curve $Z = V(f_3) \subset \mathbf{P}^2$. This description is mainly based on Atiyah's theory of the vector bundles classification over elliptic curves, in particular over Z, and on difficult computations made with the Computer Algebra System SINGULAR. The description depends on two discrete invariants — the rank and the degree of the bundle — and on a continuous invariant — the points of the curve Z.

The classification of vector bundles is of great interest, in particular of ACM bundles (i.e. those which correspond to MCM modules) over the singularities of higher dimension. In the paper [EP], matrix factorizations which define the graded MCM modules of rank 1 over $f_4 = x_1^3 + x_2^3 + x_3^3 + x_4^3$ are described. There is a finite number of such modules, which correspond to 27 lines, 27 pencils of quadrics and 72 nets of twisted cubic curves lying on the surface $Y = V(f_4) \subset \mathbb{P}^3$. From a geometrical point of view the problem is easy, but the effective description of the matrix factorizations is difficult and SINGULAR has been intensively used.

In the present paper we continue this study for the graded MCM modules of rank two. We obtain a general description of the MCM orientable modules of rank two. They are given by skew–symmetric matrix factorizations (see Theorem 6). The technique is based on the results of Herzog and Kühl (see [HK]) concerning the so–called *Bourbaki exact sequences*. The matrix factorizations of the graded, orientable, rank 2, 4–generated MCM modules are parameter families indexed by the points of the surface Y, that is, two parameter families and some finite ones in bijection with rank 1 MCM modules described in [EP] (see Theorem 8. Here an important fact is that two Gorenstein ideals of codimension 2 define the same MCM module via the associated Bourbaki sequence if and only if they belong to the same even linkage class). We also describe the non–orientable MCM modules of rank 2 over f_4. There is a finite number of such modules, which correspond somehow to the rank

1 modules described in [EP]. The graded MCM modules, non–orientable, of rank 2 are 2–syzygy over f_4 of some ideals of the form $J/(f_4)$, J being an ideal of the polynomial ring $S = K[x_1, x_2, x_3, x_4]$, ($K$ is an algebraically closed field of characteristic zero), with $f_4 \in J$, $\dim S/J = 2$, $\operatorname{depth} S/J = 1$, whose Betti numbers over S satisfy $\beta_1(J) = \beta_0(J) + 1$ and $\beta_2(J) = 1$ (see Lemma 11). This result has been essential in the description of the graded, non–orientable MCM modules. The paper highlights bijections between the classes of indecomposable, graded, non–orientable MCM modules of rank 2, 4 and 5–generated and the classes of rank 1, graded, MCM modules (see Theorem 13 and Theorem 16). Consequently, there exists a bijection between the classes of indecomposable, graded, non–orientable MCM modules of rank three, 5–generated and the classes of rank 1, graded, MCM modules (see Corollary 17). These results remind us of the theory of Atiyah and give small hope that the non-orientable case behaves in the same way for higher rank. We also show that there are no indecomposable, graded, non–orientable MCM modules of rank 2 6–generated. Consequently, there exist no indecomposable, graded, non–orientable MCM modules of rank four, 6-generated.

Until now, the description of graded rank 2 MCM modules is not too far from the theory of Atiyah. But the description of graded, rank two, 6-generated MCM modules is different (see Section 6) from what we expected, since a part of them, given by Gorenstein ideals defined by 5 general points on Y, forms a 5-parameter family (see [Mig], [IK]). However, we believe that behind these results there exists a nice theory of graded MCM modules over a cubic hypersurface in four variables which waits to be discovered.

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1 Preliminaries

Let $R_n := K[x_1, x_2, \ldots, x_n]/(f_n)$, where $f_n = x_1^3 + x_2^3 + \ldots + x_n^3$ and K is an algebraic closed field of characteristic 0. Using the classification of vector bundles over elliptic curves obtained by Atiyah [At], Laza, Pfister and Popescu [LPP] describe the matrix factorizations of the graded, indecomposable and reflexive modules over R_3. They give canonical normal forms for the matrix factorizations of all graded reflexive R_3–modules of rank 1 (see Section 3 in [LPP]) and show effectively how we can produce the indecomposable graded reflexive R_3–modules of rank ≥ 2 using SINGULAR (see Section 5 in [LPP]). We recall from [LPP] the description of the rank 1, three-generated, non–free, graded MCM R_3–modules since we shall use it in the last section of our paper. First we recall the notations. Let $P_0 = [-1 : 0 : 1] \in V(f_3)$. For each $\lambda = [\lambda_1 : \lambda_2 : 1] \in V(f_3), \lambda \neq P_0$, we set

$$\alpha_\lambda = \begin{pmatrix} 0 & x_1 - \lambda_1 x_3 & x_2 - \lambda_2 x_3 \\ x_1 + x_3 & -x_2 - \lambda_2 x_3 & -w x_3 \\ x_2 & w x_3 & (1-\lambda_1)x_3 - x_1 \end{pmatrix},$$

where $w = \frac{\lambda_2^2}{\lambda_1+1}$ and, if $\lambda = [\lambda_1 : 1 : 0] \in V(f_3)$, we set

$$\alpha_\lambda = \begin{pmatrix} 0 & x_1 - \lambda_1 x_2 & x_3 \\ x_1 + x_3 & -\lambda_1 x_1 & \lambda_1 x_1 + \lambda_1^2 x_2 \\ x_2 & x_3 - x_1 & -x_1 \end{pmatrix}.$$

Let β_λ the adjoint matrix of α_λ.

Theorem 1 ((3.7) in [LPP]). $(\alpha_\lambda, \beta_\lambda)$ *is a matrix factorization for all* $\lambda \in V(f_3)$, $\lambda \neq P_0$, *and the set of 3–generated MCM graded* R_3*–modules,*

$$\mathcal{M}_0 = \{\operatorname{Coker} \alpha_\lambda \mid \lambda \in V(f_3), \lambda \neq P_0\}$$

has the following properties:

(i) All the modules from $\mathcal{M}_0$ *have rank 1.*

(ii) Each two different modules from $\mathcal{M}_0$ *are not isomorphic.*

(iii) Every 3–generated, rank 1, non–free, graded MCM R_3*–module is isomorphic with one module from* $\mathcal{M}_0$*.*

Now we consider the case $n = 4$. In this case we do not have the support of Atiyah classification. The complete description by matrix factorizations of the rank 1, graded, indecomposable MCM modules over R_4 was given in [EP].

The aim of the present paper is to classify the rank 2, graded, indecomposable MCM modules over R_4. From now on, we shall denote $R = R_4$, $f = f_4$ and we preserve the hypothesis on K to be algebraically closed and of characteristic zero.

Let M be a rank 2 MCM module over R, and let $\mu(M)$ be the minimal number of generators of M. By Corollary 1.3 of [HK], we obtain that $\mu(M) \in \{3, 4, 5, 6\}$.

First of all we consider **the 3–generated case**. The description of the rank 1 MCM R–modules is given in [EP]. We recall the notations. For $a, b, c, d, \varepsilon \in K$ such that $a^3 = b^3 = c^3 = d^3 = -1, \varepsilon^3 = 1, \varepsilon \neq 1$, and $bcd = \varepsilon a$, we set

$$\alpha(b,c,d,\varepsilon) = \begin{pmatrix} 0 & x_1 - ax_4 & x_2 - bx_3 \\ x_1 - cx_2 & -b^2x_3 - abc^2\varepsilon^2x_4 & b^2c^2x_3 - abc\varepsilon^2x_4 \\ x_3 - dx_4 & c^2x_2 + bc^2x_3 + acx_4 & -x_1 - cx_2 - ax_4 \end{pmatrix}$$

and

$$\beta(b,c,d,\varepsilon) = \alpha(b,c,d,\varepsilon)^t,$$

that is, the transpose of $\alpha(b,c,d,\varepsilon)$. Then each of the matrices $\alpha(b,c,d,\varepsilon)$ and $\beta(b,c,d,\varepsilon)$ forms with its adjoint, $\alpha(b,c,d,\varepsilon)^*$, respectively $\beta(b,c,d,\varepsilon)^*$, a matrix factorization of f.

For $a, b, c \in K$, distinct roots of -1, and ε as above, we set

$$\eta(a,b,c,\varepsilon) = \begin{pmatrix} 0 & x_1 + x_2 & x_3 - ax_4 \\ x_1 + \varepsilon x_2 & -x_3 + cx_4 & 0 \\ x_3 - bx_4 & 0 & -x_1 - \varepsilon^2 x_2 \end{pmatrix}$$

and

$$\vartheta(a,b,c) = \begin{pmatrix} 0 & x_1 + x_3 & x_2 - ax_4 \\ x_1 - a^2bx_3 & -x_2 + cx_4 & 0 \\ x_2 - bx_4 & 0 & -x_1 + ab^2x_3 \end{pmatrix}.$$

The matrices $\eta(a,b,c,\varepsilon)$ and $\vartheta(a,b,c)$ form with their adjoint, $\eta(a,b,c,\varepsilon)^*$, respectively $\vartheta(a,b,c)^*$, a matrix factorization of f.

Theorem 2 ((3.4) in [EP]). *Let*

$$\mathcal{M} = \{\operatorname{Coker}\alpha(b,c,d,\varepsilon),\ \operatorname{Coker}\beta(b,c,d,\varepsilon) \mid b,c,d,\varepsilon \in K,$$
$$b^3 = c^3 = d^3 = -1,\ bcd = \varepsilon a,\ \varepsilon^3 = 1,\ \varepsilon \neq 1\}$$

and

$$\mathcal{N} = \{\operatorname{Coker}\eta(a,b,c,\varepsilon),\ \operatorname{Coker}\vartheta(a,b,c) \mid \varepsilon^3 = 1,\ \varepsilon \neq 1 \text{ and } (a,b,c) \text{ is a permutation of the roots of } -1\}.$$

Then the sets $\mathcal{M},\mathcal{N}$ of rank 1, 3–generated, MCM graded R–modules have the following properties:

(i) Every 3–generated, rank 1, indecomposable, graded MCM R–module is isomorphic with one module from $\mathcal{M}\cup\mathcal{N}$.

(ii) If $M = \operatorname{Coker}\alpha(b,c,d,\varepsilon)$ (or $M = \operatorname{Coker}\beta(b,c,d,\varepsilon)$) belongs to $\mathcal{M}$ and $N \in \mathcal{M}$, then $N \simeq M$ if and only if $N = \operatorname{Coker}\alpha(b\varepsilon, c\varepsilon, d\varepsilon, \varepsilon^2)$ (or $N = \operatorname{Coker}\beta(b\varepsilon, c\varepsilon, d\varepsilon, \varepsilon^2)$).

(iii) Any two different modules from $\mathcal{N}$ are not isomorphic.

(iv) Any module of $\mathcal{N}$ is not isomorphic with some module of $\mathcal{M}$.

The map $M \mapsto \Omega^1_R(M)$ is a bijection between the 3–generated, indecomposable, graded, MCM R–modules of rank two and the 3–generated, indecomposable, graded, MCM R–modules of rank 1. Thus, from the above theorem we obtain the description of the rank 2, 3–generated, indecomposable, graded MCM R–modules.

Theorem 3. *Let*

$$\mathcal{M}^* = \{\operatorname{Coker}\alpha(b,c,d,\varepsilon)^*,\ \operatorname{Coker}\beta(b,c,d,\varepsilon)^* \mid b,c,d,\varepsilon \in K,\ b^3 = c^3 = d^3 = -1, bcd = \varepsilon a,\ \varepsilon^3 = 1, \varepsilon \neq 1\}$$

and

$$\mathcal{N}^* = \{\operatorname{Coker}\eta(a,b,c,\varepsilon)^*,\ \operatorname{Coker}\vartheta(a,b,c)^* \mid \varepsilon^3 = 1,\ \varepsilon \neq 1 \text{ and } (a,b,c) \text{ is a permutation of the roots of } -1\}.$$

Then the sets $\mathcal{M}^*,\mathcal{N}^*$ of rank 2, 3–generated, MCM graded R–modules have the following properties:

(i) Every 3–generated, rank 2, indecomposable, graded MCM R–module is isomorphic with one module from $\mathcal{M}^*\cup\mathcal{N}^*$.

(ii) If $M = \operatorname{Coker}\alpha(b,c,d,\varepsilon)^*$ (or $M = \operatorname{Coker}\beta(b,c,d,\varepsilon)^*$) belongs to $\mathcal{M}^*$ and $N \in \mathcal{M}^*$, then $N \simeq M$ if and only if $N = \operatorname{Coker}\alpha(b\varepsilon, c\varepsilon, d\varepsilon, \varepsilon^2)^*$ (or $N = \operatorname{Coker}\beta(b\varepsilon, c\varepsilon, d\varepsilon, \varepsilon^2)^*$).

(iii) Any two different modules from $\mathcal{N}^*$ are not isomorphic.

(iv) Any module of $\mathcal{N}^*$ is not isomorphic with some module of $\mathcal{M}^*$.

Corollary 4. *There are 72 isomorphism classes of rank 2, indecomposable, graded MCM modules over R with three generators.*

2 Skew symmetric matrices and rank 2 orientable MCM modules

Let $\varphi = (a_{ij})_{1 \le i,j \le 2s}$ be a generic skew symmetric matrix, that is

$$a_{ii} = 0, a_{ij} = -a_{ji}, \text{ for all } i,j = 1,\dots,2s.$$

Then

$$\det(\varphi) = \operatorname{pf}(\varphi)^2,$$

where $\operatorname{pf}(\varphi)$ denotes the Pfaffian of φ (see [Bo1, §5, no. 2] or [BH, (3.4)]). Like determinants, Pfaffians can be developed along a row. Set φ_{ij} the matrix obtained from φ by deleting the i^{th} and j^{th} rows and columns. Then, for all $i = 1,\dots,2s$,

$$\operatorname{pf}(\varphi) = \sum_{\substack{j=1 \\ j \ne i}}^{2s} (-1)^{i+j}\sigma(i,j) a_{ij} \operatorname{pf}(\varphi_{ij}), \tag{1}$$

where $\sigma(i,j)$ denotes $\operatorname{sign}(j-i)$. Multiplying (1) by $\operatorname{pf}(\varphi)$, we obtain

$$\det(\varphi) = \sum_{j=1}^{2s} a_{ij} b_{ij}, \tag{2}$$

for $b_{ij} = (-1)^{i+j}\sigma(i,j)\operatorname{pf}(\varphi_{ij})\operatorname{pf}(\varphi)$ when $i \ne j$ and $b_{ii} = 0$. Since φ is a generic matrix we see from (2) that b_{ij} is exactly the algebraic complement of a_{ij} and so the transpose matrix B of (b_{ij}) is the adjoint matrix of φ. Set

$$\psi = \frac{1}{\operatorname{pf}(\varphi)} B.$$

Then

$$\varphi\psi = \psi\varphi = \operatorname{pf}(\varphi)\operatorname{Id}_{2s},$$

as it is stated also in [[JP], §3].

Proposition 5. *Let $f = x_1^3 + x_2^3 + x_3^3 + x_4^3$ and φ a skew symmetric matrix over $S = K[x_1, x_2, x_3, x_4]$ of order 4 or 6 such that $\det\varphi = f^2$, K being a field. Then $\operatorname{Coker}\varphi$ is an MCM module over $R := S/(f)$ of rank 2.*

Proof. Let ψ be given for φ as above, that is the (i,j) entry of ψ is $(-1)^{i+j}\sigma(j,i)\operatorname{pf}(\varphi_{ij})$. As above we have

$$\varphi\psi = \psi\varphi = f \cdot \operatorname{Id}_n,\ n = 4 \text{ or } 6$$

because $\operatorname{pf}(\varphi) = f$. Then (φ, ψ) is a matrix factorization which defines an MCM R–module of rank 2. □

Theorem 6. *Preserving the hypothesis of Proposition 5, the cokernel of a homogeneous skew symmetric matrix over S of order 4 or 6 of determinant f^2 defines a graded MCM R–module M of rank 2. Conversely, each non–free graded orientable MCM R–module M of rank 2 is the cokernel of a map given by a skew symmetric homogeneous matrix φ over S of order 4 or 6, whose determinant is f^2 and φ together with ψ, defined above, form the matrix factorization of M.*

Proof. According to Herzog and Kühl [HK], M must be 4 or 6 minimally generated. Suppose that M is 6–generated (the other case is similar). Then M is the second syzygy over R of a Gorenstein ideal $I \subset R$ of codimension 2, which is 5–generated by [HK]. Using the Buchsbaum–Eisenbud Theorem (see e.g. [BH], (3,4)) there exists an exact sequence

$$0 \longrightarrow S(-5) \xrightarrow{d_3} S^5(-3) \xrightarrow{d_2} S^5(-2) \xrightarrow{d_1} S \tag{3}$$

such that $J = \operatorname{Im} d_1, I = J/(f), d_2$ is a skew symmetric homogeneous matrix, d_3 is the dual of d_1, $d_3 = d_1^t$, and

$$d_1 = \Big(\operatorname{pf}\big((d_2)_1\big), -\operatorname{pf}\big((d_2)_2\big), \dots, \operatorname{pf}\big((d_2)_5\big)\Big),$$

where $(d_2)_i$ denotes the 4×4 skew symmetric matrix obtained by deleting the i^{th} row and column of d_2. (We will see at the end of this proof that, indeed, the entries of d_2 are linear forms).

