CONSTRUCTIVE NÉRON DESINGULARIZATION OF ALGEBRAS WITH BIG SMOOTH LOCUS.

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Abstract. An algorithmic proof of the General Néron Desingularization theorem and its uniform version is given for morphisms with big smooth locus. This generalizes the results for the one-dimensional case (cf. [10], [7]).

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Introduction

Motivated to generalize Artin's Approximation Theorem (cf. [2]) to excellent Henselian rings the third author developed a powerful tool, the General Néron Desingularization (cf. [14]). This result was discussed and used later by many other authors (cf. [1], [20], [19]). The proof of the Desingularization was not constructive. In this paper we want to give an algorithm to compute Néron Desingularization for an important special case. We begin recalling some standard definitions.

A ring morphism $u : A \to A'$ of Noetherian rings has *regular fibers* if for all prime ideals $P \in \operatorname{Spec} A$ the ring A'/PA' is a regular ring, i.e. its localizations are regular local rings. It has *geometrically regular fibers* if for all prime ideals $P \in \operatorname{Spec} A$ and all finite field extensions K of the fraction field of A/P the ring $K \otimes_{A/P} A'/PA'$ is regular.

A flat morphism of Noetherian rings u is *regular* if its fibers are geometrically regular. If u is regular of finite type then u is called *smooth*. A localization of a smooth algebra is called *essentially smooth*. A Henselian Noetherian local ring A is *excellent* if the completion map $A \to \hat{A}$ is regular.

Theorem 1. *(General Néron Desingularization, Popescu* [14], [15]*, André* [1]*, Swan* [20]*, Spivakovsky* [19]*) Let $u : A \to A'$ be a regular morphism of Noetherian rings and B an A-algebra of finite type. Then any A-morphism $v : B \to A'$ factors through a smooth A-algebra C, that is v is a composite A-morphism $B \to C \to A'$.*

Constructive General Néron Desingularization for the case when the rings A and A' are one-dimensional local rings, is given in [12], [10] and [7], the two dimensional case is partially done in [11]. The purpose of this paper is to find a constructive proof for the case when rings A and A' are of dimension m and the smooth locus of $B \to A'$ is big. We proceed using induction on the dimension of rings, with the induction step given in Proposition 3. In the Section 2 we prove a uniform General Néron Desingularization for m-dimensional local Cohen-Macaulay rings and some consequences of it. We also give an algorithm to find a uniform General Néron Desingularization using Singular.

1. Constructive Néron Desingularization

Let $u : A \to A'$ be a flat morphism of Noetherian local rings of dimension m. Suppose that the maximal ideal $\mathfrak{m}$ of A generates the maximal ideal of A', A' is Henselian and u is a regular morphism.

Let $B = A[Y]/I$, $Y = (Y_1, \ldots, Y_n)$. If $f = (f_1, \ldots, f_r)$, $r \leq n$ is a system of polynomials from I then we can define the ideal Δ_f generated by all $r \times r$-minors of the Jacobian matrix $(\partial f_i/\partial Y_j)$. After Elkik [4] let $H_{B/A}$ be the radical of the ideal $\sum_f ((f) : I)\Delta_f B$, where the sum is taken over all systems of polynomials f from I with $r \leq n$. $H_{B/A}$ defines the non smooth locus of B over A. B is *standard smooth* over A if there exists f in I as above such that $B = ((f) : I)\Delta_f B$.

The aim of this section is to give an algorithmic proof of the following theorem.

Theorem 2. *Any A-morphism $v : B \to A'$ such that $v(H_{B/A}A')$ is $\mathfrak{m}A'$-primary factors through a standard smooth A-algebra B'.*

To prove the above theorem we need the following proposition.

Proposition 3. *Let A and A' be Noetherian local rings of dimension m and $u : A \to A'$ be a regular morphism. Suppose that A' is Henselian. Let $B = A[Y]/I$, $Y = (Y_1, \ldots, Y_n)$, $f = (f_1, \ldots, f_r)$, $r \leq n$ be a system of polynomials from I as above, $(M_j)_{j\in[q]}$ some $r \times r$-minors [1] of the Jacobian matrix $(\partial f_i/\partial Y_{j'})$, $(N_j)_{j\in[q]} \in ((f) : I)$ and set $P := \sum_{j=1}^{q} N_j M_j$. Let $v : B \to A'$ be an A-morphism. Suppose that*