Since $f \in J$ there exists $v : S(-1) \longrightarrow S^5$ such that $d_1 v = f$ (v is given by linear forms). It is easy to see from (3) that $I = J/(f)$ has the following minimal resolution over S :

$$0 \longrightarrow S(-5) \xrightarrow{\binom{d_3}{0}} S^6(-3) \xrightarrow{(d_2,v)} S^5(-2) \xrightarrow{\tilde{d}_1} I \longrightarrow 0.$$

As in [Ei], since $fI = 0$, there exists a map $h : S^5(-5) \to S^6(-3)$ such that $(d_2, v)h = -f \cdot \mathrm{Id}_5$ and we obtain the following exact sequence

$$R^6(-5) \xrightarrow{\left(\bar{h}|{-\bar{d}_3 \atop 0}\right)} R^6(-3) \xrightarrow{(\bar{d}_2,\bar{v})} R^5(-2) \xrightarrow{\bar{d}_1} I \longrightarrow 0. \tag{4}$$

On the other hand, $\varphi = \begin{pmatrix} d_2 & v \\ -v^t & 0 \end{pmatrix}$ is a skew symmetric homogeneous matrix of order 6. Let ψ given as above. By construction ψ has the form $\begin{pmatrix} c & d_1^t \\ -d_1 & 0 \end{pmatrix}$ and so $(d_2, v)\begin{pmatrix} c \\ -d_1 \end{pmatrix} = -f \cdot \mathrm{Id}_5 = \mathrm{pf}(\varphi)\,\mathrm{Id}_5$. Taking $h = \begin{pmatrix} c \\ -d_1 \end{pmatrix}$ above, we obtain from (4) the following exact sequence:

$$R^6(-6) \xrightarrow{\varphi} R^6(-5) \xrightarrow{\psi} R^6(-3) \xrightarrow{(\bar{d}_2,\bar{v})} R^5(-2) \xrightarrow{\bar{d}_1} I \longrightarrow 0,$$

which gives

$$\mathrm{Coker}\,\varphi \cong \mathrm{Im}\,\psi = \Omega_R^2(I).$$

We have $\mathrm{pf}(\varphi) = -f$ and so $\det(\varphi) = f^2$. Therefore the entries of φ are linear forms and as a consequence, the entries of d_2 are linear forms. □

3 Orientable, rank 2, 4–generated MCM modules

Let K be an algebraically closed field of characteristic zero,
$S = K[x_1, x_2, x_3, x_4]$, $f = x_1^3 + x_2^3 + x_3^3 + x_4^3$, and $R = S/(f)$. Let M be a graded, indecomposable, 4–generated MCM R–module of rank 2. After Herzog and Kühl [HK], $M \cong \Omega_R^2(I)$, where I is a graded 3–generated Gorenstein ideal such that $\dim R/I = 1$. Then $I = J/(f)$, with $J \subset S$ a graded, 3–generated ideal containing f, $f \in mJ$ by [HK]". Let $\alpha_1, \alpha_2, \alpha_3$ be a minimal system of homogeneous generators of J. Since $\dim S/J = 1$, it follows that $\alpha_1, \alpha_2, \alpha_3$ is a regular system of elements in S.

Let $u, a, b \in K$ with $a^3 = b^3 = -1, u^2 + u + 1 = 0$ and $\sigma = (i\ j\ s)$ be a permutation of the set $\{2, 3, 4\}$ with $i < j$. Set

$$\begin{array}{rclrcl} w_{\sigma 1} & = & x_1 - a x_s, & w_{\sigma 2} & = & x_i - b x_j, \\ v_{\sigma 1} & = & x_1^2 + a x_1 x_s + a^2 x_s, & v_{\sigma 2} & = & x_i^2 + b x_i x_j + b^2 x_j^2. \end{array}$$

then we have

$$f = w_{\sigma 1} v_{\sigma 1} + w_{\sigma 2} v_{\sigma 2}.$$

Let $\lambda = [\lambda_1 : \lambda_2 : \lambda_3 : 1]$ be a point of the surface $V(f) \subset \mathbb{P}^3$. We set

$$p_{i\lambda} = x_i - \lambda_i x_4, \ \text{ and } \ q_{i\lambda} = x_i^2 + \lambda_i x_i x_4 + \lambda_i^2 x_4^2, \ \text{ for } \ 1 \leq i \leq 3.$$

Let $\lambda = [\lambda_1 : \lambda_2 : 1 : 0]$ be a point of $V(f)$. We set

$$p_{i\lambda} = x_i - \lambda_i x_3, \ q_{i\lambda} = x_i^2 + \lambda_i x_i x_3 + \lambda_i^2 x_3^2, \ \text{ for } \ 1 \leq i \leq 2$$

and

$$p_{3\lambda} = x_4, \ q_{3\lambda} = x_4^2.$$

If $\lambda = [\lambda_1 : 1 : 0 : 0] \in V(f)$, we set

$$p_{1\lambda} = x_1 - \lambda_1 x_2, \ q_{1\lambda} = x_1^2 + \lambda_1 x_1 x_2 + \lambda_1^2 x_2^2$$

and

$$p_{2\lambda} = x_3, \ p_{3\lambda} = x_4, q_{2\lambda} = x_3^2, \ q_{3\lambda} = x_4^2.$$

In all cases we have

$$f = \sum_{i=1}^{3} p_{i\lambda} q_{i\lambda}.$$

Since $f \in (\alpha_1, \alpha_2, \alpha_3)$ and eventually we are interested in $\Omega_R^2(\alpha_1, \alpha_2, \alpha_3)$, we may suppose that either α_i is in the set $\{p_{i\lambda}, q_{i\lambda}\}$ for each $1 \leq i \leq 3$, or α_i is in the set $\{w_{\sigma i}, v_{\sigma i}\}$ for each $1 \leq i \leq 2$ and $\beta = \alpha_3$ is a regular element in $R/(\alpha_1, \alpha_2)$.

Lemma 7. *Let M be a graded, indecomposable, 4–generated MCM R–module of rank 2. Then M is one of the following modules:*

(1) $\Omega_R^2(p_{1\lambda}, p_{2\lambda}, p_{3\lambda})$ *or* $\Omega_R^2(q_{1\lambda}, q_{2\lambda}, q_{3\lambda})$*, for some* $\lambda \in V(f)$*,*

(2) $\Omega_R^2(w_{\sigma 1}, v_{\sigma 2}, \beta)$ *or* $\Omega_R^2(w_{\sigma 2}, v_{\sigma 1}, \beta)$ *for some* a, b, σ *and* β *as above.*

Proof. Set

$$I_\lambda = (p_{1\lambda}, p_{2\lambda}, p_{3\lambda})$$

and

$$\varphi_\lambda = \begin{pmatrix} 0 & p_{3\lambda} & -p_{2\lambda} & -q_{1\lambda} \\ -p_{3\lambda} & 0 & -p_{1\lambda} & q_{2\lambda} \\ p_{2\lambda} & p_{1\lambda} & 0 & q_{3\lambda} \\ q_{1\lambda} & -q_{2\lambda} & -q_{3\lambda} & 0 \end{pmatrix}, \ \psi_\lambda = \begin{pmatrix} 0 & -q_{3\lambda} & q_{2\lambda} & p_{1\lambda} \\ q_{3\lambda} & 0 & q_{1\lambda} & -p_{2\lambda} \\ -q_{2\lambda} & -q_{1\lambda} & 0 & -p_{3\lambda} \\ -p_{1\lambda} & p_{2\lambda} & p_{3\lambda} & 0 \end{pmatrix}.$$

We have the following exact sequence:

$$R^3(-5) \oplus R(-6) \xrightarrow{\varphi_\lambda} R^4(-4) \xrightarrow{\psi_\lambda} R^3(-2) \oplus R(-3) \xrightarrow{A} R^3(-1) \xrightarrow{\tau} I_\lambda \longrightarrow 0,$$

where $\tau = (-p_{1\lambda}, p_{2\lambda}, p_{3\lambda})$ and A are given by the first three rows of φ_λ. Thus, $\Omega^2(I_\lambda) \cong \mathrm{Coker}(\varphi_\lambda)$ and $(\varphi_\lambda, \psi_\lambda)$ is a matrix factorization of $\Omega^2(I_\lambda)$. The ideals I_λ and $(q_{1\lambda}, q_{2\lambda}, p_{3\lambda})$ belong to the same even linkage class since

$$I_\lambda \sim (q_{1\lambda}, p_{2\lambda}, p_{3\lambda}) \sim (q_{1\lambda}, q_{2\lambda}, p_{3\lambda}).$$

For the first link we consider the regular sequence $\{p_{1\lambda}q_{1\lambda}, p_{2\lambda}, p_{3\lambda}\}$ and for the second one the sequence $\{q_{1\lambda}, p_{2\lambda}q_{2\lambda}, p_{3\lambda}\}$. Similarly, one can see that I_λ is evenly linked with the ideals $(q_{1\lambda}, p_{2\lambda}, q_{3\lambda})$ and $(p_{1\lambda}, q_{2\lambda}, q_{3\lambda})$. By [HK, Theorem 2.1], we obtain that

$$\begin{aligned} \mathrm{Coker}(\varphi_\lambda) \cong \Omega^2_R(I_\lambda) \cong \Omega^2_R(q_{1\lambda}, q_{2\lambda}, p_{3\lambda}) \cong \Omega^2_R(q_{1\lambda}, p_{2\lambda}, q_{3\lambda}) \\ \cong \Omega^2_R(p_{1\lambda}, q_{2\lambda}, q_{3\lambda}). \end{aligned}$$

Analogously, we see that

$$\begin{aligned} \mathrm{Coker}(\psi_\lambda) \cong \Omega^2_R(q_{1\lambda}, q_{2\lambda}, q_{3\lambda}) \cong \Omega^2_R(p_{1\lambda}, p_{2\lambda}, q_{3\lambda}) \cong \Omega^2_R(p_{1\lambda}, q_{2\lambda}, p_{3\lambda}) \\ \cong \Omega^2_R(q_{1\lambda}, p_{2\lambda}, p_{3\lambda}). \end{aligned}$$

Thus, the case when α_i is one of the forms $\{p_{i\lambda}, q_{i\lambda}\}$ gives (1).

Now let σ, a, b as above and $\beta \in S$ which is regular on $R/(w_{\sigma 1}, v_{\sigma 2})$. Set

$$I_{\sigma\beta}(a, b, u) = (w_{\sigma 1}, v_{\sigma 2}, \beta)$$

and

$$\varphi_{\sigma\beta}(a, b, u) = \begin{pmatrix} 0 & w_{\sigma 1} & -v_{\sigma 2} & 0 \\ -w_{\sigma 1} & 0 & -\beta & w_{\sigma 2} \\ v_{\sigma 2} & \beta & 0 & v_{\sigma 1} \\ 0 & -w_{\sigma 2} & -v_{\sigma 1} & 0 \end{pmatrix},$$

$$\psi_{\sigma\beta}(a,b,u) = \begin{pmatrix} 0 & -v_{\sigma 1} & w_{\sigma 2} & \beta \\ v_{\sigma 1} & 0 & 0 & -v_{\sigma 2} \\ -w_{\sigma 2} & 0 & 0 & -w_{\sigma 1} \\ -\beta & v_{\sigma 2} & w_{\sigma 1} & 0 \end{pmatrix}.$$

We have the following exact sequence:

$$R^4 \xrightarrow{\varphi_{\sigma\beta}(a,b,u)} R^4 \xrightarrow{\psi_{\sigma\beta}(a,b,u)} R^4 \xrightarrow{B} R^3 \xrightarrow{\tau'} I_{\sigma\beta}(a,b,u) \longrightarrow 0\,,$$

where $\tau' = (-\beta, v_{\sigma 2}, w_{\sigma 1})$ and B is the matrix given by the first three rows of $\varphi_{\sigma\beta}(a,b,u)$. Thus,

$$\Omega^2_R\big(I_{\sigma\beta}(a,b,u)\big) \cong \mathrm{Coker}\big(\varphi_{\sigma\beta}(a,b,u)\big).$$

As above, we see that

$$\Omega^2_R\big(I_{\sigma\beta}(a,b,u)\big) \cong \Omega^2_R(w_{\sigma 2}, v_{\sigma 1}, \beta)$$

and

$$\Omega^2_R(w_{\sigma 1}, w_{\sigma 2}, \beta) \cong \Omega^2_R(v_{\sigma 1}, v_{\sigma 2}, \beta) \cong \mathrm{Coker}\big(\psi_{\sigma\beta}(a,b,u)\big).$$

Thus, the case when α_i is one of the forms $\{w_{\sigma i}, v_{\sigma i}\}$ for $i \leq 2$ gives (2). □

Let

$$\mathcal{M} = \big\{\mathrm{Coker}(\varphi_\lambda), \mathrm{Coker}(\psi_\lambda) \mid \lambda \in V(f)\big\}.$$

For a, b, σ as above, set

$$\varphi_\sigma(a,b,u) = \varphi_{\sigma, x_j x_s}(a,b,u),\ \ \psi_\sigma(a,b,u) = \psi_{\sigma, x_j x_s}(a,b,u),$$

that is, $\beta = x_j x_s$. Let

$$\mathcal{P} = \big\{\mathrm{Coker}\big(\varphi_\sigma(a,b,u)\big), \mathrm{Coker}(\psi_\sigma(a,b,u)) \mid a, b, \sigma \text{ as above }\big\}.$$

Theorem 8. *The set $\mathcal{M} \cup \mathcal{P}$ contains only non–isomorphic, indecomposable, graded, orientable, 4–generated MCM R–modules of rank 2 and every indecomposable, graded, orientable, 4–generated MCM R–module of rank 2 is isomorphic with one module of $\mathcal{M} \cup \mathcal{P}$.*

Proof. Applying Lemma 7, we must show in the case (2) that β can be taken $x_j x_s$. Since $v_{\sigma 1} - w_{\sigma 1}(x_1 + 2a x_s) = 3a^2 x_s^2$, adding in $\varphi_{\sigma\beta}(a,b,u)$ multiples of the last row to the second one and multiples of the first column to the third one, we may suppose the entry $(2,3)$ of the form $\gamma + x_s\delta$, with γ, δ depending only on x_j, x_i. These transformations modify the entries $(2,2), (3,3)$ which are now possibly non–zero. Adding similar multiples of the last column to the second one and multiples of the first row to the third one, we obtain $\varphi_{\sigma,\beta}(a,b,u)$ of the same type as before but with $\beta = \gamma + x_s\delta$. We may reduce to consider $\delta \notin K$. Indeed, if $\delta \in K$, then, acting on the rows and columns of $\varphi_{\sigma\beta}(a,b,u)$, we obtain that $M = \mathrm{Coker}(\varphi_{\sigma\beta}(a,b,u))$ is decomposable or belongs to the set $\mathcal{M}$. Now let δ be not constant. Similarly, adding in $\varphi_{\sigma\beta}(a,b,u)$ multiples of the first row to the second one and multiples of the last column to the third one we may suppose that the entry $(2,3)$ has the form $\varepsilon x_j x_s$ with $\varepsilon \in K$. These transformations modify the entries $(2,2), (3,3)$. After similar transformations, we obtain $\varphi_{\sigma\beta}(a,b,u)$ of the same type as before but with $\beta = \varepsilon x_j x_s$. If $\varepsilon = 0$ we see that $\varphi_{\sigma\beta}(a,b,u)$ is a direct sum of two 2×2–matrices, which contradicts the indecomposability of $M = \mathrm{Coker}\big(\varphi_{\sigma\beta}(a,b,u)\big)$. So $\varepsilon \neq 0$. Divide the second and the third column of $\varphi_{\sigma\beta}(a,b,u)$ with ε, and multiply the first and the last row of $\varphi_{\sigma\beta}(a,b,u)$ with ε. We reduce to the case $\varepsilon = 1$, that is $\beta = x_j x_s$.

Now we show that two different modules from $\mathcal{M} \cup \mathcal{P}$ are not isomorphic. Note that the Fitting ideals of φ_λ (respectively ψ_λ) modulo $(x_1, \ldots, x_4)^2$ have the form $(p_{1\lambda}, p_{2\lambda}, p_{3\lambda})$ and the Fitting ideals of $\varphi_\sigma(a,b,u)$ (respectively $\psi_\sigma(a,b,u)$) modulo $(x_1, \ldots, x_4)^2$ have the form $(w_{\sigma 1}, w_{\sigma 2})$ and these ideals are all different. Thus,

$$\big\{\mathrm{Coker}(\varphi_\lambda) \mid \lambda \in V(f)\big\} \cup \big\{\mathrm{Coker}\big(\varphi_\sigma(a,b,u)\big) \mid \sigma, a, b \text{ as above } \big\}$$

contains only non–isomorphic modules (similarly for ψ's). It follows that, if $N, P \in \mathcal{M} \cup \mathcal{P}$ are isomorphic and different, then $N \simeq \Omega^1_R(P)$.

If $N = \mathrm{Coker}(\varphi_\lambda)$, for $\lambda \in V(f)$, then this is not possible since the ideals $(p_{1\lambda}, p_{2\lambda}, p_{3\lambda})$ and $(q_{1\lambda}, q_{2\lambda}, q_{3\lambda})$ are not in the same even linkage class. Indeed, by the proof of (1) in Lemma 7, $(p_{1\lambda}, p_{2\lambda}, p_{3\lambda})$ is evenly linked with $(q_{1\lambda}, q_{2\lambda}, p_{3\lambda})$ and this last ideal is obviously directly linked with $(q_{1\lambda}, q_{2\lambda}, q_{3\lambda})$. If $N = \mathrm{Coker}\big(\varphi_\sigma(a,b,u)\big)$ for some σ, a, b, and $N \simeq \Omega^1_R(N)$, then the ideals $(w_{\sigma 1}, v_{\sigma 2}, x_j x_s)$ and $(w_{\sigma 1}, w_{\sigma 2}, x_j x_s)$ are evenly linked. But these ideals are directly linked by the regular sequence $\{w_{\sigma 1}, v_{\sigma 2} w_{\sigma 2}, x_j x_s\}$, contradiction!

It remains to show that $\mathcal{M} \cup \mathcal{P}$ contains only indecomposable modules. If $N \in \mathcal{M}$, let us say $N = \mathrm{Coker}(\varphi_\lambda)$ for $\lambda = [\lambda_1 : \lambda_2 : \lambda_3 : 1]$, we see that

N/x_4N is exactly the module corresponding to the matrix

$$\begin{pmatrix} 0 & x_3 & -x_2 & -x_1^2 \\ -x_3 & 0 & -x_1 & x_2^2 \\ x_2 & x_1 & 0 & x_3^2 \\ x_1^2 & -x_2^2 & -x_3^2 & 0 \end{pmatrix}$$

whose cokernel is the special module M_2 (see [LPP] for the special module of rank 2 which corresponds to the special bundle from Atiyah classification). Thus, N/x_4N is indecomposable and, by Nakayama's Lemma, N is indecomposable. Now let $N \in \mathcal{P}$, $N = \mathrm{Coker}\big(\psi_\sigma(a,b,u)\big)$. By the permutation of the rows and the columns of $\psi_\sigma(a,b,u)$, we may suppose that it has the form:

$$\begin{pmatrix} w_{\sigma 1} & -v_{\sigma 2} & x_j x_s & 0 \\ w_{\sigma 2} & v_{\sigma 1} & 0 & x_j x_s \\ 0 & 0 & v_{\sigma 1} & v_{\sigma 2} \\ 0 & 0 & -w_{\sigma 2} & w_{\sigma 1} \end{pmatrix}.$$

Suppose N is decomposable. Then $\psi_\sigma(a,b,u)$ is equivalent with a direct sum of two matrices of order 2 A_1, A_2. Let B_1, B_2 be the submatrices of the $\psi_\sigma(a,b,u)$ given by the first two lines and columns, respectively the last two lines and columns. Certainly A_1, A_2, B_1, B_2 define some maximal Cohen-Macaulay modules of rank one N_1, N_2, T_1, T_2, and due to the particular form of $\psi_\sigma(a,b,u)$ we have the following exact sequence

$$0 \to T_1 \to N_1 \oplus N_2 = N \to T_2 \to 0.$$

Note that $\psi_\sigma(a,b,u)$ is modulo x_j or x_s the sum of B_1, B_2. Thus $T_i/x_jT_i \cong N_i/x_jN_i$ for $i = 1,2$ and similarly for x_s. Since we have the whole description of rank one maximal Cohen-Macaulay modules we can see that A_i is equivalent with B_i modulo x_j and modulo x_s only when A_i is equivalent with B_i. Thus $T_i \cong N_i$ for $i = 1,2$ and so $N \cong T_1 \oplus T_2$. By a subtle result of Miyata ([Mi]) this happens only if the above exact sequence splits. This means that there exist two matrices A, B of order two such that

$$x_j x_s \cdot \mathrm{Id}_2 = \begin{pmatrix} w_{\sigma 1} & -v_{\sigma 2} \\ w_{\sigma 2} & v_{\sigma 1} \end{pmatrix} A + B \begin{pmatrix} v_{\sigma 1} & v_{\sigma 2} \\ -w_{\sigma 2} & w_{\sigma 1} \end{pmatrix},$$

which is impossible. □

Remarks 9. (1) There exists a bijection between

$$\mathcal{P}_1 = \big\{\mathrm{Coker}\big(\varphi_\sigma(a,b,u)\big) \mid \sigma, a, b\big\}$$

and the 2–generated, non–free, MCM R–modules, which remind us of Atiyah's classification. Thus, $\mathcal{P}_1$ contains 27 modules corresponding to 27 lines and 27 pencils of conics of $V(f)$.
Similarly, $\mathcal{P}_2 = \big\{\mathrm{Coker}\big(\psi_\sigma(a,b,u)\big) \mid \sigma, a, b\big\}$ contains 27 modules.