1. *there exist an element $d \in A$ such that $d \equiv P$ modulo I and*
2. *there exist a smooth A-algebra D and an A-morphism $\omega : D \to A'$ such that $\operatorname{Im} v \subset \operatorname{Im} \omega + d^{2e+1}A'$ and for $\bar{A} = A/(d^{2e+1})$ (defining e by $(0 :_A d^e) = (0 :_A d^{e+1})$) the map $\bar{v} = \bar{A} \otimes_A v : \bar{B} = B/d^{2e+1}B \to \bar{A}' = A'/d^{2e+1}A'$ factors through $\bar{D} = D/d^{2e+1}D$.*

Then there exist a B-algebra B' which is standard smooth over A such that v factors through B'.

Proof. Let $\delta : B \otimes_A D \cong D[Y]/ID[Y] \to A'$ be the A-morphism given by $b \otimes \lambda \to v(b)\omega(\lambda)$. First we show that δ factors through a special $B \otimes_A D$-algebra E of finite type.

Let the map $\bar{B} \to \bar{D}$ is given by $Y \to y' + d^{2e+1}D$. Thus $I(y') \equiv 0$ modulo $d^{2e+1}D$. Since $\bar{v}$ factors through $\bar{\omega}$ we see that $\bar{\omega}(y' + d^{2e+1}D) = \bar{y}$. Set $\tilde{y} = \omega(y')$. We get $y - \tilde{y} = v(Y) - \tilde{y} \in d^{2e+1}A'^n$, let us say $y - \tilde{y} = d^{e+1}\nu$ for $\nu \in d^e A'^n$.

We have $M_j = \det H_j$, where H_j is the matrix $(\partial f_i/\partial Y_{j'})_{i\in[r],j'\in[n]}$ completed with some $(n-r)$ rows from 0, 1. Since $d \equiv P$ modulo I we get $P(y') \equiv d$ modulo d^{2e+1} in D because $I(y') \equiv 0$ modulo $d^{2e+1}D$. Thus $P(y') = ds$ for some $s \in D$ with $s \equiv 1$ modulo d. Let G'_j be the adjoint matrix of H_j and $G_j = N_j G'_j$. We have $G_j H_j = H_j G_j = M_j N_j \mathrm{Id}_n$ and so

$$ds\mathrm{Id}_n = P(y')\mathrm{Id}_n = \sum_{j=1}^{q} G_j(y')H_j(y').$$

[1] We use the notation $[q] = \{1, \ldots, q\}$.

But H_j is the matrix $(\partial f_i/\partial Y_{j'})_{i\in[r],j'\in[n]}$ completed with some $(n-r)$ rows from 0, 1. Especially we obtain

$$(\partial f/\partial Y)G_j = (M_j N_j \mathrm{Id}_r|0). \tag{1}$$

Then $t_j := H_j(y')\nu \in d^e A'^n$ satisfies

$$G_j(y')t_j = M_j(y')N_j(y')\nu = ds\nu$$

and so

$$s(y-\tilde{y}) = d^e \sum_{j=1}^{q} \omega(G_j(y'))t_j.$$

Let

$$h = s(Y-y') - d^e \sum_{j=1}^{q} G_j(y')T_j, \tag{2}$$

where $T_j = (T_1,\dots,T_r,T_{j,r+1},\dots,T_{j,n})$ are new variables. The kernel of the map $\varphi : D[Y,T] \to A'$ given by $Y \to y$, $T_j \to t_j$ contains h. Since

$$s(Y-y') \equiv d^e \sum_{j=1}^{q} G_j(y')T_j \text{ modulo } h$$

and

$$f(Y) - f(y') \equiv \sum_{j'} (\partial f/\partial Y_{j'})((y')(Y_{j'} - y'_{j'})$$

modulo higher order terms in $Y_{j'} - y'_{j'}$, by Taylor's formula we see that for $p = \max_i \deg f_i$ we have

$$s^p f(Y) - s^p f(y') \equiv \sum_{j'} s^{p-1} d^e (\partial f/\partial Y_{j'})(y') \sum_{j=1}^{q} G_{jj'}(y')T_{jj'} + d^{2e}Q \tag{3}$$

modulo h where $Q \in T^2 D[T]^r$. We have $f(y') = d^{e+1}b$ for some $b \in d^e D^r$. Then

$$g_i = s^p b_i + s^p T_i + d^{e-1}Q_i, \qquad i \in [r] \tag{4}$$

is in the kernel of φ. Indeed, we have $s^p f_i = d^{e+1} g_i$ modulo h because of (3) and $P(y') = ds$. Thus $d^{e+1}\varphi(g) = d^{e+1}g(t) \in (h(y,t), f(y)) = (0)$ and $g(t) \in d^e A'^r$ and so $g(t) \in (0 :_{A'} d^{e+1}) \cap d^e A' = 0$ because $(0 :_{A'} d^e) = (0 :_{A'} d^{e+1})$, the map u being flat. Set $E = D[Y,T]/(I,g,h)$ and let $\psi : E \to A'$ be the map induced by φ. Clearly, v factors through ψ because v is the composed map $B \to B \otimes_A D \cong D[Y]/I \to E \xrightarrow{\psi} A'$.

Now we show that there exist $s', s'' \in E$ such that $E_{ss's''}$ is standard smooth over A and ψ factors through $E_{ss's''}$.

Note that the $r\times r$-minor s' of $(\partial g/\partial T)$ given by the first r-variables T is from $s^{rp} + (T) \subset 1 + (d,T)$ because $Q \in (T)^2$. Then $V = (D[Y,T]/(h,g))_{ss'}$ is smooth over D. We claim that $I \subset (h,g)D[Y,T]_{ss's''}$ for some other $s'' \in 1 + (d,T)D[Y,T]$. Indeed, we have $PI \subset (h,g)D[Y,T]_s$ and so $P(y' + s^{-1}d^e \sum_{j=1}^{q} G_j(y')T_j)I \subset (h,g)D[Y,T]_s$. Since $P(y' + s^{-1}d^e \sum_{j=1}^{q} G_j(y')T_j) \in P(y') + d^e(T)D[Y,T]_s$ we get

$P(y'+s^{-1}d^e\sum_{j=1}^q G_j(y')T_j)=ds''$ for some $s''\in 1+(T)D[Y,T]_s$. It follows that $s''I\subset ((h,g):d)D[Y,T]_{ss'}$. Thus $s''IV\subset (0:_V d)\cap d^eV=0$ because $(0:_V d)\cap d^eV=0$, and V is flat over D and so over A. This shows our claim. It follows that $I\subset (h,g)D[Y,T]_{ss's''}$. Thus $E_{ss's''}\cong V_{s''}$ is a B-algebra which is also standard smooth over D and A.

As $\omega(s)\equiv 1$ modulo d and $\psi(s'),\psi(s'')\equiv 1$ modulo (d,t), $d,t\in \mathfrak{m}A'$ we see that $\omega(s),\psi(s'),\psi(s'')$ are invertible because A' is local. Thus ψ (and so v) factors through the standard smooth A-algebra $B'=E_{ss's''}$. □

Proof of Theorem 2

We choose $\gamma_1,\gamma_2,\ldots,\gamma_m\in v(H_{B/A})A'\cap A$ such that γ_k for $k\in[m]$ is a system of parameters in A, and $\gamma_k=\sum_{i=1}^q v(b_i)z_i^{(k)}$, where $z_i^{(k)}\in A'$, $b_i\in H_{B/A}$. Set $B_0=B[Z^{(1)},\ldots,Z^{(m)}]/(f^{(1)},\ldots,f^{(m)})$, where $f^{(k)}=-\gamma_k+\sum_{i=1}^q b_iZ_i^{(k)}\in B[Z^{(k)}]$, $Z^{(k)}=(Z_1^{(k)},\ldots,Z_q^{(k)})$, and let $v_0:B_0\to A'$ be the map of B-algebras given by $Z^{(k)}\to z^{(k)}$. Changing B by B_0 we may suppose that $\gamma_k\in H_{B/A}$.

As in [10] we need the following lemma.

Lemma 4. (1) ([13, Lemma 3.4]) *Let B_1 be the symmetric algebra $S_B(I/I^2)$ of I/I^2 over*[2] *B. Then $H_{B/A}B_1\subset H_{B_1/A}$ and $(\Omega_{B_1/A})_\gamma$ is free over $(B_1)_\gamma$ for any $\gamma\in H_{B/A}$.*

(2) ([20, Proposition 4.6]) *Suppose that $(\Omega_{B/A})_\gamma$ is free over B_γ. Let $I'=(I,Y')\subset A[Y,Y']$, $Y'=(Y_1',\ldots,Y_n')$. Then $(I'/I'^2)_\gamma$ is free over B_γ.*

(3) ([16, Corollary 5.10]) *Suppose that $(I/I^2)_\gamma$ is free over B_γ. Then a power of γ is in $((g):I)\Delta_g$ for some $g=(g_1,\ldots g_r)$, $r\le n$ in I.*

Using (1) of Lemma 4 we reduce our proof to the case when $\Omega_{B_{\gamma_k}/A}$ for all $k\in[m]$ are free over B_{γ_k} respectively.