(2) $\mathcal{M}$ is a kind of "blowing up" of $M_2, \Omega^1_R(M_2)$ from [LPP] (see the proof of Theorem 8). Note also that $\mathcal{M}$ consists of two classes of modules parameterized by the points of $V(f)$, which is also in Atiyah's idea.

(3) The matrices φ defining the modules of $\mathcal{M} \cup \mathcal{P}$ are skew symmetric as our Theorem 6 predicted.

4 Non–orientable, rank 2, 4–generated MCM modules

Let M be a graded non–orientable, rank 2, MCM R–module, without free direct summands. We should like to express M as a 2–syzygy of an ideal I, $M \cong \Omega^2_R(I)$, with $\mu(M) = \mu(I)+1$ (this is known in orientable case by [HK], see here Section 3).

The following proposition can be found in [B, Korollar 2].

Proposition 10. *Let (A, m) be a Noetherian normal local domain with $\dim A \geq 2$ and N a finite torsion–free A–module. Then there exists a finite free submodule $F \subset N$ such that N/F is isomorphic with an ideal of A and the canonical map $F/mF \to N/mN$ is injective.*

Applying Proposition 10, we obtain the following exact sequence:

$$0 \to R \longrightarrow M \longrightarrow I \longrightarrow 0 \tag{5}$$

for an ideal $I \subset R$, which induces an exact sequence

$$0 \longrightarrow K = R/m \longrightarrow M/mM \longrightarrow I/mI \longrightarrow 0.$$

Thus $\mu(M) = \mu(I) + 1$.

As we know in the orientable case to obtain MCM R–modules of rank 2 we must choose I such that $\mathrm{Ext}^1_R(I, R)$ is a cyclic R–module or, more precisely, such that R/I is Gorenstein. In the non–orientable case one can also show that $\mathrm{Ext}^1_R(I, R)$ must be a cyclic R–module, but this is not very helpful since it is hard to check this condition for arbitrary I. Below we shall state an easier condition.

Let $J \subset S = K[X_1, \ldots, X_4]$ be an ideal such that $f \in mJ$ and $I = J/(f)$.

Lemma 11. *Let*

$$0 \longrightarrow S^{s_3} \xrightarrow{d_3} S^{s_2} \xrightarrow{d_2} S^{s_1} \xrightarrow{d_1} J \longrightarrow 0$$

be a minimal free S–resolution of an ideal J with depth $S/J = 1$.

If $\operatorname{rank} \Omega^2_R\big(J/(f)\big) = 2$ *and* $\mu\Big(\Omega^2_R\big(J/(f)\big)\Big) = \mu(I) + 1$ *then* $s_1 = s_2 \leq 5$ *and* $s_3 = 1$.

Proof. As in the proof of Theorem 6, we obtain a minimal free resolution of $I = J/(f)$ over S in the following way:

Let $v : S \to S^{s_1}$ be an S–linear map such that $jd_1v = f\,\mathrm{Id}_S$, where $j : J \to S$ is the inclusion. Let $\tilde{d}_1$ be the composite map $S^{s_1} \xrightarrow{d_1} J \to J/(f) = I$. Then the following sequence

$$0 \longrightarrow S^{s_3} \xrightarrow{\binom{d_3}{0}} S^{s_2+1} \xrightarrow{(d_2,v)} S^{s_1} \xrightarrow{\bar{d}_1} I \longrightarrow 0$$

is exact and forms a minimal free resolution of I over S. Since

$$f \cdot S^{s_1} \subset \mathrm{Im}(d_2, v),$$

there exists an S–linear map $h : S^{s_1} \to S^{s_2+1}$ such that

$$(d_2, v)h = f\,\mathrm{Id}_{S^{s_1}}$$

and we obtain the following exact sequence

$$R^{s_3+s_1} \xrightarrow{\left(\bar{h}\middle|\begin{smallmatrix}\bar{d}_3\\0\end{smallmatrix}\right)} R^{s_2+1} \xrightarrow{(\bar{d}_2,\bar{v})} R^{s_1} \xrightarrow{\bar{d}_1} I \longrightarrow 0,$$

which is part of a minimal free R–resolution of I. Thus, $M = \Omega_R^2(I)$ is the image of the first map above and so $s_1 + s_3 = s_2 + 1 = s_1 + 1$ because $\mu(M) = \mu\big(\Omega_R^1(M)\big) = \mu(I) + 1$ by hypothesis. It follows that $s_3 = 1$, $s_1 = s_2$. As $\mu(M) \leq 3\,\mathrm{rank}_R M = 6$ we obtain $s_1 \leq 5$. □

Let $\det N$ be the corresponding class of the bidual $(\wedge^n N)^{**}$, $n = \mathrm{rank}\, N$, in $Cl(R)$ for a torsion free R–module N. Since det is an additive function, we obtain $\det(M) = 0$ if and only if $\det(I) = 0$. Thus, M is non–orientable if and only if I is non–orientable, that is, $\mathrm{codim}(J) \leq 1$ for all ideals $J \subset R$ isomorphic with I, according to [HK]. Since M has rank 2, we obtain $\mathrm{codim}(I) = 1$. Thus, $\dim R/I = 2$ and, from (5), we obtain $\mathrm{depth}\, R/I = 1$, that is, R/I is not Cohen–Macaulay. Also from (5) we obtain $\Omega_R^2(M) \simeq \Omega_R^2(I)$ and so $M \simeq \Omega_R^2(I)$.

Proposition 12. *Each graded, non–orientable, rank 2, s–generated MCM R–module is the second syzygy $\Omega_R^2(I)$ of an $(s-1)$–generated graded ideal $I \subset R$ with* $\mathrm{depth}\, R/I = 1$ *and* $\dim R/I = 2$.

As in Section 3, let $u, a, b \in K$, with

$$a^3 = b^3 = -1,\ u^2 + u + 1 = 0,$$

$\sigma = (i\ j\ s)$ be a permutation of the set $\{2, 3, 4\}$ with $i < j$ and set

$$\begin{array}{rclrcl} w_{\sigma 1} & = & x_1 - a x_s, & w_{\sigma 2} & = & x_i - b x_j, \\ v_{\sigma 1} & = & x_1^2 + a x_1 x_s + a^2 x_s^2, & v_{\sigma 2} & = & x_i^2 + b x_i x_j + b^2 x_j^2. \end{array}$$

We have

$$v_{\sigma 1} = v'_{\sigma 1} v''_{\sigma 1},\ v_{\sigma 2} = v'_{\sigma 2} v''_{\sigma 2}$$

for

$$\begin{array}{rclrcl} v'_{\sigma 1} & = & x_1 - u a x_s, & v''_{\sigma 1} & = & x_1 + (1+u) a x_s, \\ v'_{\sigma 2} & = & x_i - u b x_j, & v''_{\sigma 2} & = & x_i + (1+u) b x_j. \end{array}$$

Set

$$\begin{aligned}
I_{1\sigma}(a,b,u) &= (x_s v'_{\sigma 2}, v_{\sigma 2}, w_{\sigma 1}),\\
I_{2\sigma}(a,b,u) &= (x_j v''_{\sigma 1}, v_{\sigma 1}, w_{\sigma 2}),\\
I_{3\sigma}(a,b,u) &= (x_s v''_{\sigma 2}, v_{\sigma 2}, v_{\sigma 1}),\\
I_{4\sigma}(a,b,u) &= (x_j v'_{\sigma 1}, v_{\sigma 1}, v_{\sigma 2}),\\
I_{5\sigma}(a,b,u) &= (x_s v''_{\sigma 2}, v_{\sigma 2}, w_{\sigma 1}),\\
I_{6\sigma}(a,b,u) &= (x_j v'_{\sigma 1}, v_{\sigma 1}, w_{\sigma 2}),\\
I_{7\sigma}(a,b,u) &= (x_s v'_{\sigma 2}, v_{\sigma 2}, v_{\sigma 1}),\\
I_{8\sigma}(a,b,u) &= (x_j v''_{\sigma 1}, v_{\sigma 2}, v_{\sigma 1})
\end{aligned}$$

Set

$$\varphi_{1\sigma}(a,b,u) = \begin{pmatrix} w_{\sigma 1} & -w_{\sigma 2} & 0 & x_s \\ v_{\sigma 2} & v_{\sigma 1} & x_s v'_{\sigma 2} & 0 \\ 0 & 0 & w_{\sigma 1} & -v''_{\sigma 2} \\ 0 & 0 & -w_{\sigma 2} v'_{\sigma 2} & -v_{\sigma 1} \end{pmatrix},$$

$$\varphi_{2\sigma}(a,b,u) = \begin{pmatrix} w_{\sigma 1} & -w_{\sigma 2} & 0 & -x_j \\ v_{\sigma 2} & v_{\sigma 1} & x_j v''_{\sigma 1} & 0 \\ 0 & 0 & w_{\sigma 2} & -v'_{\sigma 1} \\ 0 & 0 & -w_{\sigma 1} v''_{\sigma 1} & -v_{\sigma 2} \end{pmatrix},$$

$$\psi_{3\sigma}(a,b,u) = \begin{pmatrix} w_{\sigma 1} & -w_{\sigma 2} & -x_s & 0 \\ v_{\sigma 2} & v_{\sigma 1} & 0 & -x_s v''_{\sigma 2} \\ 0 & 0 & -v'_{\sigma 2} & w_{\sigma 1} \\ 0 & 0 & -v_{\sigma 1} & -w_{\sigma 2} v''_{\sigma 2} \end{pmatrix},$$

$$\psi_{4\sigma}(a,b,u) = \begin{pmatrix} w_{\sigma 1} & -w_{\sigma 2} & x_j & 0 \\ v_{\sigma 1} & v_{\sigma 2} & 0 & -x_j v'_{\sigma 1} \\ 0 & 0 & -v''_{\sigma 1} & w_{\sigma 2} \\ 0 & 0 & -v_{\sigma 2} & -w_{\sigma 1} v'_{\sigma 1} \end{pmatrix},$$

$$\varphi_{3\sigma}(a,b,u) = \begin{pmatrix} v_{\sigma 1} & w_{\sigma 2} & 0 & -x_s \\ -v_{\sigma 2} & w_{\sigma 1} & x_s v''_{\sigma 2} & 0 \\ 0 & 0 & -w_{\sigma 2} v''_{\sigma 2} & -w_{\sigma 1} \\ 0 & 0 & v_{\sigma 1} & -v'_{\sigma 2} \end{pmatrix},$$

$$\varphi_{4\sigma}(a,b,u) = \begin{pmatrix} v_{\sigma 1} & w_{\sigma 2} & x_j v'_{\sigma 1} & 0 \\ -v_{\sigma 2} & w_{\sigma 1} & 0 & -x_j \\ 0 & 0 & -w_{\sigma 1} v'_{\sigma 1} & -w_{\sigma 2} \\ 0 & 0 & v_{\sigma 2} & 0 - v''_{\sigma 1} \end{pmatrix},$$

$$\psi_{1\sigma}(a,b,u) = \begin{pmatrix} v_{\sigma 1} & w_{\sigma 2} & 0 & x_s \\ -v_{\sigma 2} & w_{\sigma 1} & -x_s v'_{\sigma 2} & 0 \\ 0 & 0 & v_{\sigma 1} & -v''_{\sigma 2} \\ 0 & 0 & -w_{\sigma 2} v'_{\sigma 2} & -w_{\sigma 1} \end{pmatrix},$$

$$\psi_{2\sigma}(a,b,u) = \begin{pmatrix} v_{\sigma 1} & w_{\sigma 2} & 0 & x_j \\ -v_{\sigma 2} & w_{\sigma 1} & -x_j v''_{\sigma 1} & 0 \\ 0 & 0 & v_{\sigma 2} & -v'_{\sigma 1} \\ 0 & 0 & -w_{\sigma 1} v''_{\sigma 1} & -w_{\sigma 2} \end{pmatrix}.$$

Theorem 13. *(1) For each $1 \le t \le 4$, the pair $\big(\varphi_{t\sigma}(a,b,u), \psi_{t\sigma}(a,b,u)\big)$ forms a matrix factorization of $\Omega^2_R\big(I_{t\sigma}(a,b,u)\big)$.*

(2) The set

$$\mathcal{N} = \big\{\mathrm{Coker}\big(\varphi_{t\sigma}(a,b,u)\big),\ \mathrm{Coker}\big(\psi_{t\sigma}(a,b,u)\big) \mid 1 \le t \le 4,\ \sigma, a, b, u\big\}$$

contains only graded, indecomposable, non–orientable, 4–generated MCM R–modules of rank 2.

(3) Every indecomposable, graded, non–orientable, 4–generated MCM module over R of rank 2 is isomorphic with one module of $\mathcal{N}$.

(4) The modules of $\mathcal{N}$ are pairwise isomorphic. In particular, there exist 216 isomorphism classes of indecomposable, graded, non–orientable, 4–generated MCM modules over R of rank 2.

Proof. (1) It is easy to check that

$$\varphi_{t\sigma}(a,b,u) \cdot \psi_{t\sigma}(a,b,u) = f \cdot \mathrm{Id}_4$$

and the following sequence is exact:

$$R(-6)^4 \xrightarrow{\varphi_{1\sigma}(a,b,u)} R(-5)^2 \otimes R(-4)^2 \xrightarrow{\psi_{1\sigma}(a,b,u)} R(-3)^4 \xrightarrow{A_1} R(-2)^2 \otimes R(-1) \longrightarrow I_{1\sigma}(a,b,u) \longrightarrow 0,$$

where A_1 is the $3 \times 4-$matrix formed by the first three rows of $\varphi_{1\sigma}(a, b, u)$. Thus, (1) holds for $t = 1$, the other cases being similar.

(2) Clearly $I_{1\sigma}(a, b, u) \subset (v'_{\sigma 2}, w_{\sigma 1})$ and so $\dim R/I_{1\sigma}(a, b, u) = 2$. As x_s is zero–divisor in $R/I_{1\sigma}(a, b, u)$ we see that $\operatorname{depth} R/I_{1\sigma}(a, b, u) = 1$ and, by Proposition 12, $\Omega^2_R(I)$ is non–orientable, 4–generated of rank 2. Note that the module $\operatorname{Coker}\big(\varphi_{1\sigma}(a, b, u)\big)$, as in the last part of the proof of Theorem 8, is indecomposable because there exist no two matrices A, B of order two such that

$$\begin{pmatrix} 0 & x_s \\ x_s v'_{\sigma 2} & 0 \end{pmatrix} = \begin{pmatrix} w_{\sigma 1} & -w_{\sigma 2} \\ v_{\sigma 2} & v_{\sigma 1} \end{pmatrix} A + B \begin{pmatrix} w_{\sigma 1} & -v''_{\sigma 2} \\ -w_{\sigma 2} v'_{\sigma 2} & -v_{\sigma 1} \end{pmatrix}.$$

Similarly, the cases $t > 1$ follows.

(3) Now let M be an indecomposable, graded, non–orientable, 4–generated MCM R–module of rank 2. By Proposition 12, there exists a graded ideal $I \subset R$ with $\dim R/I = 2$, $\operatorname{depth} R/I = 1$, which is 3–generated and such that $M \simeq \Omega^2_R(I)$. Then $I = J/(f)$ with $J \subset S = K[x_1, x_2, x_3, x_4]$ is a 3–generated ideal containing f. We have still $f \in mJ$, though we are not in the orientable case (see $[EP_1]$ for details). Let $\alpha_1, \alpha_2, \alpha_3$ be a minimal system of homogeneous generators of J. If f does not belong to the ideal generated by two α_t, then, as in Section 3, $f = \sum_{t=1}^{3} p_t q_t$ and, after a renumbering, we may suppose that α_t is necessarily either p_t or q_t, for all $1 \le t \le 3$. Then $\alpha_1, \alpha_2, \alpha_3$ is a regular system of elements in S and so $R/I = S/J$ is Cohen–Macaulay which is false.

Thus, we may suppose $f \in (\alpha_1, \alpha_2)$. Then there exist $a, b \in K$ with $a^3 = b^3 = -1$, and $\sigma = (i\ j\ s)$ a permutation of the set $\sigma = \{2, 3, 4\}$, $i < j$, such that α_t is necessarily either $w_{\sigma t}$ or $v_{\sigma t}$, for $t = 1, 2$. If $\alpha_1 = w_{\sigma 1}, \alpha_2 = w_{\sigma 2}$, then $R/(\alpha_1, \alpha_2)$ is a domain and $\alpha_1, \alpha_2, \alpha_3$ must be a regular system of elements in S and so, again, $R/I = S/J$ is Cohen–Macaulay, contradiction!

We have the following cases:

Case I: $\alpha_1 = w_{\sigma 1}$

Then α_2 must be $v_{\sigma 2}$ and we have

$$(\alpha_1, \alpha_2) = (v'_{\sigma 2}, w_{\sigma 1}) \cap (v''_{\sigma 2}, w_{\sigma 1}).$$

It follows that a zero–divisor of $R/(\alpha_1, \alpha_2)$ must be either in $(v'_{\sigma 2}, w_{\sigma 1})$ or in $(v''_{\sigma 2}, w_{\sigma 1})$. As we know, α_3 is a zero–divisor in $R/(\alpha_1, \alpha_2)$ and so $\alpha_3 \in (v'_{\sigma 2}, w_{\sigma 1})$ or $\alpha_3 \in (v''_{\sigma 2}, w_{\sigma 1})$.