Let B_1 be given by Lemma 1. The inclusion $B\subset B_1$ has a retraction w which maps I/I^2 to zero. For the reduction we change B,v by B_1,vw.

Using (2) of Lemma 4 we may reduce to the case when $(I/I^2)_{\gamma_k}$ is free over B_{γ_k} for all $k\in[m]$.

Since $\Omega_{B_{\gamma_k}/A}$ is free over B_{γ_k} we see using Lemma 2 that changing I with $(I,Y')\subset A[Y,Y']$ we may suppose that $(I/I^2)_{\gamma_k}$ is free over B_{γ_k}.

Now using Using (3) of Lemma 4 we will reduce further to the case when a power of γ_k is in $((f^{(k)}):I)\Delta_{f^{(k)}}$ for some $f^{(k)}=(f_1^{(k)},\ldots f_{r_k}^{(k)})$, $r_k\le n$ from I.

We reduced to the case when $(I/I^2)_{\gamma_k}$ is free over B_{γ_k}. Then it is enough to use Lemma 3.

Replacing B_1 by B we may assume that a power d_k of γ_k for all $k\in[m]$ has the form $d_k\equiv P_k=\sum_{i=1}^{q_k}M_i^{(k)}L_i^{(k)}$ modulo I, for some $r_k\times r_k$ minors $M_i^{(k)}$ of $(\partial f^{(k)}/\partial Y)$ and $L_i^{(k)}\in ((f^{(k)}):I)$.

The Jacobian matrix $(\partial f^{(k)}/\partial Y)$ can be completed with $(n-r_k)$ rows from A^n obtaining a square n matrix $H_i^{(k)}$ such that $\det H_i^{(k)}=M_i^{(k)}$.

[2] Let M b e a finitely represented B-module and $B^m\xrightarrow{(a_{ij})}B^n\to M\to 0$ a presentation then $S_B(M)=B[T_1,\ldots,T_n]/J$ with $J=(\{\sum_{i=1}^n a_{ij}T_i\}_{j=1,\ldots,m})$.

This is easy using just the integers $0, 1$. Set $d = d_m$, $f = f^{(m)}$, $r = r_m$, $q = q_m$, $M_i = M_i^{(m)}$, $N_i = N_i^{(m)}$, $\bar{A} = A/d^{2e+1}$, $\bar{B} = \bar{A} \otimes_A B$, $\bar{A}' = A'/(d^{2e+1}A')$, $\bar{v} = \bar{A} \otimes_A v$. Then we have $d \equiv \sum_j M_j N_j$ modulo I. Now we will use the induction on m.

Case I: $m = 0$

If $m = 0$ then A and A' are Artinian local rings and $u : A \to A'$ is a regular morphism. Then we are done by Corollary 3.3 [13].

Case II: $m > 0$

Suppose by the induction hypothesis that we have a standard smooth $\bar{A}$-algebra $\bar{D} \cong (\bar{A}[Z]/(\bar{g}))_{\bar{h}\bar{M}}$, for $Z = (Z_1, \ldots, Z_p), \bar{g} = (\bar{g}_1, \ldots, \bar{g}_q)$ with $q \leq p$, $\bar{h} \in \bar{A}[Z]$ and $\bar{M}$ a $q \times q$-minor of $(\frac{\partial \bar{g}}{\partial Z})$, such that the map $\bar{v} : \bar{B} \to \bar{A}'$ factors through $\bar{D}$, let us say $\bar{v}$ is the composite map $\bar{B} \to \bar{D} \xrightarrow{\bar{\omega}} \bar{A}'$.