I(a) Suppose
$$\alpha_3 \in (v'_{\sigma 2}, w_{\sigma 1}).$$
Subtracting from α_3 a multiple of $w_{\sigma 1}$, we may take $\alpha_3 = v'_{\sigma 2}\beta$ for a form β of S. Note that the matrices

$$\varphi = \begin{pmatrix} 0 & w_{\sigma 1} & -v''_{\sigma 2} & 0 \\ -w_{\sigma 1} & 0 & -\beta & w_{\sigma 2} \\ v_{\sigma 2} & \beta v'_{\sigma 2} & 0 & v_{\sigma 1} \\ 0 & -w_{\sigma 2}v'_{\sigma 2} & -v_{\sigma 1} & 0 \end{pmatrix},$$

$$\psi = \begin{pmatrix} 0 & -v_{\sigma 1} & w_{\sigma 2} & \beta \\ v_{\sigma 1} & 0 & 0 & -v''_{\sigma 2} \\ -w_{\sigma 2}v'_{\sigma 2} & 0 & 0 & -w_{\sigma 1} \\ -\beta v'_{\sigma 2} & v_{\sigma 2} & w_{\sigma 1} & 0 \end{pmatrix}$$

give the following exact sequence:

$$\longrightarrow R^4 \xrightarrow{\varphi} R^4 \xrightarrow{\psi} R^4 \xrightarrow{B_1} R^3 \longrightarrow I \longrightarrow 0,$$

where B_1 is given by the first three rows of φ. Thus. (φ, ψ) is a matrix factorization of $\Omega^2_R(I) \simeq M$. Adding in φ multiples of the first row to the second one and adding multiples of the fourth column to the third one, we may suppose that the entry $(2,3)$ of φ depends only on x_1, x_s. These transformations modify also the entries $(2,2)$ and $(3,3)$, which are now not zero. Adding similar multiples of the first column to the second one and of the fourth row to the third one, we obtain φ of the same type as before but with β depending only on x_1, x_s. Since $v_{\sigma 1} - w_{\sigma 1}(x_1 + 2ax_s) = 3ax_s^2$, adding in φ multiples of the first column to the third one and multiples of the fourth row to the second row, we may suppose that the entry $(2,3)$ has the form λx_s for some $\lambda \in K$. These transformations modify also the entries $(3,3)$ and $(2,2)$, which are now not zero. Adding similar multiples of the first row to the third one and of the fourth column to the second column, we obtain φ of the same type as before but with $\beta = \lambda x_s$. If $\lambda = 0$, then, clearly, φ is the direct sum of two 2–matrices which contradicts that M is indecomposable. So. $\lambda \neq 0$. Now we divide the second and the third column of φ by λ and multiply the first and the fourth row by λ. The new φ is as before but with $\lambda = 1$, that is $\varphi = \varphi_{1\sigma}(a, b, u)$.

I(b) Suppose
$$\alpha_3 \in (v''_{\sigma 2}, w_{\sigma 1}).$$

Then we may take $\alpha_3 = v''_{\sigma 2}\beta$, for a form β. With a similar proof as above, we obtain $M \simeq \text{Coker}\big(\psi_{3\sigma}(a,b,u)\big)$.

Case II: $\alpha_2 = w_{\sigma 2}$**.**

Then $\alpha_1 = v_{\sigma 1}$. It follows that $(\alpha_1, \alpha_2) = (v'_{\sigma 1}, w_{\sigma 2}) \cap (v''_{\sigma 1}, w_{\sigma 2})$. We have the following two subcases:

II(a) $\alpha_3 \in (v'_{\sigma 1}, w_{\sigma 2})$. We may suppose $\alpha_3 = v'_{\sigma 1}\beta$, for a form β and we obtain that $M \simeq \text{Coker}\big(\psi_{4\sigma}(a,b,u)\big)$.

II(b) $\alpha_3 \in (v''_{\sigma 1}, w_{\sigma 2})$. In this subcase we may take $\alpha_3 = v''_{\sigma 1}\beta$, for a form β and we obtain that $M \simeq \text{Coker}\big(\varphi_{2\sigma}(a,b,u)\big)$.

Case III: $\alpha_1 = v_{\sigma 1}, \alpha_2 = v_{\sigma 2}$**.**

Then $(\alpha_1, \alpha_2) = (v'_{\sigma 1}, v'_{\sigma 2}) \cap (v'_{\sigma 1}, v''_{\sigma 2}) \cap (v''_{\sigma 1}, v'_{\sigma 2}) \cap (v''_{\sigma 1}, v''_{\sigma 2})$. We proceed as in the above cases, taking α_3 from one prime ideal of the above decomposition of $\big(\alpha_1, \alpha_2)\big)$, let us say $\alpha_3 \in (v'_{\sigma 1}, v'_{\sigma 2})$, that is $\alpha_3 = v'_{\sigma 1}\beta + v'_{\sigma 2}\gamma$ for some $\beta, \gamma \in S$. Suppose that one cannot reduce the problem to the case $\beta = 0$ or $\gamma = 0$, this implies, for example, that $v'_{\sigma 1}$ does not divide γ and $v'_{\sigma 2}$ does not divide β. Then $\Omega^1_S\big((\alpha_1, \alpha_2, \alpha_3)\big) \subset S^3$ contains the columns of the following matrix

$$\begin{pmatrix} v_{\sigma 2} & \alpha_3 & 0 & v''_{\sigma 2}\beta \\ -v_{\sigma 1} & 0 & \alpha_3 & v''_{\sigma 1}\gamma \\ 0 & -v_{\sigma 1} & -v_{\sigma 2} & -v''_{\sigma 1}v''_{\sigma 2} \end{pmatrix}$$

and we can see that $\mu(\Omega^1_S\big((\alpha_1, \alpha_2, \alpha_3)\big) \geq 4$, which contradicts Lemma 11. Thus we may suppose, let us say $\alpha_3 = v'_{\sigma 1}\beta$, where β is not a multiple of $v''_{\sigma 1}$. Now we may proceed as in the above cases and we obtain, in order, $M \simeq \text{Coker}\big(\varphi_{4\sigma}(a,b,u)\big)$, $M \simeq \text{Coker}\big(\varphi_{3\sigma}(a,b,u)\big)$, $M \simeq \text{Coker}\big(\psi_{1\sigma}(a,b,u)\big)$, and $M \simeq \text{Coker}\big(\psi_{2\sigma}(a,b,u)\big)$.

(4) The matrices

$$\varphi_{t\sigma}(a,b,u),\ \psi_{t\sigma}(a,b,u), 1 \leq t \leq 4,\ \sigma, a, b, u,$$

are equivalent in pairs. Namely:

$$\varphi_{1\sigma}(a,b,u) \sim \psi_{3\sigma}(a,b,u^2),\ \varphi_{2\sigma}(a,b,u) \sim \psi_{4\sigma}(a,b,u^2),$$

$$\varphi_{3\sigma}(a,b,u) \sim \psi_{1\sigma}(a,b,u^2),\ \varphi_{4\sigma}(a,b,u) \sim \psi_{2\sigma}(a,b,u^2).$$

We shall prove that the matrices of the set

$$\mathcal{N}' = \big\{\varphi_{t\sigma}(a,b,u),\ |\ 1 \leq t \leq 4,\ \sigma, a, b, u\big\}$$

are pairwise non–equivalent. We shall consider the matrices which are obtained from the matrices of $\mathcal{N}'$, reducing their entries modulo $\mathfrak{m}^2$. If $A, B \in \mathcal{N}'$ are equivalent, then there exist P, Q, two invertible 4×4–matrices with the entries in $K[x_1, x_2, x_3, x_4]$ such that $PA = BQ$. Let $\widetilde{A}$ and $\widetilde{B}$ be the matrices obtained from A, respectively B, by reducing modulo $\mathfrak{m}^2$ their entries. From the equality $PA = BQ$, we obtain that there exist two invertible scalar matrices $\widetilde{P}, \widetilde{Q} \in \mathcal{M}_4(K)$ such that $\widetilde{P}\widetilde{A} = \widetilde{B}\widetilde{Q}$. This means that the matrices $\widetilde{A}, \widetilde{B}$ are also equivalent by some scalar invertible matrices. We construct the "reduced" matrices $\tilde{\varphi}_{t\sigma}(a, b, u)$ and for all t. We see that the matrices $\tilde{\varphi}_{1\sigma}(a, b, u), \tilde{\varphi}_{2\sigma}(a, b, u)$, have the entries of the rows 2 and 4 zero and the rest of the matrices have the entries of the columns 1 and 3 zero. First, we choose two matrices $\widetilde{A}, \widetilde{B}$, one of them with the rows 2 and 4 zero and the other with the columns 1 and 3 zero. Suppose that $\widetilde{A} \sim \widetilde{B}$. It results that there are two invertible scalar 4×4–matrices U, V such that

$$\widetilde{A}U = V\widetilde{B}.$$

From this equality we obtain that the rows 2 and 4 in the matrix $V\widetilde{B}$ are zero. Looking at the two possibilities to choose the matrix $\widetilde{B}$, we see that the non–zero elements of the columns 2 and 4 in $\widetilde{B}$ are linear independent. From the above equality we get that V is not invertible.

Hence, we could find two equivalent matrices in the set $\mathcal{N}'$ only if both have the rows 2 and 4 zero or the columns 1 and 3 zero. It is clear that we may reduce the study of the equivalent matrices $\widetilde{A} = \tilde{\varphi}_{1\sigma}(a, b, u), \widetilde{B} = \tilde{\varphi}_{2\sigma}(a, b, u)$, which have the rows 2 and 4 zero. Let $U, V \in \mathcal{M}_{4\times 4}(K)$ be invertible matrices such that $\widetilde{A}U = V\widetilde{B}$. We may transform the reduced matrices $\widetilde{A}, \widetilde{B}$ such that the last two rows are zero. Let

$$\widetilde{A} = \begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix}, \widetilde{B} = \begin{pmatrix} B_1 & B_2 \\ 0 & 0 \end{pmatrix}, U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}, V = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix},$$

be the decomposition of our matrices in 2×2 blocks. Comparing the elements in the above equality, we obtain contradiction with the fact that U is invertible.

In the same way we check that if $\tilde{\varphi}_{1\sigma}(a, b, u)$ and $\tilde{\varphi}_{1\tau}(n, p, v)$ are different, then they are not equivalent. □

Let $M(\sigma, a, b)$, $M'(\tau, n, p)$ be two rank one MCM-modules corresponding to lines and $N(\sigma, a, b)$, $N'(\tau, n, p)$ be two rank one MCM-modules corresponding to conics (that is $\mathrm{Coker}\big(\varphi_\sigma(a, b)\big)$, $\mathrm{Coker}\big(\psi_\sigma(a, b)\big)$ by [EP]).

Remark 14. There exists an indecomposable extension in $\mathrm{Ext}^1_R(M(\sigma,a,b), M'(\tau,n,p))$ only if $\sigma=\tau$. In this case, for fixed $M(\sigma,a,b)$ there exists 4 non–orientable MCM–modules, which are extensions E of the form

$$0\to M'_i\to E\to M(\sigma,a,b)\to 0\ .$$

for some M'_i, $i=1,2$ of type $M'(\sigma,n,p)$. So we have 4×27 non–orientable MCM–modules. Similarly, taking now extensions F of the form

$$0\to N_i\to F\to N(\sigma,a,b)\to 0, i=1,2$$

we obtain another 4×27 non–orientable MCM–modules. Thus all are $216 = 8\times 27$.

5 Non–orientable, rank 2, 5–generated MCM modules

As in Section 3, let $u,a,b\in K$, with

$$a^3=b^3=-1,\ u^2+u+1=0,$$

$\sigma=(i\ j\ s)$ be a permutation of the set $\{2,3,4\}$ with $i<j$ and set

$$\begin{array}{rclrcl} w_{\sigma 1} & = & x_1-ax_s, & w_{\sigma 2} & = & x_i-bx_j,\\ v_{\sigma 1} & = & x_1^2+ax_1x_s+a^2x_s^2, & v_{\sigma 2} & = & x_i^2+bx_ix_j+b^2x_j^2. \end{array}$$

We have

$$v_{\sigma 1}=v'_{\sigma 1}v''_{\sigma 1},\ v_{\sigma 2}=v'_{\sigma 2}v''_{\sigma 2}$$

for

$$\begin{array}{rclrcl} v'_{\sigma 1} & = & x_1-uax_s, & v''_{\sigma 1} & = & x_1+(1+u)ax_s,\\ v'_{\sigma 2} & = & x_i-ubx_j, & v''_{\sigma 2} & = & x_i+(1+u)bx_j. \end{array}$$

Consider the following ideals:

Set

$$\begin{aligned}
J_{1\sigma}(a,b,u) &= (v_{\sigma 1}, v_{\sigma 2}, v'_{\sigma 1}v'_{\sigma 2}, v''_{\sigma 1}v''_{\sigma 2}),\\
J_{2\sigma}(a,b,u) &= (v_{\sigma 1}, v_{\sigma 2}, v'_{\sigma 1}v'_{\sigma 2}, v''_{\sigma 1}v'_{\sigma 2}),\\
J_{3\sigma}(a,b,u) &= (v_{\sigma 1}, v_{\sigma 2}, v'_{\sigma 1}v'_{\sigma 2}, v''_{\sigma 1}(v'_{\sigma 2}+v''_{\sigma 2})),\\
J_{4\sigma}(a,b,u) &= (v_{\sigma 1}, v_{\sigma 2}, v'_{\sigma 1}v''_{\sigma 2}, v''_{\sigma 1}v'_{\sigma 2}),\\
J_{5\sigma}(a,b,u) &= (v_{\sigma 1}, v_{\sigma 2}, v'_{\sigma 1}v''_{\sigma 2}, v''_{\sigma 1}v''_{\sigma 2}),\\
J_{6\sigma}(a,b,u) &= (v_{\sigma 1}, v_{\sigma 2}, v'_{\sigma 1}v''_{\sigma 2}, v''_{\sigma 1}(v'_{\sigma 2}+v''_{\sigma 2})),\\
J_{7\sigma}(a,b,u) &= (w_{\sigma 1}v'_{\sigma 1}, v_{\sigma 2}, v'_{\sigma 1}v'_{\sigma 2}, w_{\sigma 1}v''_{\sigma 2}),\\
J_{8\sigma}(a,b,u) &= (w_{\sigma 1}v'_{\sigma 1}, v_{\sigma 2}, v'_{\sigma 1}v'_{\sigma 2}, w_{\sigma 1}v'_{\sigma 2}),\\
J_{9\sigma}(a,b,u) &= (w_{\sigma 1}v'_{\sigma 1}, v_{\sigma 2}, v'_{\sigma 1}v'_{\sigma 2}, w_{\sigma 1}(v'_{\sigma 2}+v''_{\sigma 2})),\\
J_{10\sigma}(a,b,u) &= (w_{\sigma 1}v'_{\sigma 1}, v_{\sigma 2}, v'_{\sigma 1}v''_{\sigma 2}, w_{\sigma 1}v'_{\sigma 2}),\\
J_{11\sigma}(a,b,u) &= (w_{\sigma 1}v'_{\sigma 1}, v_{\sigma 2}, v'_{\sigma 1}v''_{\sigma 2}, w_{\sigma 1}v''_{\sigma 2}),\\
J_{12\sigma}(a,b,u) &= (w_{\sigma 1}v'_{\sigma 1}, v_{\sigma 2}, v'_{\sigma 1}v''_{\sigma 2}, w_{\sigma 1}(v'_{\sigma 2}+v''_{\sigma 2})),
\end{aligned}$$

and denote by $\mathcal{J}$ the set of these ideals.

Set:

$$\rho_{1\sigma}(a,b,u) = \begin{pmatrix} 0 & w_{\sigma 1} & -v'_{\sigma 2} & -v''_{\sigma 2} & 0\\ v'_{\sigma 1} & w_{\sigma 2} & 0 & 0 & -v''_{\sigma 2}v''_{\sigma 1}\\ -v''_{\sigma 2} & 0 & v''_{\sigma 1} & 0 & 0\\ 0 & 0 & 0 & v'_{\sigma 1} & v_{\sigma 2}\\ 0 & 0 & 0 & -w_{\sigma 2} & w_{\sigma 1}v''_{\sigma 1} \end{pmatrix},$$

$$\omega_{1\sigma}(a,b,u) = \begin{pmatrix} -w_{\sigma 2}v''_{\sigma 1} & w_{\sigma 1}v''_{\sigma 1} & -w_{\sigma 2}v'_{\sigma 2} & 0 & v''_{\sigma 2}v''_{\sigma 1}\\ v_{\sigma 1} & v_{\sigma 2} & v'_{\sigma 1}v'_{\sigma 2} & v''_{\sigma 2}v''_{\sigma 1} & 0\\ -w_{\sigma 2}v''_{\sigma 2} & w_{\sigma 1}v''_{\sigma 2} & w_{\sigma 1}v'_{\sigma 1} & 0 & (v''_{\sigma 2})^2\\ 0 & 0 & 0 & w_{\sigma 1}v''_{\sigma 1} & -v_{\sigma 2}\\ 0 & 0 & 0 & w_{\sigma 2} & v'_{\sigma 1} \end{pmatrix},$$

$$\rho_{2\sigma}(a,b,u) = \begin{pmatrix} 0 & w_{\sigma 1} & -v'_{\sigma 2} & 0 & 0\\ v'_{\sigma 1} & w_{\sigma 2} & 0 & 0 & -v''_{\sigma 1}v'_{\sigma 2}\\ -v''_{\sigma 2} & 0 & v''_{\sigma 1} & -v''_{\sigma 1} & 0\\ 0 & 0 & 0 & v'_{\sigma 1} & v_{\sigma 2}\\ 0 & 0 & 0 & -w_{\sigma 2} & w_{\sigma 1}v''_{\sigma 1} \end{pmatrix},$$

$$\omega_{2\sigma}(a,b,u) = \begin{pmatrix} -w_{\sigma 2}v''_{\sigma 1} & w_{\sigma 1}v''_{\sigma 1} & -w_{\sigma 2}v'_{\sigma 2} & 0 & v''_{\sigma 1}v'_{\sigma 2} \\ v_{\sigma 1} & v_{\sigma 2} & v'_{\sigma 1}v'_{\sigma 2} & v'_{\sigma 2}v''_{\sigma 1} & 0 \\ -w_{\sigma 2}v''_{\sigma 2} & w_{\sigma 1}v''_{\sigma 2} & w_{\sigma 1}v'_{\sigma 1} & v''_{\sigma 1}w_{\sigma 1} & 0 \\ 0 & 0 & 0 & w_{\sigma 1}v''_{\sigma 1} & -v_{\sigma 2} \\ 0 & 0 & 0 & w_{\sigma 2} & v'_{\sigma 1} \end{pmatrix},$$

$$\rho_{3\sigma}(a,b,u) = \begin{pmatrix} 0 & w_{\sigma 1} & -v'_{\sigma 2} & -v''_{\sigma 2} & 0 \\ v'_{\sigma 1} & w_{\sigma 2} & 0 & 0 & -v''_{\sigma 1}(v'_{\sigma 2}+v''_{\sigma 2}) \\ -v''_{\sigma 2} & 0 & v''_{\sigma 1} & -v''_{\sigma 1} & 0 \\ 0 & 0 & 0 & v'_{\sigma 1} & v_{\sigma 2} \\ 0 & 0 & 0 & -w_{\sigma 2} & w_{\sigma 1}v''_{\sigma 1} \end{pmatrix},$$

$$\omega_{3\sigma}(a,b,u) = \begin{pmatrix} -w_{\sigma 2}v''_{\sigma 1} & w_{\sigma 1}v''_{\sigma 1} & -w_{\sigma 2}v'_{\sigma 2} & 0 & v''_{\sigma 1}(v'_{\sigma 2}+v''_{\sigma 2}) \\ v_{\sigma 1} & v_{\sigma 2} & v'_{\sigma 1}v'_{\sigma 2} & v''_{\sigma 1}(v'_{\sigma 2}+v''_{\sigma 2}) & 0 \\ -w_{\sigma 2}v''_{\sigma 2} & w_{\sigma 1}v''_{\sigma 2} & w_{\sigma 1}v'_{\sigma 1} & v''_{\sigma 1}w_{\sigma 1} & (v''_{\sigma 2})^2 \\ 0 & 0 & 0 & w_{\sigma 1}v''_{\sigma 1} & -v_{\sigma 2} \\ 0 & 0 & 0 & w_{\sigma 2} & v'_{\sigma 1} \end{pmatrix}.$$

Replacing $v'_{\sigma 2}$ by $v''_{\sigma 2}$ and conversely, we get other three pairs of matrices, $\rho_{i\sigma}(a,b,u),\ \omega_{i\sigma}(a,b,u),\ i=4,5,6$. Next, replacing $w_{\sigma 1}$ by $v''_{\sigma 1}$ and conversely, we get other three pairs of matrices, $\rho_{i\sigma}(a,b,u),\ \omega_{i\sigma}(a,b,u),\ i=7,8,9$, and, finally, performing the both changes, we get the pairs of matrices $\rho_{i\sigma}(a,b,u),\ \omega_{i\sigma}(a,b,u),\ i=10,11,12$.

Now let us consider the following ideals:

$$
\begin{aligned}
T_{1\sigma}(a,b,u) &= (w_{\sigma 2}v''_{\sigma 1}, v_{\sigma 1}, w_{\sigma 2}v'_{\sigma 2}, x_s v''_{\sigma 1}),\\
T_{2\sigma}(a,b,u) &= (w_{\sigma 2}v''_{\sigma 1}, v_{\sigma 1}, w_{\sigma 2}v'_{\sigma 2}, x_j v''_{\sigma 1}),\\
T_{3\sigma}(a,b,u) &= (w_{\sigma 2}v''_{\sigma 1}, v_{\sigma 1}, w_{\sigma 2}v'_{\sigma 2}, (x_j+x_s) v''_{\sigma 1}),\\
T_{4\sigma}(a,b,u) &= (w_{\sigma 2}v''_{\sigma 1}, v_{\sigma 1}, w_{\sigma 2}v''_{\sigma 2}, x_s v''_{\sigma 1}),\\
T_{5\sigma}(a,b,u) &= (w_{\sigma 2}v''_{\sigma 1}, v_{\sigma 1}, w_{\sigma 2}v''_{\sigma 2}, x_j v''_{\sigma 1}),\\
T_{6\sigma}(a,b,u) &= (w_{\sigma 2}v''_{\sigma 1}, v_{\sigma 1}, w_{\sigma 2}v''_{\sigma 2}, (x_j+x_s) v''_{\sigma 1}),\\
T_{7\sigma}(a,b,u) &= (w_{\sigma 2}w_{\sigma 1}, w_{\sigma 1}v'_{\sigma 1}, w_{\sigma 2}v'_{\sigma 2}, x_s w_{\sigma 1}),\\
T_{8\sigma}(a,b,u) &= (w_{\sigma 2}w_{\sigma 1}, w_{\sigma 1}v'_{\sigma 1}, w_{\sigma 2}v'_{\sigma 2}, x_j w_{\sigma 1}),\\
T_{9\sigma}(a,b,u) &= (w_{\sigma 2}w_{\sigma 1}, w_{\sigma 1}v'_{\sigma 1}, w_{\sigma 2}v'_{\sigma 2}, (x_j+x_s) w_{\sigma 1}),\\
T_{10\sigma}(a,b,u) &= (w_{\sigma 2}w_{\sigma 1}, w_{\sigma 1}v'_{\sigma 1}, w_{\sigma 2}v''_{\sigma 2}, x_s w_{\sigma 1}),\\
T_{11\sigma}(a,b,u) &= (w_{\sigma 2}w_{\sigma 1}, w_{\sigma 1}v'_{\sigma 1}, w_{\sigma 2}v''_{\sigma 2}, x_j w_{\sigma 1}),\\
T_{12\sigma}(a,b,u) &= (w_{\sigma 2}w_{\sigma 1}, w_{\sigma 1}v'_{\sigma 1}, w_{\sigma 2}v''_{\sigma 2}, (x_j+x_s) w_{\sigma 1}),
\end{aligned}
$$

We denote by $\mathcal{T}$ the set of these ideals and set:

$$
\mu_{1\sigma}(a,b,u) = \begin{pmatrix} 0 & -v'_{\sigma 1} & v'_{\sigma 2} & 0 & -x_s \\ w_{\sigma 1} & w_{\sigma 2} & 0 & x_s & 0 \\ -v''_{\sigma 2} & 0 & v''_{\sigma 1} & 0 & 0 \\ 0 & 0 & 0 & v'_{\sigma 1} & -w_{\sigma 2} \\ 0 & 0 & 0 & v_{\sigma 2} & w_{\sigma 1}v''_{\sigma 1} \end{pmatrix},
$$

$$
\nu_{1\sigma}(a,b,u) = \begin{pmatrix} w_{\sigma 2}v''_{\sigma 1} & v_{\sigma 1} & w_{\sigma 2}v'_{\sigma 2} & -x_s v''_{\sigma 1} & 0 \\ -v''_{\sigma 1}w_{\sigma 1} & v_{\sigma 2} & -v'_{\sigma 2}w_{\sigma 1} & 0 & -x_s \\ w_{\sigma 2}v''_{\sigma 2} & v'_{\sigma 1}v''_{\sigma 2} & -w_{\sigma 1}v'_{\sigma 1} & -x_s v''_{\sigma 2} & 0 \\ 0 & 0 & 0 & w_{\sigma 1}v''_{\sigma 1} & w_{\sigma 2} \\ 0 & 0 & 0 & -v_{\sigma 2} & v'_{\sigma 1} \end{pmatrix},
$$

$$
\mu_{2\sigma}(a,b,u) = \begin{pmatrix} 0 & -v'_{\sigma 1} & v'_{\sigma 2} & 0 & -x_j \\ w_{\sigma 1} & w_{\sigma 2} & 0 & x_j & 0 \\ -v''_{\sigma 2} & 0 & v''_{\sigma 1} & 0 & 0 \\ 0 & 0 & 0 & v'_{\sigma 1} & -w_{\sigma 2} \\ 0 & 0 & 0 & v_{\sigma 2} & w_{\sigma 1}v''_{\sigma 1} \end{pmatrix},
$$

$$\nu_{2\sigma}(a,b,u)=\begin{pmatrix} w_{\sigma 2}v''_{\sigma 1} & v_{\sigma 1} & w_{\sigma 2}v'_{\sigma 2} & -x_j v''_{\sigma 1} & 0\\ -v''_{\sigma 1}w_{\sigma 1} & v_{\sigma 2} & -v'_{\sigma 2}w_{\sigma 1} & 0 & -x_j\\ w_{\sigma 2}v''_{\sigma 2} & v'_{\sigma 1}v''_{\sigma 2} & -w_{\sigma 1}v'_{\sigma 1} & -x_j v''_{\sigma 2} & 0\\ 0 & 0 & 0 & w_{\sigma 1}v''_{\sigma 1} & w_{\sigma 2}\\ 0 & 0 & 0 & -v_{\sigma 2} & v'_{\sigma 1}\end{pmatrix},$$

$$\mu_{3\sigma}(a,b,u)=\begin{pmatrix} 0 & -v'_{\sigma 1} & v'_{\sigma 2} & 0 & -(x_j+x_s)\\ w_{\sigma 1} & w_{\sigma 2} & 0 & x_j+x_s & 0\\ -v''_{\sigma 2} & 0 & v''_{\sigma 1} & 0 & 0\\ 0 & 0 & 0 & v'_{\sigma 1} & -w_{\sigma 2}\\ 0 & 0 & 0 & v_{\sigma 2} & w_{\sigma 1}v''_{\sigma 1}\end{pmatrix},$$

$$\nu_{3\sigma}(a,b,u)=\begin{pmatrix} w_{\sigma 2}v''_{\sigma 1} & v_{\sigma 1} & w_{\sigma 2}v'_{\sigma 2} & -(x_j+x_s)v''_{\sigma 1} & 0\\ -v''_{\sigma 1}w_{\sigma 1} & v_{\sigma 2} & -v'_{\sigma 2}w_{\sigma 1} & 0 & -(x_j+x_s)\\ w_{\sigma 2}v''_{\sigma 2} & v'_{\sigma 1}v''_{\sigma 2} & -w_{\sigma 1}v'_{\sigma 1} & -(x_j+x_s)v''_{\sigma 2} & 0\\ 0 & 0 & 0 & w_{\sigma 1}v''_{\sigma 1} & w_{\sigma 2}\\ 0 & 0 & 0 & -v_{\sigma 2} & v'_{\sigma 1}\end{pmatrix},$$

Replacing $v''_{\sigma 2}$ by $v'_{\sigma 2}$ and conversely, we get other three pairs of matrices, $\mu_{i\sigma}(a,b,u),\ \nu_{i\sigma}(a,b,u),\ i=4,5,6$. Next, replacing $w_{\sigma 1}$ by $v''_{\sigma 1}$ and conversely, we get other three pairs of matrices, $\mu_{i\sigma}(a,b,u),\ \nu_{i\sigma}(a,b,u),\ i=7,8,9$, and, finally, performing the both changes, we get the pairs of matrices $\mu_{i\sigma}(a,b,u),\ \nu_{i\sigma}(a,b,u),\ i=10,11,12$.

Clearly, the pair of matrices $(\rho_{i\sigma}(a,b,u),\omega_{i\sigma}(a,b,u))$ forms a matrix factorization of $\Omega^2_R(J_{i\sigma}(a,b,u)/(f))$ for $1\le i\le 12$, and the pair $(\mu_{i\sigma}(a,b,u),\nu_{i\sigma}(a,b,u))$ forms a matrix factorization of $\Omega^2_R(T_{i\sigma}(a,b,u)/(f))$ for $1\le i\le 12$.

Lemma 15. *Let M be a graded non–orientable, rank 2, 5–generated MCM R–module, without free direct summands. Then there exists an ideal $J\in\mathcal{J}\cup\mathcal{T}$ such that $f\in J$ and $M\cong\Omega^2\big(J/(f)\big)$ or $M^*\cong\Omega^2\big(J/(f)\big)$, where M^* is the dual of M. Conversely, for every $J\in\mathcal{J}\cup\mathcal{T}$, the module $\Omega^2\big(J/(f)\big)$ is a non–orientable, rank two, 5–generated MCM R–module without free direct summands.*

Proof. The second statement follows easily, as we already have the matrix factorizations above of those ideals. Let M be as above. As in the beginning of Section 4 we see that $M \cong \Omega^2\big(J/(f)\big)$, for J an ideal of S containing f, with $\mu(J) = 4$, dim $S/J = 2$, depth S/J=1 and $\mu\big(\Omega^1_S(J)\big) = 5$. We may also suppose $J = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ with $f \in (\alpha_1, \alpha_2)$, where α_t is necessarily either $w_{\sigma t}$ or $v_{\sigma t}$ for $t = 1, 2$ for some a, b and a certain permutation σ as above. Clearly we cannot have, simultaneously, $\alpha_t = w_{\sigma t}$ because then (α_1, α_2) is a prime ideal and one cannot find α_3, α_4 zero divisors, as we need. We treat the following cases:

Case I: $\alpha_1 = w_{\sigma 1}$

Then we have $\alpha_2 = v_{\sigma 2}$ and (α_1, α_2) is the intersection of the prime ideals $(v'_{\sigma 2}, w_{\sigma 1})$, $(v''_{\sigma 2}, w_{\sigma 1})$. Since α_3, α_4 must be zero divisors in $S/(\alpha_3, \alpha_4)$ we have the following possibilities:

(I1) $\alpha_3 = v'_{\sigma 2}\beta$, $\alpha_4 = v'_{\sigma 2}\gamma$, (I2) $\alpha_3 = v''_{\sigma 2}\beta$,$\alpha_4 = v''_{\sigma 2}\gamma$,
(I3) $\alpha_3 = v'_{\sigma 2}\beta$, $\alpha_4 = v''_{\sigma 2}\gamma$, (I4) $\alpha_3 = v''_{\sigma 2}\beta$, $\alpha_4 = v'_{\sigma 2}\gamma$,

for some homogeneous β, γ from $\mathfrak{m} = (x_1, x_2, x_3, x_4)$. In the first case we see that the relations given by the columns of the following matrix:

$$\begin{pmatrix} v_{\sigma 2} & \alpha_3 & \alpha_4 & 0 & 0 \\ -w_{\sigma 1} & 0 & 0 & \gamma & \beta \\ 0 & -w_{\sigma 1} & 0 & 0 & -v''_{\sigma 2} \\ 0 & 0 & -w_{\sigma 1} & -v''_{\sigma 2} & 0 \end{pmatrix},$$

are elements in $\Omega^1_S(J) \subset S^4$. Clearly these columns are part of the minimal system of generators of $\Omega^1_S(J)$ because $w_{\sigma 1}, v''_{\sigma 2}$ form a regular system in S. The subcase (I2) is similar, this contradicts Lemma 11.

Suppose now (I3) holds. Then the relations given by the columns of the following matrix

$$\begin{pmatrix} v_{\sigma 2} & v'_{\sigma 2}\beta & v''_{\sigma 2}\beta & 0 & 0 \\ -w_{\sigma 1} & 0 & 0 & \beta & \gamma \\ 0 & -w_{\sigma 1} & 0 & -v''_{\sigma 2} & 0 \\ 0 & 0 & -w_{\sigma 1} & 0 & -v'_{\sigma 2} \end{pmatrix},$$

are part of a minimal set of generators of $\Omega^1_S(J)$ (note that $w_{\sigma 1}, v''_{\sigma 2}, v'_{\sigma 2}$ form a regular system in S). Contradiction! Case (I4) is similar.

Case II: $\alpha_1 = v_{\sigma 1}$, $\alpha_2 = v_{\sigma 2}$

Since $(\alpha_1, \alpha_2) = (v'_{\sigma 1}, v'_{\sigma 2}) \cap (v'_{\sigma 1}, v''_{\sigma 2}) \cap (v''_{\sigma 1}, v''_{\sigma 2}) \cap (v''_{\sigma 1}, v'_{\sigma 2})$, we see that the zero divisors of $S/(\alpha_1, \alpha_2)$ must be in one of the prime ideals of the above decomposition. Suppose $\alpha_3 \in (v'_{\sigma 1}, v'_{\sigma 2})$. If $\alpha_3 = \beta_1 v'_{\sigma 1} + \beta_2 v'_{\sigma 2}$ then, as in the proof of Case III of Proposition 13, we see that there are at least four minimal relations between first three α. Then all α have at least five minimal relations. Contradiction! Thus, α_3 as well α_4 are multiples of one $v'_{\sigma t}, v''_{\sigma t}$. So we have the following possibilities:

(II1)	$\alpha_3 = v'_{\sigma 1}\beta,\ \alpha_4 = v'_{\sigma 1}\gamma,$	(II2)	$\alpha_3 = v''_{\sigma 1}\beta,\ \alpha_4 = v''_{\sigma 1}\gamma,$
(II3)	$\alpha_3 = v'_{\sigma 2}\beta,\ \alpha_4 = v'_{\sigma 2}\gamma,$	(II4)	$\alpha_3 = v''_{\sigma 2}\beta,\ \alpha_4 = v''_{\sigma 2}\gamma,$
(II5)	$\alpha_3 = v'_{\sigma 1}\beta,\ \alpha_4 = v''_{\sigma 1}\gamma,$	(II6)	$\alpha_3 = v'_{\sigma 1}\beta,\ \alpha_4 = v''_{\sigma 2}\gamma,$
(II7)	$\alpha_3 = v'_{\sigma 2}\beta, \alpha_4 = v''_{\sigma 1}\gamma,$	(II8)	$\alpha_3 = v'_{\sigma 2}\beta, \alpha_4 = v''_{\sigma 2}\gamma,$
(II9)	$\alpha_3 = v'_{\sigma 1}\beta, \alpha_4 = v'_{\sigma 2}\gamma,$	(II10)	$\alpha_3 = v''_{\sigma 1}\beta, \alpha_4 = v''_{\sigma 2}\gamma.$

Subcase: $\alpha_3 = v'_{\sigma 1}\beta,\ \alpha_4 = v'_{\sigma 1}\gamma,\ (v_{\sigma 2}v''_{\sigma 1}, \gamma) \cong 1,\ (v_{\sigma 2}v''_{\sigma 1}, \beta) \cong 1$

We see that the relations given by the columns of the following matrix

$$\begin{pmatrix} v_{\sigma 2} & \beta & \gamma & 0 & 0 \\ -v_{\sigma 1} & 0 & 0 & \alpha_3 & \alpha_4 \\ 0 & -v''_{\sigma 1} & 0 & -v_{\sigma 2} & 0 \\ 0 & 0 & -v''_{\sigma 1} & 0 & -v_{\sigma 2} \end{pmatrix},$$

are part of a minimal system of generators of $\Omega^1_S(J)$, which must be false. Indeed, it is easy to see that the last four columns are part of a minimal system of generators of $\Omega^1_S(J)$. If the first column belongs to the module generated by the last four, then there exist $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in S$ such that:

$$\begin{aligned} v_{\sigma 2} &= \lambda_1\beta + \lambda_2\gamma, \\ -v_{\sigma 1} &= \lambda_3 v'_{\sigma 1}\beta + \lambda_4 v''_{\sigma 1}\gamma, \\ 0 &= \lambda_1 v''_{\sigma 1} + \lambda_3 v_{\sigma 2}, \\ 0 &= \lambda_2 v''_{\sigma 1} + \lambda_4 v_{\sigma 2}. \end{aligned}$$

It follows that $v_{\sigma 2} \mid \lambda_1$ and $v_{\sigma 2} \mid \lambda_2$ and so we obtain $1 \in (\beta, \gamma)$. Contradiction! If $(v_{\sigma 2}v''_{\sigma 1}, \beta) \ncong 1$, then we are in the subcase (II5), (II6), ... In the same way we treat (II2), (II3), (II4).