Now let $g \in A[Z]^q$ be a lifting of $\bar{g}$ and M the $q \times q$-minor of $(\frac{\partial g}{\partial Z})$ corresponding $\bar{M}$. Take $h \in A[Z]$ such that h lifts $\bar{h}$. Then $D \cong (A[Z]/(g))_{hM}$ is a standard smooth A-algebra and by the *Implicit Function Theorem* the map $\bar{\omega}$ can be lifted to $\omega : D \to A'$ since A' is Henselian. It follows that $\operatorname{Im} v \subset \operatorname{Im} \omega + d^{2e+1}A'$. Applying Proposition 3 we get a B-algebra C smooth over A such that v factors through C, $B \to C \to A'$.

2. A uniform Néron Desingularization

Let $u : A \to A'$ be a regular morphism of Cohen-Macaulay local rings of dimension m. Suppose that the maximal ideal $\mathfrak{m}$ of A generates the maximal ideal of A', A' is Henselian and A, A' have the same completions.

Let $B = A[Y]/I$, $Y = (Y_1, \ldots, Y_n)$, and for $i \in [m]$ let $f^{(i)} = (f_1^{(i)}, \ldots, f_{r_i}^{(i)})$, $r_i \leq n$ be a system of polynomials from I. Let M_i be an $r_i \times r_i$-minor of the Jacobian matrix $(\partial f^{(i)}/\partial Y)$ and $N_i \in ((f^{(i)}) : I)$, $P_i = N_i M_i$. Let $v : B \to A'/\mathfrak{m}^{3k+c}A'$ be an A-morphism for some $k, c \in \mathbb{N}$. Suppose that $v(N_1M_1, \ldots, N_mM_m)A'/\mathfrak{m}^{3k+c}A' \supset \mathfrak{m}^k A'/\mathfrak{m}^{3k+c}A'$. Let $y' \in A^n$ be a lifting of $v(Y)$ to A and let $d_i = P_i(y')$. Then $(d_1, \ldots, d_m)A'/\mathfrak{m}^{3k+c}A' \supset \mathfrak{m}^k A'/\mathfrak{m}^{3k+c}A'$. Note that $\mathfrak{m}^k \subset (d_1, \ldots, d_m)A + \mathfrak{m}^{3k+c} \subset (d_1, \ldots, d_m)A + \mathfrak{m}^{3(3k+c)+c} \subset \ldots$. Thus $\mathfrak{m}^k \subset (d_1, \ldots, d_m)A$ and it follows that $(d_1, \ldots, d_m)A' \supset \mathfrak{m}^k A'$. Since A is Cohen-Macaulay we get $d = \{d_1, \ldots, d_m\}$ regular sequence in A. Note that $(d_1, \ldots, d_m)$ is the ideal corresponding to $v(P_1, \ldots, P_m)A'$ by the isomorphism $A/\mathfrak{m}^{3k+c} \cong A'/\mathfrak{m}^{3k+c}A'$.

Theorem 5. *There exists a B-algebra C which is standard smooth over A with the following properties.*

1. *Every A-morphism $v' : B \to A'$ with $v' \equiv v$ modulo $(d_1^3, \ldots, d_m^3)A'$ factors through C.*
2. *Every A-morphism $v' : B \to A'$ with $v' \equiv v$ modulo $\mathfrak{m}^{3k}A'$ factors through C.*

(3) *There exists an A-morphism $w : C \to A'$ which makes the following diagram commutative*

$$\begin{array}{ccccc} B & \longrightarrow & C & \xrightarrow{w} & A' \\ \downarrow & & & & \downarrow \\ A/\mathfrak{m}^{3k+c} & \longrightarrow & A/\mathfrak{m}^{c} & \longrightarrow & A'/\mathfrak{m}^{c}A' \end{array}$$

Proof. Let $v' : B \to A'$ be an A-morphism with $v' \equiv v$ modulo $(d_1^3, \ldots, d_m^3)A'$. We apply induction on m.

Case I: $m = 1$

If $m = 1$ then A and A' are Noetherian local rings of dimension 1 and $u : A \to A'$ is a regular morphism. Then we are done by Theorem 2 [7], with $e = 1$.