Subcase: $\alpha_3 = v'_{\sigma 1}\beta,\ \alpha_4 = v''_{\sigma 1}\gamma$

We see that the relations given by the columns of the following matrix

$$\begin{pmatrix} v_{\sigma 2} & \beta & \gamma & 0 & 0 \\ -v_{\sigma 1} & 0 & 0 & \alpha_3 & \alpha_4 \\ 0 & -v''_{\sigma 1} & 0 & -v_{\sigma 2} & 0 \\ 0 & 0 & -v'_{\sigma 1} & 0 & -v_{\sigma 2} \end{pmatrix},$$

are elements in $\Omega^1_S(J)$. The columns two and three, together with the last two columns divided by $(\beta, v_{\sigma 2})$, respectively $(\gamma, v_{\sigma 2})$, are part of a minimal system of generators. Since $\mu\big(\Omega^1_S(J)\big) = 4$, we see that the first column is a linear combination of the others, as above. Thus, there exist $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in S$ such that:

$$\begin{aligned} v_{\sigma 2} &= \lambda_1 \beta + \lambda_2 \gamma, \\ -v_{\sigma 1} &= \lambda_3 v'_{\sigma 1} \beta/(\beta, v_{\sigma 2}) + \lambda_4 v''_{\sigma 1} \gamma/(\gamma, v_{\sigma 2}), \\ 0 &= \lambda_1 v''_{\sigma 1} + \lambda_3 v_{\sigma 2}/(\beta, v_{\sigma 2}), \\ 0 &= \lambda_2 v'_{\sigma 1} + \lambda_4 v_{\sigma 2}/(\gamma, v_{\sigma 2}). \end{aligned}$$

It follows that $v_{\sigma 2}/(\beta, v_{\sigma 2}) | \lambda_1$ and $v_{\sigma 2}/(\gamma, v_{\sigma 2}) | \lambda_2$ and so we obtain $1 \in (\beta, \gamma)$, which is false, as above, if $(\beta, v_{\sigma 2}) \cong 1$, $(\gamma, v_{\sigma 2}) \cong 1$. Clearly β, γ cannot be multiples of $v_{\sigma 2}$ because otherwise J is only 3–generated. The analysis of the possibilities $(\beta, v_{\sigma 2}) = v'_{\sigma 2}$ and $(\beta, v_{\sigma 2}) = v''_{\sigma 2}$ will lead to the conclusion that $J \in \mathcal{J}$. In this way one can discuss all the above cases. $\square$

Theorem 16. *Let*

$$\mathcal{E}_1 = \{\mathrm{Coker}\big(\rho_{i\sigma}(a,b,u)\big),\ \mathrm{Coker}\big(\mu_{i\sigma}(a,b,u)\big)\big) \mid \ \sigma, a, b, u,\ i = \overline{1,6}\},$$

$\mathcal{E}_2$ be the set of the duals of the modules from the set $\mathcal{E}_1$, and $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$.

(1) The set $\mathcal{E}$ contains only indecomposable, graded, non–orientable, 5–generated MCM R–modules of rank 2.

(2) Every indecomposable, graded, non–orientable, 5–generated MCM module over R of rank 2 is isomorphic with one module of $\mathcal{E}$.

(3) There are 648 isomorphism classes of indecomposable, graded, non–orientable MCM modules over R of rank 2, with five generators.

Proof. (1) For the proof of indecomposability we may proceed as in the last part of the proof of Theorem 8. For example, let N be the module $\mathrm{Coker}\big(\rho_{1\sigma}(a,b,u)\big)$ and suppose that it decomposes. Then $\rho_{1\sigma}(a,b,u)$ is equivalent with a direct sum of two matrices: A_1, of order three and A_2, of order two. Let B_1, B_2 be the submatrices of $\rho_{1\sigma}(a,b,u)$ given by the first three lines and columns, respectively the last two lines and columns. Certainly A_1, A_2, B_1, B_2 define some maximal Cohen-Macaulay modules of rank one that we denote, respectively, by N_1, N_2, T_1, T_2, and due to the particular form of $\rho_{1\sigma}(a,b,u)$ we have the following exact sequence

$$0 \to T_1 \to N_1 \oplus N_2 = N \to T_2 \to 0.$$

Since $\rho_{1\sigma}(a,b,u)$ is modulo x_j the sum of B_1, B_2, $T_i/x_jT_i \cong N_i/x_jN_i$ for $i = 1, 2$. Looking at the description of rank one maximal Cohen-Macaulay modules we can see that A_i is equivalent with B_i modulo x_j only when A_i is equivalent with B_i. Thus $T_i \cong N_i$ for $i = 1, 2$ and so $N \cong T_1 \oplus T_2$. By [Mi], this happens only if the above exact sequence splits, that is impossible.
(2) It is enough to observe that the matrices

$$\rho_{i\sigma}(a,b,u), a, b, u, \sigma, i = \overline{1, 12}$$

and

$$\mu_{i\sigma}(a,b,u), a, b, u, \sigma, i = \overline{1, 12},$$

are pairwise equivalent. Indeed, one may show that

$$\rho_{7\sigma}(a,b,u) \sim \rho_{4\sigma}(a,b,u),\ \rho_{8\sigma}(a,b,u) \sim \rho_{5\sigma}(a,b,u),\ \rho_{9\sigma}(a,b,u) \sim \rho_{6\sigma}(a,b,u),$$

and

$$\rho_{10\sigma}(a,b,u) \sim \rho_{1\sigma}(a,b,u),\ \rho_{11\sigma}(a,b,u) \sim \rho_{2\sigma}(a,b,u),\ \rho_{12\sigma}(a,b,u) \sim \rho_{3\sigma}(a,b,u).$$

One may find, in each case, a pair of some permutations matrices U_i, V_i such that

$$U_i\rho_{i\sigma}(a,b,u) = \rho_{(i-3)\sigma}(au,b,u)V_i,\ i = 7, 8, 9,$$

and

$$U_i\rho_{i\sigma}(a,b,u) = \rho_{(i-9)\sigma}(au,b,u)V_i,\ i = 10, 11, 12.$$

In a similar way one may group in pairs the matrices $\mu_{i\sigma}(a,b,u)$.
(3) It is a laborious task to prove the that the modules of the list $\mathcal{E}$ are pairwise non–isomorphic. One can use the following procedure in SINGULAR:

```
LIB"matrix.lib";
option(redSB);
proc isomorph5(matrix X, matrix Y)
{
matrix U[5][5]=u(1..25);
matrix V[5][5]=v(1..25);
matrix C=U*X-Y*V;
```

```
ideal I=flatten(C);
ideal J=det(U)-1,det(V)-1;
for (int j=1;j<=size(I);j++)
{
J=J+transpose(coef(I[j],x(1)*x(2)*x(3)*x(4)))[2];
}
ideal K=std(J);
return(K);
}
```

□

Corollary 17. *Let*

$$\mathcal{F}_1 = \{\operatorname{Coker}\big(\omega_{i\sigma}(a,b,u)\big),\ \operatorname{Coker}\big(\nu_{i\sigma}(a,b,u)\big) \mid \sigma, a, b, u,\ i = \overline{1,6}\},$$

$\mathcal{F}_2$ be the set of the duals of the modules from the set $\mathcal{F}_1$, and $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$.

(1) The set $\mathcal{F}$ contains only indecomposable, graded, non–orientable, 5–generated MCM R–modules of rank 3.

(2) Every indecomposable, graded, non–orientable, 5–generated MCM module over R of rank 3 is isomorphic with one module of $\mathcal{F}$.

(3) There are 648 isomorphism classes of indecomposable, graded, non–orientable MCM modules over R of rank 3, with 5 generators.

Proof. The map $M \mapsto \Omega^1_R(M)$ is a bijection between the 5–generated, indecomposable, graded, MCM R–modules of rank 2 and the 5–generated, indecomposable, graded, MCM R–modules of rank 3. □

Remark 18. For each 2-gen MCM module M (line or conic) there exist two non-isomorphic 3-gen MCM modules P_1, P_2 and 3 non-isomorphic extensions for each:

$$0 \to P_i \to E_{ij} \to M \to 0,$$

$i = 1, 2,\ j = 1, 2, 3$. So there are 6×54 MCM of type E_{ij}. Taking the duals we get another 6×54 MCM. Thus all are $648 = 12 \times 54$.

Lemma 19. *There exist no graded, indecomposable, non–orientable, rank 2, 6–generated MCM modules.*

Proof. Suppose there exist such MCM module M. Then $M \cong \Omega^2_R\big(J/(f)\big)$ for a certain 5–generated ideal $J = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ of S as hinted at in the first part of Section 4. Then any four elements from the α_t must generate an ideal J'' in $\mathcal{J} \cup \mathcal{T}$ because, otherwise, $\mu(\Omega^1_S\big(J''/(f)\big) > 4$ and so, obviously $\mu(\Omega^1_S(J/(f)) > 5$. So we may suppose $\alpha_t = v_{\sigma t}$ for $t = 1, 2$ and after some permutations $\alpha_3 = v'_{\sigma 1} v''_{\sigma 2}$. Set $J' = (\alpha_1, \alpha_2, \alpha_3)$. If $(J', \alpha_4) \in \mathcal{J}$. and $(J', \alpha_5) \in \mathcal{J}$ then there are 4 minimal relations of (J', α_4) and 4 minimal relations of (J', α_5) over S, among them at least 6 minimal relations of J over S which contradicts Lemma 11. In the same way we treat the other cases. □

Corollary 20. *There exist no indecomposable, graded, non–orientable, rank 4, 6–generated MCM modules.*

6 Orientable, rank 2, 6–generated MCM modules

Let $S = K[x_1, x_2, x_3, x_4]$, and $R = S/(f)$, $f = x_1^3 + x_2^3 + x_3^3 + x_4^3$.

We have proved that a non–free graded orientable 6–generated MCM R–module corresponds to a skew symmetric homogeneous matrix over S of order 6, whose determinant is f^2.

Let Λ be such a matrix. Notice that Λ has linear entries and the matrix $\underline{\Lambda} := \Lambda|_{x_4=0}$, obtained from Λ by restricting the entries to $x_4 = 0$, is a homogeneous matrix over $S_3 = K[x_1, x_2, x_3]$, whose determinant is f_3^2, where $f_3 = x_1^3 + x_2^3 + x_3^3$. Therefore, Coker$\underline{\Lambda}$ defines a graded rank 2, 6–generated MCM over $R_3 = S_3/(f_3)$. These modules were explicitly described in [LPP].

Lemma 21. *Let M be a non–free graded orientable 6–generated MCM module over R. Then the restriction of M to the curve defined by $f = x_4 = 0$ splits into a direct sum of a 3–generated MCM of rank 1 and its dual. Especially, there exists $\lambda \in V(f_3) \setminus \{P_0\}$ and a skew symmetric matrix $\Gamma \in \mathcal{M}_{6\times 6}(K)$, such that M is the cokernel of a map given by the matrix $\Lambda = x_4 \cdot \Gamma + \begin{pmatrix} 0 & -\alpha_\lambda^t \\ \alpha_\lambda & 0 \end{pmatrix}$.*

(The same notations as in [LPP] and in Preliminaries.)

Proof. Let Λ_1 be a skew symmetric homogeneous matrix over S, corresponding to M, and denote $\underline{\Lambda_1} = \Lambda_1|_{x_4=0}$. Suppose that the MCM S_3–module corresponding to $\underline{\Lambda_1}$ is indecomposable. Then we can generate it as described in Theorem 4.2 and Lemma 5.4 from [LPP]. Denote with D the matrix which we obtain by this means.

Since $D \sim \underline{\Lambda_1}$, and $\underline{\Lambda_1}$ is skew symmetric, there exist two invertible matrices $U, V \in \mathcal{M}_{6\times 6}(K)$ such that $U \cdot D \cdot V + (U \cdot D \cdot V)^t = 0$. Therefore, there exists $T \in \mathcal{M}_{6\times 6}(K)$ an invertible matrix such that $T \cdot D + (TD)^t = 0$. (Take $T = (V^t)^{-1} \cdot U$.)

With the help of Singular, we find that, in fact, there is no invertible matrix T such that $T \cdot D$ is skew symmetric. Therefore, the module corresponding to $\underline{\Lambda_1}$ should decompose.

```
//First, we generate the matrix D

LIB"matrix.lib";
option(redSB);
proc reflexivHull(matrix M)
{
  module N=mres(transpose(M),3)[3];
  N=prune(transpose(N));
  return(matrix(N));
}

proc tensorCM(matrix Phi, matrix Psi)
{
   int s=nrows(Phi);
   int q=nrows(Psi);
   matrix A=tensor(unitmat(s),Psi);
   matrix B=tensor(Phi,unitmat(q));
   matrix R=concat(A,B,U);
   return(reflexivHull(R));
}

proc M2(ideal I)
{
   matrix A=syz(transpose(mres(I,3)[3]));
```

```
   return(transpose(A));
}

ring R=0,(x(1..3)),(c,dp);
qring S=std(x(1)^3+x(2)^3+x(3)^3);
ideal I=maxideal(1);
matrix C=M2(I);

ring R1=(0,a),(x(1..3),e,b),lp;
ideal I=x(1)^3+x(2)^3+x(3)^3,(a-1)^3+b3+1,e*b+a2-3*a+3,e*a-b2;
qring S1=std(I);

matrix B[3][3]=        0,  x(1)-(a-1)*x(3),         x(2)-b*x(3),
               x(1)+x(3),     -x(2)-x(3)*b,              -x(3)*e,
                    x(2),           x(3)*e,  -x (1)+(-a+2)*x(3);
matrix C=imap(S,C); matrix D=tensorCM(C,B);

//We check the existence of the invertible matrix T

ring R2=0,(x(1..3),a,e,b,t(1..36)),dp;
ideal I=x(1)^3+x(2)^3+x(3)^3,(a-1)^3+b3+1,e*b+a2-3*a+3,e*a-b2;
qring S2=std(I);
matrix D=imap(S1,D);
matrix T[6][6]=t(1..36);
matrix A=T*D+transpose(T*D); ideal I=flatten(A);
ideal I1=transpose(coeffs(I,x(1)))[2];
ideal I2=transpose(coeffs(I,x(2)))[2];
ideal I3=transpose(coeffs(I,x(3)))[2];
ideal J=I1+I2+I3+ideal(det(T)-1);
ideal L=std(J);
L;

L[1]=1
```

//Therefore, there does not exist an invertible matrix T such that $T \cdot D$ skew symmetric.

So, after some linear transformations, $\underline{\Lambda_1}$ decomposes into two matrices of order three and rank 1 with determinant $f_3 = x_1^3+x_2^3+x_3^3$, which correspond to two points λ_1, λ_2 in $V(f_3) \smallsetminus \{P_0\}$, $P_0 = [-1 : 0 : 1]$. Let us denote them by A and B. We can consider $A = \alpha_{\lambda_1}$, $B = \alpha_{\lambda_2}$.

Since $\underline{\Lambda_1}$ is skew symmetric, there exists an invertible matrix $U \in \mathcal{M}_{6\times 6}(K)$ such that $U \cdot \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ is skew symmetric. Therefore, if we consider $U = \left(\begin{smallmatrix} U_1 & U_2 \\ U_3 & U_4 \end{smallmatrix}\right)$, we have the following equalities:

$$\begin{cases} U_1 \cdot A + (U_1 \cdot A)^t & = & 0 \\ U_4 \cdot B + (U_4 \cdot B)^t & = & 0 \\ U_2 \cdot B + \ A^t \cdot U_3^t & = & 0 \\ U_3 \cdot A + \ B^t \cdot U_2^t & = & 0 \end{cases}$$

So $U_1 \cdot \alpha_{\lambda_1}$ and $U_4 \cdot \alpha_{\lambda_2}$ are skew symmetric, so they have only zeros on the main diagonal. Since the entries of the second and third line and column of α_{λ_1} and α_{λ_2} are linearly independent, we easily obtain that $U_1 = U_4 = 0$. Therefore, U_2 and U_3 are invertible matrices and $B = -U_2^{-1} \cdot A^t \cdot U_3^t$.

We have obtained $\underline{\Lambda_1} \sim \begin{pmatrix} \alpha_{\lambda_1} & 0 \\ 0 & \alpha_{\lambda_1}^t \end{pmatrix} \sim \begin{pmatrix} 0 & -\alpha_{\lambda_1}^t \\ \alpha_{\lambda_1} & 0 \end{pmatrix}$.

Therefore, there exists $\Gamma \in \mathcal{M}_{6\times 6}(K)$ skew symmetric, and $\lambda \in V(f_3) \smallsetminus \{P_0\}$ such that $\Lambda_1 \sim \Lambda = x_4 \cdot \Gamma + \begin{pmatrix} 0 & -\alpha_\lambda^t \\ \alpha_\lambda & 0 \end{pmatrix}$. We can write $\Gamma = \begin{pmatrix} \Gamma_1 & -\Gamma_2^t \\ \Gamma_2 & \Gamma_3 \end{pmatrix}$, $\Gamma_i \in \mathcal{M}_{3\times 3}(K)$, $i = 1, 2, 3$, Γ_1 and Γ_3 skew symmetric. □

Remark 22 *(Notation).* For any $\lambda = [a : b : c] \in V(f_3) \smallsetminus \{P_0\}$ there exists a unique point in $V(f_3) \smallsetminus \{P_0\}$ which we denote as λ^t, such that $\alpha_\lambda^t \sim \alpha_{\lambda^t}$. We find $\lambda^t = [c : b : a]$.[2]

For $\lambda = [a : b : 1]$ we denote with U_λ and V_λ two invertible matrices such that $U_\lambda \cdot \alpha_\lambda^t = \alpha_{\lambda^t} \cdot V_\lambda$.

If $a \neq 0$, then we can take $U_\lambda = \begin{pmatrix} b^2 & b(a+1) & -(a+1)^2 \\ -(a+1)^2 & b^2 & -b(a+1) \\ b(a+1) & (a+1)^2 & b^2 \end{pmatrix}$ and $V_\lambda = U_\lambda^t$.

If $a = 0$, then we can take $U_\lambda = \begin{pmatrix} -b^2 & -b & 1 \\ -2b & 1 & b^2 \\ 2b^2 & 2b & 1 \end{pmatrix}$ and $V_\lambda = \begin{pmatrix} 1 & -2b & 2b^2 \\ -b & -b^2 & -1 \\ -b & -b^2 & 2 \end{pmatrix}$.