Case II: $m > 1$

Now let $\bar{A} = A/(d_1^3, \ldots, d_{m-1}^3)$, and consider the map $\bar{v'} = \bar{A} \otimes_A v' : \bar{B} = \bar{A} \otimes_A B \to \bar{A}' = \bar{A} \otimes_A A'$. By the induction hypothesis there exists a standard smooth algebra $\bar{D} \cong (\bar{A}[Z]/(\bar{g}))_{\bar{h}\bar{M}}$, for $Z = (Z_1, \ldots, Z_p), \bar{g} = (\bar{g}_1, \ldots, \bar{g}_q)$ with $q \leq p$, $\bar{h} \in \bar{A}[Z]$ and $\bar{M}$ a $q \times q$-minor of $(\frac{\partial \bar{g}}{\partial Z})$, such that the map $\bar{v'}$ factors through $\bar{D}$, say $\bar{v'}$ is the composite map $\bar{B} \to \bar{D} \xrightarrow{\bar{\omega'}} A'$.

Now let $g \in A[Z]^q$ be a lifting of $\bar{g}$ and M the $q \times q$-minor of $(\frac{\partial g}{\partial Z})$ corresponding $\bar{M}$. Take $h \in A[Z]$ such that h lifts $\bar{h}$. Then $D \cong (A[Z]/(g))_{hM}$ is a standard smooth A-algebra and by the *Implicit Function Theorem* the map $\bar{\omega'}$ can be lifted to $\omega' : D \to A'$ since A' is Henselian. It follows that $\operatorname{Im} v' \subset \operatorname{Im} \omega' + d_m^3 A'$. Applying Proposition 3 (with $e = 1$) we get a B-algebra C standard smooth over A such that v' factors through C. This proves (1) which obviously implies (2).

Now for (3) take the map $\hat{w} : C \cong (D[Y,T]/(I, g, h))_{ss'} \to A'/\mathfrak{m}^c A'$ given by $(Y, T) \to (y', 0)$. Then the composite map $B \to C \xrightarrow{\hat{w}} A'/\mathfrak{m}^c A'$ is lifted by v. Since C is standard smooth, we may lift $\hat{w}$ to an A-morphism $w : C \to A'$ by the Implicit Function Theorem. Clearly, w makes the above diagram commutative. □

Example 6. *(Rond)* Let k be a field, $A = k[[x]]$, $x = (x_1, x_2, x_3)$, $B = A[Y]/(f)$, $Y = (Y_1, \ldots, Y_4)$, $f = Y_1 Y_2 - Y_3 Y_4$. Then $\Delta_f = H_{B/A} = (Y)$. Let $p \in \mathbf{N}$ and set $y_1' = x_1^p$, $y_2' = x_2^p$, $y_3' = x_1 x_2 - x_3^p$. Then there exists $y_4' \in A$ such that $f(y') \equiv 0$ modulo $(x)^{p^2}$. It follows that $d_1 = x_1^p$, $d_2 = x_2^p$ and $d_3 = x_3^{p^2}$ belongs to $\Delta_f(y')$ because $x_1^p x_2^p - x_3^{p^2} = y_3'(x_1^{p-1} x_2^{p-1} + x_1^{p-2} x_2^{p-2} x_3^p + \ldots x_3^{p^2})$. For $k = 2p + p^2 - 2$ we have $(x)^k \subset (d_1, d_2, d_3) \subset H_{B/A}(y')$. If $f(y') \equiv 0$ modulo $(x)^{3k+p+1}$ then by Theorem 5 (3) we could get $y \in A^4$ such that $f(y) = 0$ and $y \equiv y'$ modulo $(x)^{p+1}$. But this is not the case, since $f(y') \equiv 0$ modulo $(x)^{p^2}$ and we cannot apply the quoted theorem. Thus it is not a surprise that [18, Remark 4.7] says that there exist no $y \in A^4$ such that $f(y) = 0$ and $y \equiv y'$ modulo $(x)^{p+1}$.

Corollary 7. *(Elkik) Let $(A, \mathfrak{m})$ be a Cohen-Macaulay Henselian local ring of dimension m and $B = A[Y]/I$, $Y = (Y_1, \ldots, Y_n)$ an A-algbra of finite type. Then for every $k \in \mathbf{N}$ there exist two integers $m_0, p \in \mathbf{N}$ such that if $y' \in A^n$ satisfies $m^k \subset H_{B/A}(y')$ and $I(y') \equiv 0$ modulo $\mathfrak{m}^m$ for some $m > m_0$ then there exists $y \in A^n$ such that $I(y) = 0$ and $y \equiv y'$ modulo $\mathfrak{m}^{m-p}$.*

Proof. Suppose that $A' = A$. In the notation of Theorem 5 given k set $m_0 = p = 3k$ and suppose that $y' \in A^n$ satisfies $m^k \subset H_{B/A}(y')$ and $I(y') \equiv 0$ modulo $\mathfrak{m}^m$ for some $m > m_0$. Let $v : B \to A/\mathfrak{m}^m$ be given by $Y \to y'$. Set $c = m - p$. By Theorem 5 there exists a smooth A-algebra C and a map $w : C \to A$ which makes the above digram commutative. Let y be the image of Y by the composite map $B \to C \xrightarrow{w} A$. Then $I(y) = 0$ and $y \equiv y'$ modulo $\mathfrak{m}^c = \mathfrak{m}^{m-p}$. □

Corollary 8. *With the assumptions and notation of the Theorem 5, let $\rho : B \to C$ be the structural algebra map. Then ρ induces bijections ρ^* given by $\rho^*(w) = w \circ \rho$, between*

(1) $\{w \in \mathrm{Hom}_A(C, A') : w \circ \rho \equiv v \text{ modulo } (d_1^3, \ldots, d_m^3)A'\}$ *and* $\{v' \in \mathrm{Hom}_A(B, A') : v' \equiv v \text{ modulo } (d_1^3, \ldots, d_m^3)A'\}$
(2) $\{w \in \mathrm{Hom}_A(C, A') : w \circ \rho \equiv v \text{ modulo } \mathfrak{m}^{3k}A'\}$ *and* $\{v' \in \mathrm{Hom}_A(B, A') : v' \equiv v \text{ modulo } \mathfrak{m}^{3k}A'\}$

Proof. We will use induction on the dimension of A.

(1) **Case I:** $m = 1$

If $m = 1$ then A and A' are Cohen-Macaulay local rings of dimension 1. Then we are done by Corollary 8 [7].

(2) **Case II:** $m > 1$

By Theorem 5, (1), ρ^* is surjective. Let Now let $\bar{A} = A/(d_1^3, \ldots, d_{m-1}^3)$, and the map $\bar{v'} = \bar{A} \otimes_A v' : \bar{B} = \bar{A} \otimes_A B \to \bar{A}' = \bar{A} \otimes_A A'$. By the induction hypothesis there exists a standard smooth algebra $\bar{D}$ such that the maps $\bar{w}$ and $\bar{w}'$ restricted to $\bar{D}$ coincide. This implies that $w \mid_D$ and $w' \mid_D$ lift the same map $\bar{w} \mid_{\bar{D}}$. Thus $w \mid_D = w' \mid_D$ by uniqueness in the Implicit Function Theorem.

By construction $C = E_{ss'}$, $E = D[Y, T]/(I, g, h)$ and $H_m(y')(w(Y) - w'(Y)) \equiv d_m^2(w(T) - w'(T))$ modulo h. Thus $d_m^2(w(T) - w'(T)) = 0$ and so $w|_E = w'|_E$ because d_m is regular in A' since d_m is regular in A and u is flat. It follows that $w = w'$

(3) Apply Theorem 5 (2) for the surjectivity. The injectivity follows from above. □

Corollary 9. *With the assumptions and notation of the above Corollary, the following statements hold:*

(1) *If there exists an A-morphism $\tilde{v} : B \to A'$ with $\tilde{v} \equiv v$ modulo $(d_1^3, \ldots, d_m^3)A'$, then there exists a unique A-morphism $\tilde{w} : C \to A'$ such that $\tilde{w} \circ \rho = \tilde{v}$.*
(2) *If there exists an A-morphism $\tilde{v} : B \to A'$ with $\tilde{v} \equiv v$ modulo $\mathfrak{m}^{3k}A'$, then there exists a unique A-morphism $\tilde{w} : C \to A'$ such that $\tilde{w} \circ \rho = \tilde{v}$.*

For the proof take $\tilde{w} = \rho^{*-1}(\tilde{v})$, where ρ^* is defined in the Corollary 8.
By construction, C has the form $(D[T]/(g))_{Mh}$, where $M = \det(\partial g_i/\partial T_j)_{i,j\in[r]}$ and $h = s' \in A[T]$ satisfies $\tilde{w}(h) \notin \mathfrak{m}A'$.

Lemma 10. *There exist canonical bijections*

(1) $(d_1^3,\dots,d_m^3)A'^{n-r} \to \{w' \in \mathrm{Hom}_A(C,A') : w' \equiv \tilde{w}$ *modulo* $(d_1^3,\dots,d_m^3)A'\}$.
(2) $\mathfrak{m}^{3k}A'^{n-r} \to \{w' \in \mathrm{Hom}_A(C,A') : w' \equiv \tilde{w}$ *modulo* $\mathfrak{m}^{3k}A'\}$.