Notice that $\lambda^t = \lambda$ for $\lambda = [1 : b : 1] \in V(f_3)$, and $\lambda \neq \lambda^t$ for all other $\lambda \in V(f_3) \smallsetminus \{P_0\}$.

[2]If λ corresponds to the 3–generated rank 1 MCM N, then λ^t corresponds to its dual $N^\vee$.

Remark 23.
For any $\lambda = [1 : b : 0] \in V(f_3) \smallsetminus \{P_0\}$ and any $\Lambda = x_4 \cdot \Gamma + \left(\begin{smallmatrix} 0 & -\alpha_\lambda^t \\ \alpha_\lambda & 0 \end{smallmatrix}\right)$ skew symmetric with $\det \Lambda = f^2$, we have $\lambda^t = [0 : b : 1]$ in $V(f_3) \smallsetminus \{P_0\}$ and $\Lambda' = x_4 \Gamma' + \left(\begin{smallmatrix} 0 & -\alpha_{\lambda^t}^t \\ \alpha_{\lambda^t} & 0 \end{smallmatrix}\right)$ skew symmetric with $\det \Lambda' = f^2$ such that $\Lambda \sim \Lambda'$.

Indeed, take $\Lambda' = U \cdot \Lambda \cdot U^t$ where $U = \left(\begin{smallmatrix} 0 & T_1 \\ T_2 & 0 \end{smallmatrix}\right)$, $T_1 = \frac{1}{\sqrt{3}} \left(\begin{smallmatrix} b & b^2 & 0 \\ 0 & 1 & -b^2 \\ 2 & 0 & 1 \end{smallmatrix}\right)$ and $T_2 = \frac{1}{\sqrt{3}} \left(\begin{smallmatrix} -1 & b & b \\ 2 & b^2 & b^2 \\ -2b^2 & 1 & -2 \end{smallmatrix}\right)$.

Therefore, Coker Λ and Coker Λ' define two isomorphic MCM modules. This is the reason why we may only consider the case $\lambda = [a : b : 1] \in V(f_3) \smallsetminus \{P_0\}$, from now on. □

Remark 24. Consider $\lambda = [a : b : 1] \in V(f_3) \smallsetminus \{P_0\}$ and $\Lambda = x_4 \cdot \Gamma + \left(\begin{smallmatrix} 0 & -\alpha_\lambda^t \\ \alpha_\lambda & 0 \end{smallmatrix}\right)$ as in Lemma 21. Then there exists $\overline{\Lambda} = x_4 \cdot \overline{\Gamma} + \left(\begin{smallmatrix} \alpha_\lambda & 0 \\ 0 & \alpha_{\lambda^t} \end{smallmatrix}\right)$ with $\det \overline{\Lambda} = f^2$ such that $\overline{\Lambda} \sim \Lambda$.

Indeed, consider $\overline{\Lambda} = \left(\begin{smallmatrix} 0 & \mathrm{Id} \\ -U_\lambda & 0 \end{smallmatrix}\right) \cdot \Lambda \cdot \left(\begin{smallmatrix} \mathrm{Id} & 0 \\ 0 & V_\lambda^{-1} \end{smallmatrix}\right)$.

We obtain $\overline{\Gamma} = \left(\begin{smallmatrix} \Gamma_2 & \Gamma_3 \cdot V_\lambda^{-1} \\ -U_\lambda \cdot \Gamma_1 & U_\lambda \cdot \Gamma_2^t \cdot V_\lambda^{-1} \end{smallmatrix}\right)$. □

Lemma 25. *Consider* $\Lambda = x_4 \cdot \Gamma + \left(\begin{smallmatrix} 0 & -\alpha_\lambda^t \\ \alpha_\lambda & 0 \end{smallmatrix}\right)$ *as above. Then the MCM module* M *corresponding to* Λ *is indecomposable if and only if* $\Gamma_1 \neq 0$ *or* $\Gamma_3 \neq 0$.

Proof. Suppose M is indecomposable. If $\Gamma_1 = \Gamma_3 = 0$, then $\left(\begin{smallmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{smallmatrix}\right) \cdot \Lambda = \left(\begin{smallmatrix} x_4 \cdot \Gamma_2 + \alpha_\lambda & 0 \\ 0 & -x_4 \cdot \Gamma_2^t - \alpha_\lambda^t \end{smallmatrix}\right)$, so Λ decomposes after some linear transformation. This contradicts the indecomposability of $M =$ Coker Λ, so we must have $\Gamma_1 \neq 0$ or $\Gamma_3 \neq 0$.

Now, let us suppose $\Gamma_1 \neq 0$ or $\Gamma_3 \neq 0$ and prove that M is indecomposable. Suppose M decomposes. Then there exists a matrix $\left(\begin{smallmatrix} T_1 & 0 \\ 0 & T_2 \end{smallmatrix}\right)$ equivalent to Λ with T_1, T_2 two matrices of order three and rank 1, with $\det T_1 = \det T_2 = f$ and $T_1|_{x_4=0} = \alpha_{\lambda_1}$, $T_2|_{x_4=0} = \alpha_{\lambda_2}$, where $\lambda_1, \lambda_2 \in V(f_3) \smallsetminus \{P_0\}$.

Since Λ is skew symmetric, after some linear transformations, $\left(\begin{smallmatrix} \alpha_{\lambda_1} & 0 \\ 0 & \alpha_{\lambda_2} \end{smallmatrix}\right)$ should also become skew symmetric. As we saw in the proof of Lemma 21, this gives $\alpha_{\lambda_2} \sim \alpha_{\lambda_1}^t$, so $\lambda_2 = \lambda_1^t$.

Using Remark 23, there exist $U, V \in \mathcal{M}_{6\times 6}(K)$ invertible matrices such that $U \cdot \overline{\Lambda} \cdot V = \left(\begin{smallmatrix} T_1 & 0 \\ 0 & T_2 \end{smallmatrix}\right) = x_4 \cdot \left(\begin{smallmatrix} N_1 & 0 \\ 0 & N_2 \end{smallmatrix}\right) + \left(\begin{smallmatrix} \alpha_{\lambda_1} & 0 \\ 0 & \alpha_{\lambda_1^t} \end{smallmatrix}\right)$.

Therefore,

$$\begin{cases} U \cdot \begin{pmatrix} \alpha_\lambda & 0 \\ 0 & \alpha_{\lambda^t} \end{pmatrix} & = \begin{pmatrix} \alpha_{\lambda_1} & 0 \\ 0 & \alpha_{\lambda_1^t} \end{pmatrix} \cdot V^{-1} \qquad (1) \\ U \cdot \begin{pmatrix} \Gamma_2 & \Gamma_3 \cdot V_\lambda^{-1} \\ -U_\lambda \cdot \Gamma_1 & U_\lambda \cdot \Gamma_2^t \cdot V_\lambda^{-1} \end{pmatrix} & = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix} \cdot V^{-1} \qquad (2)\,. \end{cases}$$

Let us consider $U = \left(\begin{smallmatrix} U_1 & U_2 \\ U_3 & U_4 \end{smallmatrix}\right)$ and $V^{-1} = \left(\begin{smallmatrix} V_1 & V_2 \\ V_3 & V_4 \end{smallmatrix}\right)$ with $U_i, V_i \in \mathcal{M}_{3\times 3}(K)$, $i = 1, \ldots, 4$.

The first system of equations gives:

$$\begin{cases} U_1 \cdot \alpha_\lambda & = & \alpha_{\lambda_1} \cdot V_1 \\ U_2 \cdot \alpha_{\lambda^t} & = & \alpha_{\lambda_1} \cdot V_2 \\ U_3 \cdot \alpha_\lambda & = & \alpha_{\lambda_1^t} \cdot V_3 \\ U_4 \cdot \alpha_{\lambda^t} & = & \alpha_{\lambda_1^t} \cdot V_4 \end{cases}$$

By comparing the coefficients of x_1, x_2, x_3 on the left–hand side and right–hand side of the above equalities, we obtain easily:

$$U_i = V_i = K_i \cdot \mathrm{Id}_3 \ \text{ with } K_i \in K,\ i = 1, \ldots, 4\,.$$

Moreover, if $\lambda \neq \lambda_1$, then $K_1 = K_4 = 0$ and if $\lambda \neq \lambda_1^t$, then $K_2 = K_3 = 0$. Since U is invertible, we have $\lambda = \lambda_1$ or $\lambda = \lambda_1^t$.

We know that $\alpha_{\lambda_1} = T_1|_{x_4=0}$ where T_1 is a matrix of order three over $S = K[x_1, x_2, x_3, x_4]$ of rank 1 and with determinant f. So Coker T_1 is a graded 3–generated rank 1 MCM R–module. In [EP], all the isomorphism classes of such modules are given explicitly. We obtain $\alpha_{\lambda_1} \sim \alpha|_{x_4=0}$ or $\alpha_{\lambda_1} \sim \alpha^t|_{x_4=0}$ or $\alpha_{\lambda_1} \sim \eta|_{x_4=0}$ or $\alpha_{\lambda_1} \sim \nu|_{x_4=0}$.

With the help of computers, we obtain that none of the above matrices is equivalent to $\alpha_{[1:\ell:1]}$, therefore, $\lambda_1 \neq \lambda_1^t$.

```
LIB"matrix.lib";
option(redSB);

ring r=0,(x(1..3),l,a,b,c,d,e,v(1..9),u(1..9)),dp;
```

```
ideal I=x(1)^3+x(2)^3+x(3)^3,
        l^3+2,
        a3+1,b3+1,c3+1,d3+1,e2+e+1,bcd-e*a;
qring s=std(I);

proc isomorf(matrix X,matrix Y)
{
   matrix U[3][3]=u(1..9);
   matrix V[3][3]=v(1..9);
   matrix C=U*X-Y*V;
   ideal I=flatten(C);
   ideal I1=transpose(coeffs(I,x(1)))[2];
   ideal I2=transpose(coeffs(I,x(2)))[2];
   ideal I3=transpose(coeffs(I,x(3)))[2];
   ideal J=I1+I2+I3+ideal(det(U)-1,det(V)-1);
   ideal L=std(J);
   return(L);
}

matrix A[3][3]=0,            x(1)-x(3),   x(2)-l*x(3),
               x(1)+x(3), -x(2)-l*x(3), -1/2*l^2*x(3),
                    x(2), 1/2*l^2*x(3),         -x(1);

//This is the matrix corresponding to the point (1:l:1)

//We now write the matrices corresponding to the rank 1 3-
generated MCM modules, restricted to x(4)=0

matrix alpha[3][3]=0,      x(1),                         -x(3)*b+x(2),
                   -x(2)*c+x(1),           -x(3)*b^2,  x(3)*b^2*c^2,
                           x(3), x(3)*b*c^2+x(2)*c^2,  -x(2)*c-x(1);

matrix alphat=transpose(alpha);

matrix eta[3][3]=0,x(1)+x(2),      x(3),
                 x(1)+e*x(2),     -x(3),                0,
                        x(3),         0,    -x(1)-e^2*x(2);

matrix nu[3][3]=0,    x(1)+x(3),   x(2),
                x(1)-a^2*b*x(3),  -x(2),                0,
```

```
                          x(2),      0,  -x(1)+a*b^2*x(3);

isomorf(alpha,A); L[1]=1
isomorf(alphat,A); L[1]=1
isomorf(eta,A); L[1]=1
isomorf(teta,A); L[1]=1
```

//Therefore none is isomorphic to $\alpha_{[1:\ell:1]}$ and this means $\lambda_1 \neq \lambda_1^t$.

If $\lambda = \lambda_1 \neq \lambda_1^t$ as a solution of the system (1), we obtain: $U = V = \left(\begin{smallmatrix} K_1\cdot \mathrm{Id} & 0 \\ 0 & K_4\cdot \mathrm{Id}\end{smallmatrix}\right)$, $K_1 \cdot K_4 \neq 0$.

Replacing U and V in (2), we obtain: $\begin{cases} K_1 \cdot \Gamma_3 \cdot V_\lambda = 0 \\ K_4 \cdot U_\lambda \cdot \Gamma_1 = 0. \end{cases}$

Since $K_1 \neq 0$, $K_4 \neq 0$ and U_λ, V_λ are invertible matrices, we obtain $\Gamma_1 = \Gamma_3 = 0$, which is a contradiction to our hypothesis.

If $\lambda = \lambda_1^t \neq \lambda_1$, we obtain, as a solution of (1): $U = V = \left(\begin{smallmatrix} 0 & K_2\cdot \mathrm{Id} \\ K_3\,\mathrm{Id} & 0\end{smallmatrix}\right)$, $K_2 \cdot K_3 \neq 0$.

Replacing U and V in (2), we obtain: $\begin{cases} K_2 \cdot U_\lambda \cdot \Gamma_1 = 0 \\ K_3 \cdot \Gamma_3 \cdot V_\lambda = 0. \end{cases}$ Therefore, we must have again $\Gamma_1 = \Gamma_3 = 0$. □

For each $\lambda = [a : b : 1] \in V(f_3) \setminus \{P_0\}$, we define a family of skew symmetric homogeneous indecomposable matrices of order six over $S = K[x_1, x_2, x_3, x_4]$ with determinant f^2:

$$\mathcal{M}_\lambda := \left\{ \Lambda_{(\lambda,\Gamma)} = x_4 \cdot \Gamma + \begin{pmatrix} 0 & -\alpha_\lambda^t \\ \alpha_\lambda & 0 \end{pmatrix},\ \det \Lambda_{(\lambda,\Gamma)} = f^2, \right.$$
$$\left. \Gamma = \begin{pmatrix} \Gamma_1 & -\Gamma_2^t \\ \Gamma_2 & \Gamma_3 \end{pmatrix},\quad \Gamma_1, \Gamma_3 \text{ skew symmetric, } \begin{matrix} \Gamma_1 \neq 0 \\ \text{or}\quad \Gamma_3 \neq 0 \end{matrix} \right\}.$$

Notice that, as in the proof of Lemma 25, if $\Lambda_{(\lambda,\Gamma)} \sim \Lambda_{(\lambda',\Gamma')}$, then $\lambda' = \lambda$ or $\lambda' = \lambda^t$.

Lemma 26. *Let* $\lambda = [a : b : 1] \in V(f_3) \setminus \{P_0\}$ *with* $a \neq 1$.

(1) *Inside the family $\mathcal{M}_\lambda$, two matrices, Λ and Λ', are equivalent if and only if there exists $k \in K^*$ such that $\Lambda' = U_k \cdot \Lambda \cdot U_k^t$, $U_k = \begin{pmatrix} k\,\mathrm{Id} & 0 \\ 0 & \frac{1}{k}\,\mathrm{Id} \end{pmatrix}$.*

This condition means: $\begin{cases} \Gamma_2' = \Gamma_2 \\ \Gamma_1' = k^2 \cdot \Gamma_1 \\ \Gamma_3' = \frac{1}{k^2} \cdot \Gamma_3. \end{cases}$

(2) *A matrix Λ from $\mathcal{M}_\lambda$ is equivalent to a matrix Λ' from $\mathcal{M}_{\lambda'}$, $\lambda' \neq \lambda$ if and only if $\lambda' = [1 : b : a]$ and $\Lambda' = U_k \cdot \Lambda \cdot U_k^t$, where $k \in K^*$ and $U_k = \begin{pmatrix} 0 & k \cdot U_\lambda^{-1} \\ -\frac{1}{k} U_\lambda & 0 \end{pmatrix}$.*

Proof. We assume $a \neq 0$. The case $a = 0$ is treated similarly. Two matrices, $\Lambda = \Lambda_{(\lambda,\Gamma)}$ and $\Lambda' = \Lambda_{(\lambda',\Gamma')}$, are equivalent if and only if $\overline{\Lambda}$ and $\overline{\Lambda}'$ are equivalent (see Remark 24).

If U and V are two invertible matrices such that $U \cdot \overline{\Lambda} = \overline{\Lambda}' \cdot V$, as in the proof of Lemma 25, we obtain

$$U = V = \begin{pmatrix} K_1\,\mathrm{Id} & K_2\,\mathrm{Id} \\ K_3\,\mathrm{Id} & K_4\,\mathrm{Id} \end{pmatrix} \quad \text{with} \quad \begin{aligned} &K_1 = K_4 = 0 \text{ if } \lambda \neq \lambda' \text{ and} \\ &K_2 = K_3 = 0 \text{ if } \lambda' \neq \lambda^t\,. \end{aligned}$$

Since $U \cdot \overline{\Lambda} = \overline{\Lambda}' \cdot V$, we have:

$$\begin{pmatrix} 0 & \mathrm{Id} \\ -U_{\lambda'} & 0 \end{pmatrix}^{-1} \cdot U \cdot \begin{pmatrix} 0 & \mathrm{Id} \\ -U_\lambda & 0 \end{pmatrix} \cdot \Lambda \cdot \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & V_\lambda^{-1} \end{pmatrix} \cdot U^{-1} \cdot \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & V_{\lambda'}^{-1} \end{pmatrix}^{-1} = \Lambda'. \tag{$*$}$$

(1) If $\lambda = \lambda'$ then $\lambda' \neq \lambda^t$, so $U = \begin{pmatrix} K_1\,\mathrm{Id} & 0 \\ 0 & K_4\,\mathrm{Id} \end{pmatrix}$ with $K_1 \neq 0$, $K_4 \neq 0$. So $(*)$ implies: $\begin{pmatrix} K_4 \cdot \mathrm{Id} & 0 \\ 0 & K_1\,\mathrm{Id} \end{pmatrix} \cdot \Lambda \cdot \begin{pmatrix} \frac{1}{K_1}\,\mathrm{Id} & 0 \\ 0 & \frac{1}{K_4}\,\mathrm{Id} \end{pmatrix} = \Lambda'$. For $k = \sqrt{\frac{K_4}{K_1}}$ and $U_k = \begin{pmatrix} k\,\mathrm{Id} & 0 \\ 0 & \frac{1}{k} \cdot \mathrm{Id} \end{pmatrix}$ we have $\Lambda' = U_k \cdot \Lambda \cdot U_k^t$.

(2) If $\lambda' = \lambda^t$ then $\lambda' \neq \lambda$, so $U = \begin{pmatrix} 0 & K_2\,\mathrm{Id} \\ K_3\,\mathrm{Id} & 0 \end{pmatrix}$, $K_2 \neq 0$, $K_3 \neq 0$. Replacing U in $(*)$ we obtain:

$$\Lambda' = \begin{pmatrix} 0 & -K_3 U_{\lambda^t}^{-1} \\ -K_2 U_\lambda & 0 \end{pmatrix} \cdot \Lambda \cdot \begin{pmatrix} 0 & \frac{1}{K_3} V_{\lambda^t} \\ \frac{1}{K_2} V_\lambda^{-1} & 0 \end{pmatrix}.$$

Since $a \neq 0$ and $a \neq 1$, $V_\lambda = U_\lambda^t$, $\lambda^t = \left[\frac{1}{a} : \frac{1}{b} : 1\right]$, $U_{\lambda^t} = \frac{1}{a^2} \cdot U_\lambda$, $V_{\lambda^t} = \frac{1}{a^2} U_\lambda^t$ (see Remark 22).

So $\Lambda' = \begin{pmatrix} 0 & -K_3 a^2 U_\lambda^{-1} \\ -K_2 U_\lambda & 0 \end{pmatrix} \cdot \Lambda \cdot \begin{pmatrix} 0 & \frac{1}{K_3} \cdot \frac{1}{a^2} \cdot U_\lambda^t \\ \frac{1}{K_2} (U_\lambda^{-1})^t & 0 \end{pmatrix} = \begin{pmatrix} 0 & k U_\lambda^{-1} \\ -\frac{1}{k} U_\lambda & 0 \end{pmatrix} \cdot$
$\Lambda \cdot \begin{pmatrix} 0 & k U_\lambda^{-1} \\ -\frac{1}{k} U_\lambda & 0 \end{pmatrix}^t$, where $k^2 = -a^2 \cdot \frac{K_3}{K_2}$. □

In a similar way, we can prove the following lemma:

Lemma 27. *Let $\lambda = [1 : b : 1] \in V(f_3) \smallsetminus \{P_0\}$.*

(1) Inside the family $\mathcal{M}_\lambda$, two matrices Λ and Λ' are equivalent if and only if $\Lambda' = T \cdot \Lambda \cdot T^t$, where

$$T = \begin{pmatrix} K_4 \cdot \mathrm{Id} & K_3 \cdot U_\lambda^{-1} \\ K_2 \cdot U_\lambda & K_1 \cdot \mathrm{Id} \end{pmatrix}, \ K_1, K_2, K_3, K_4 \in K \text{ such that } K_1 K_4 - K_2 K_3 = 1\,.$$

(2) No $\lambda \in V(f_3) \smallsetminus \{P_0, [1 : b : 1]\}$ exists, such that a matrix from $\mathcal{M}_\lambda$ is equivalent to a matrix from $\mathcal{M}_{[1:b:1]}$.

Now let us see "how large" the family $\mathcal{M}_\lambda$ is for a given λ in $V(f_3) \smallsetminus \{P_0\}$. For $\Lambda = \Lambda_{(\lambda,\Gamma)}$ in $\mathcal{M}_\lambda$, we denote:

$$\Gamma_1 = \begin{pmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} 0 & a_4 & a_5 \\ -a_4 & 0 & a_6 \\ -a_5 & -a_6 & 0 \end{pmatrix}.$$

The condition $\det \Lambda = f^2$ provides 10 equations in the above 15 parameters. Six of these equations are linear in the entries of Γ_2 and form a linear system of dimension three.

(1) If $b = 0$ the solution of this system is:

$$\begin{cases} a_7 = -a_{12} \cdot (a^2 + 1) \\ a_8 = a_{10} = a_{15} = 0 \\ a_9 = a_{11} - a_{13} \\ a_{14} = a^2 \cdot a_{12}\,. \end{cases}$$

(2) If $b \neq 0$, the system has the following solution:

$$\begin{cases} a_8 &= -\frac{b}{a+1} a_7 + a_{15} \\ a_9 &= -\frac{a-1}{b(a+1)} a_7 - \frac{a^2}{b^2} \cdot a_{15} \\ a_{10} &= \frac{b}{a+1} \cdot a_7 \\ a_{12} &= -\frac{a^2+3}{(a+1)^2} \cdot a_7 + \frac{b}{a+1} a_{11} + \frac{1-a}{b(a+1)} a_{15} \\ a_{13} &= \frac{a-1}{b(a+1)} \cdot a_7 + a_{11} + \frac{a^2}{b^2} \cdot a_{15} \\ a_{14} &= \frac{2(1-a)}{(a+1)^2} \cdot a_7 - \frac{b}{a+1} \cdot a_{11} + \frac{a-1}{b(a+1)} \cdot a_{15}\,. \end{cases}$$

The other four equations are linear in the entries of Γ_1 with coefficients in $K[a_4, \ldots, a_{15}]$ and have dimension five:

```
LIB"matrix.lib";
option(redSB);

ring r=0,(x(4),x(1),x(2),x(3),e,a,b,a(1..15)),dp;
ideal ii=a3+b3+1,e*b+a2-a+1,e*a+e-b2;
qring s=std(ii);

matrix B[10][1];
B[1,1]=x(4)*a(1);
B[2,1]=x(4)*a(2);
B[3,1]=-x(4)*a(7);
B[4,1]=-x(4)*a(10)-(x(1)+x(3));
B[5,1]=x(4)*a(3);
B[6,1]=-x(4)*a(8)-(x(1)-a*x(3));
B[7,1]=-x(4)*a(11)+x(2)+b*x(3);
B[8,1]=-x(4)*a(9)-x(2)+b*x(3);
B[9,1]=-x(4)*a(12)+e*x(3);
B[10,1]=x(4)*a(4);

matrix V[1][5];
V[1,1]=-x(4)*a(13)-x(2);
V[1,2]=-x(4)*a(14)-e*x(3);
V[1,3]=-x(4)*a(15)+x(1)+(a-1)*x(3);
```

```
V[1,4]=x(4)*a(5); V[1,5]=x(4)*a(6);

poly p1=B[5,1]*B[10,1]-B[6,1]*B[9,1]+B[7,1]*B[8,1];
poly p2=B[2,1]*B[10,1]-B[3,1]*B[9,1]+B[4,1]*B[8,1];
poly p3=B[1,1]*B[10,1]-B[3,1]*B[7,1]+B[4,1]*B[6,1];
poly p4=B[1,1]*B[9,1]-B[2,1]*B[7,1]+B[4,1]*B[5,1];
poly p5=B[1,1]*B[8,1]-B[2,1]*B[6,1]+B[3,1]*B[5,1];

poly g=V[1,1]*p1-V[1,2]*p2+V[1,3]*p3-V[1,4]*p4+V[1,5]*p5;
poly f=x(4)^3+x(1)^3+x(2)^3+x(3)^3; g=g-f;

//For our skew symmetric matrix the condition g=f is equivalent
//to det Λ = f^2.

matrix H=coef(g,x(4)*x(1)*x(2)*x(3));
for(int j=1;j<=13;j++)
{H[1,j]=0;}

ideal I=H; I=interred(I);

I[1]=a(9)-a(11)+a(13) I[2]=a(8)+a(10)-a(15) I[3]=a(7)+a(12)+a(14)
I[4]=a*a(10)-e*a(11)+b*a(12)+2*e*a(13)+2*b*a(14)-2*a*a(15)+a(10)
     +a(15)
I[5]=2*e*a(10)+2*b*a(11)-2*a*a(12)-b*a(13)-a*a(14)-e*a(15)
     +a(12)+2*a(14)
I[6]=a(3)*a(4)-a(2)*a(5)+a(1)*a(6)+a(11)^2+a(10)*a(12)-a(11)*a(13)
     +a(13)^2-a(10)*a(14)-2*a(12)*a(15)-a(14)*a(15)
I[7]=a(1)*a(4)+a(3)*a(5)+a(2)*a(6)-a(10)^2+a(11)*a(12)+a(12)*a(13)
     +2*a(11)*a(14)-a(13)*a(14)+a(10)*a(15)-a(15)^2
I[8]=2*e^2*a(12)+2*a*b*a(12)-3*b^2*a(13)+2*e^2*a(14)-a*b*a(14)
     -3*e*b*a(15)-6*e*a(11)-b*a(12)+12*e*a(13)+2*b*a(14)-6*a*a(15)
I[9]=a(3)*a(5)*a(10)-a(2)*a(6)*a(10)-a(2)*a(5)*a(11)-a(1)*a(6)*a(11)
      +a(1)*a(5)*a(12)+a(3)*a(6)*a(12)-a(2)*a(5)*a(13)
      +2*a(1)*a(6)*a(13)+a(13)^3+a(2)*a(4)*a(14)+a(3)*a(6)*a(14)
      +a(10)*a(11)*a(14)+a(12)^2*a(14)-2*a(10)*a(13)*a(14)
      +a(12)*a(14)^2+a(3)*a(5)*a(15)+2*a(2)*a(6)*a(15)
      +a(11)*a(14)*a(15)-2*a(13)*a(14)*a(15)-a(15)^3-1
I[10]=2*e*a(2)*a(4)-2*e*a(1)*a(5)+2*b*a(2)*a(5)-2*a*a(3)*a(5)
      +2*b*a(1)*a(6)-4*a*a(2)*a(6)-2*b*a(11)^2+2*a*a(11)*a(12)
      +2*e*a(12)^2+5*b*a(11)*a(13)-4*a*a(12)*a(13)-2*b*a(13)^2
```

```
-2*b*a(10)*a(14)-a*a(11)*a(14)+2*a*a(13)*a(14)-2*e*a(14)^2
+3*e*a(11)*a(15)-6*e*a(13)*a(15)-2*b*a(14)*a(15)+6*a*a(15)^2
+4*a(3)*a(5)+2*a(2)*a(6)-a(11)*a(12)+2*a(12)*a(13)
+2*a(11)*a(14)-4*a(13)*a(14)-6*a(15)^2

ideal J=I[1],I[2],I[3],I[4],I[5],I[8];
```

//This is the ideal generated by the linear equations in the //entries of Γ_2.

```
ideal JJ=std(J); dim(JJ); 14

ideal J1=I[6],I[7],I[9],I[10];

//This is the ideal generated by the other four equations.

ideal JJ1=std(J1); dim(JJ1); 16
```

Let us summarize the results.

Let M be an indecomposable graded rank 2, 6–generated MCM and $\overline{M}$ the restriction of M to the elliptic curve on our surface defined by $f = x_4 = 0$. Then $\overline{M} \cong N_\lambda \oplus N_\lambda^\vee$ for a suitable 3–generated rank 1 MCM $N_\lambda = \operatorname{coker}(\alpha_\lambda)$, $\lambda \in V(f, x_4) \smallsetminus \{[-1:0:1:0]\} \cong V(f_3) \smallsetminus \{[-1:0:1]\} =: C$. If $\lambda = [a:b:c]$ and $\lambda^t := [c:b:a]$, then $N_\lambda^\vee \cong N_{\lambda^t}$, in particular, there exist skew–symmetric 3×3–matrices Γ_1, Γ_3 with constant entries not being zero simultaneously and a 3×3–matrix Γ_2 such that $M = \operatorname{coker}(\Lambda)$ for $\Lambda = x_4 \begin{pmatrix} \Gamma_1 & -\Gamma_2^t \\ \Gamma_2 & \Gamma_3 \end{pmatrix} + \begin{pmatrix} 0 & -\alpha_\lambda^t \\ \alpha_\lambda & 0 \end{pmatrix}$, $\Gamma_1 = \begin{pmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{pmatrix}$, $\Gamma_3 = \begin{pmatrix} 0 & a_4 & a_5 \\ -a_4 & 0 & a_6 \\ -a_5 & -a_6 & 0 \end{pmatrix}$, $\Gamma_2 = \begin{pmatrix} a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{pmatrix}$ and $\det(\Lambda) = f^2$.

Let $\mathbb{A}^{15}$ be the 15–dimensional affine space with the coordinates $(a_1, \ldots, a_{15})$ and G be the subgroup of $\mathrm{Sl}_2(K)$ generated by the matrices $g_k = \begin{pmatrix} 0 & k \\ -\frac{1}{k} & 0 \end{pmatrix}$, $k \in K \smallsetminus \{0\}$. Consider the action of G on $\mathbb{A}^{15}$: $G \times \mathbb{A}^{15} \to \mathbb{A}^{15}$, $(g_k, \underline{a}) \to \underline{a}' = (k^2 a_1, k^2 a_2, k^2 a_3, \frac{1}{k^2} a_4, \frac{1}{k^2} a_5, \frac{1}{k^2} a_6, a_7, .., a_{15})$. Denote $\mathbb{A} = \mathbb{A}^{15}/G$.

A point $(\lambda; \underline{a}) \in C \times \mathbb{A}$ corresponds to a matrix $\Lambda = x_4 \begin{pmatrix} \Gamma_1 & -\Gamma_2^t \\ \Gamma_2 & \Gamma_3 \end{pmatrix} + \begin{pmatrix} 0 & -\alpha_\lambda^t \\ \alpha_\lambda & 0 \end{pmatrix}$. The group G acts on $C \times \mathbb{A}$ in the following way: let $(\lambda; a) \in C \times \mathbb{A}$ correspond to $\Lambda = x_4 \begin{pmatrix} \Gamma_1 & -\Gamma_2^t \\ \Gamma_2 & \Gamma_3 \end{pmatrix} + \begin{pmatrix} 0 & -\alpha_\lambda^t \\ \alpha_\lambda & 0 \end{pmatrix}$, then $g_k(\lambda; a) = (\lambda^t; b)$, where $(\lambda^t; b)$

corresponds to $U_k \Lambda U_k^t$. Let $\mathcal{M} \subseteq C \times \mathbb{A}$ be the G–invariant closed subset defined by $\det(\Lambda) = f^2$. Let $\pi : \mathcal{M} \to C$ be the canonical projection. If $\lambda = [1 : b : 1]$ with $b^3 = -2$, then $\mathrm{Sl}_2(K)$ acts on $\pi^{-1}(\lambda)$ via the representation $\mathrm{Sl}_2(K) \to \left\{ \begin{pmatrix} K_1 \mathrm{Id} & K_2 U_\lambda^{-1} \\ K_3 U_\lambda & K_4 \mathrm{Id} \end{pmatrix}, \ K_1K_4 - K_2K_3 = 1 \right\}$, $\begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} \mapsto \begin{pmatrix} K_1 \mathrm{Id} & K_2 U_\lambda^{-1} \\ K_3 U_\lambda & K_4 \mathrm{Id} \end{pmatrix}$.

Theorem 28. *(1) Every indecomposable graded rank 2, 6–generated MCM is represented by a point in* $\mathcal{M}$.

(2) $\mathcal{M} \smallsetminus \pi^{-1}(\{[1 : b : 1] \mid b^3 = -2\})/G$ *is the moduli space of isomorphism classes of indecomposable graded rank 2, 6–generated MCM* M *such that the restriction to* $V(f, x_4)$, $\overline{M} \cong N_\lambda \oplus N_\lambda^\vee$ *for* N_λ *being not self–dual. This moduli space is 5–dimensional.*

(3) $Sl_2(K)$ *acts on* $\pi^{-1}(\{[1 : b : 1] \mid b^3 = -2\})$ *and* $\pi^{-1}(\{[1 : b : 1] \mid b^3 = -2\})/Sl_2(K)$ *is the moduli space of isomorphism classes of indecomposable graded rank 2, 6–generated MCM* M *such that the restriction to* $V(f, x_4)$, $\overline{M} \cong N_\lambda \oplus N_\lambda$ *for* N_λ *being self–dual.*

Remark 29. It is well known that the ideal defining 5 general points in $\mathbf{P}_K^3$ (this means any four from them are not on a hyperplane) is Gorenstein . Restricting to the 5 general points on the surface $V(f)$ we get a family of Gorenstein ideals whose isomorphism classes of 2-syzygies over R (they are indecomposable, graded, rank 2, 6-generated MCM modules) form a 5-parameter family (see [Mig], [IK]).

Here we give an example. Let $[1 : 0 : 0 : -1]$, $[1 : 0 : -1 : 0]$, $[1 : -1 : 0 : 0]$, $[1 : -u : 0 : 0]$, $[1 : -u : 1 : -u]$, $u^2 + u + 1 = 0$ be 5 general points on $V(f)$ and I the ideal defined by these points in R. I is generated by the following quadratic forms: $x_2x_4 + ux_3x_4$, $-ux_2x_3 + ux_3x_4$, $x_1x_4 + x_4^2 - (1-u)x_3x_4$, $u(x_1 + x_3)x_3 + 2x_3x_4$, $-x_3x_4 - x_1^2 + ux_1x_2 - u^2x_2^2 + x_3^2 + x_4^2$. Then the second syzygy of I over R is the cokernel of a skew symmetric matrix A defined by
$A[1,1] = A[2,2] = A[3,3] = A[4,4] = A[5,5] = A[6,6] = 0$,
$A[1,2] = (-3u-2)x_3 + (2u-1)x_4 = -A[2,1]$,
$A[1,3] = -ux_1 + (-2u+1)x_2 + (u+1)x_3 + ux_4 = -A[3,1]$,
$A[1,4] = (u-2)x_1 - x_2 + (-3u-4)x_3 + (2u-1)x_4 = -A[4,1]$,
$A[1,5] = (u+1)x_3 - ux_4 = -A[5,1]$,
$A[1,6] = -ux_1 + (u+1)x_2 + (1/7u + 3/7)x_3 + (-3/7u - 2/7)x_4 = -A[6,1]$,
$A[2,3] = (u-2)x_1 - x_2 + x_3 + (-u+2)x_4 = -A[3,2]$,
$A[2,4] = (3u+2)x_1 + (2u+3)x_2 + 4ux_3 + x_4 = -A[4,2]$,

$A[2,5] = (-3u-1)x_3 + (u-2)x_4 = -A[5,2]$,
$A[2,6] = (-u-2)x_1 + (-u+1)x_2 + (-u-1)x_3 + ux_4 = -A[6,2]$,
$A[3,4] = -3x_3 = -A[4,3]$,
$A[3,5] = (u+1)x_3 = -A[5,3]$,
$A[3,6] = (-6/7u - 4/7)x_3 + x_4 = -A[6,3]$,
$A[4,5] = (-3u-1)x_3 = -A[5,4]$,
$A[4,6] = -ux_3 + ux_4 = -A[6,4]$,
$A[5,6] = -x_1 - ux_2 = -A[6,5]$.

This matrix is equivalent to $\Lambda_{(\lambda,\Gamma)} = x_4 \cdot \Gamma + \begin{pmatrix} 0 & -\alpha_\lambda^t \\ \alpha_\lambda & 0 \end{pmatrix}$ for $\lambda = (0 : u+1 : 1)$ and Γ given by:
$a_1 = -\frac{4}{3}u - \frac{2}{3}$,
$a_2 = -u - 1$,
$a_3 = \frac{2}{3}u + \frac{1}{3}$,
$a_4 = -\frac{1}{4}u + 1$,
$a_5 = -u - \frac{1}{2}$,
$a_6 = -\frac{3}{2}u + \frac{3}{4}$,
$a_7 = -\frac{1}{2}$,
$a_8 = -\frac{1}{2}u$,
$a_9 = \frac{1}{2}u$,
$a_{10} = -\frac{1}{2}u - \frac{1}{2}$,
$a_{11} = u + \frac{3}{2}$,
$a_{12} = u + 1$,
$a_{13} = \frac{1}{2}u + \frac{3}{2}$,
$a_{14} = -u - \frac{1}{2}$,
$a_{15} = -u - \frac{1}{2}$.

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