Proof. Note that $\{w' \in \mathrm{Hom}_A(C,A') : w' \equiv \tilde{w} \text{ modulo } (d_1^3,\dots,d_m^3)A'\}$ is in bijection with the set of all $t \in A'^n$ such that $g(t) = 0$ and $t \equiv \tilde{w}(T)$ modulo $(d_1^3,\dots,d_m^3)A'^n$.

Set $V = (T_1,\dots,T_r)$, $Z = (T_{r+1},\dots,T_n)$. Thus $g(U, w'(Z)) = 0$ has a unique solution (namely $U = w'(V)$) in $\tilde{w}(Z) + (d_1^3,\dots,d_m^3)A'^{n-r}$ by the Implicit Function Theorem. Consequently, $w'(V)$ is uniquely defined by $w'(Z)$, that is by the restriction $w'|_{A[Z]}$.

Therefore, $\{w' \in \mathrm{Hom}_A(C,A') : w' \equiv \tilde{w} \text{ modulo } (d_1^3,\dots,d_m^3)A'^n\}$ is in bijection with $\{w'' \in \mathrm{Hom}_A(A[Z],A') : w'' \equiv \tilde{w}|_{A[Z]} \text{ modulo } (d_1^3,\dots,d_m^3)A'^{n-r}\}$, the latter set being in bijection with $\tilde{w}(Z) + (d_1^3,\dots,d_m^3)A'^{n-r}$, that is with $(d_1^3,\dots,d_m^3)A'^{n-r}$. The proof of (2) goes similarly. □

Theorem 11. *With the assumptions and notation of Corollary 8 there exist canonical bijections*

(1)
$$(d_1^3,\dots,d_m^3)A'^{n-r} \to \{v' \in \mathrm{Hom}_A(B,A') : v' \equiv v \textit{ modulo } (d_1^3,\dots,d_m^3)A'\}.$$
(2)
$$\mathfrak{m}^{3k}A'^{n-r} \to \{v' \in \mathrm{Hom}_A(B,A') : v' \equiv v \textit{ modulo } \mathfrak{m}^{3k}A'\}.$$

For the proof apply Corollary 8 and the above lemma.

3. Algorithms

In this section we present the algorithms corresponding to the results of Sections 1 and 2. We will use in our algorithm for uniform desingularization the following algorithm for the one dimensional case (cf. [7]):

Algorithm 1 UniformNeronDesingularizationDim1

Input: A, B, v, k, f, N given by the following data. $A = K[x]_{(x)}/J$, $J = (h_1, \dots, h_{p'})$, $h_i \in K[x]$, $x = (x_1, \dots, x_t)$, $\dim(A) = 1$, K a field. $B = A[Y]/I$, $I = (g_1, \dots, g_l)$, $g_i \in K[x, Y]$, $Y = (Y_1, \dots, Y_n)$, integer k, c, $f = (f_1, \dots, f_r), f_i \in I$, $v : B \to A/m_A^c$ defined by $y' \in K[x]^n$, $N \in (f_1, \dots, f_r) : I$.

Output: (D, π) given by the following data. $D = (A[Z]/(g))_{hM}$ a standard smooth algebra, $Z = (Z_1, \dots, Z_p)$, $g = (g_1, \dots, g_q)$, $q \le p$, $h \in A[Z]$, M a $q \times q$ minor of $(\partial g/\partial Z)$, $\pi : B \to D$ given by $\pi(Y)$ and factorizing v or the message "$y', N, f_1, \dots, f_r$ are not well chosen."

Compute $M = \det((\partial f_i/\partial Y_j)_{i,j \in [r]})$, $P := NM$ and $d := P(y')$
$f := (f_1, \dots, f_r)$
Compute e such that $(0 :_A d^e) = (0 :_A d^{e+1})$
if $I(y') \not\subseteq (x)^{(2e+1)k} + J$ or $(x)^k \not\subseteq (d) + J$ **then**
  return "y', N, $(f_1, \dots, f_r)$ are not well chosen"
end if
Complete $(\partial f_i/\partial Y_j)_{i \le r}$ by $(0|(\mathrm{Id}_{n-r}))$ to obtain a square matrix H
Compute G' the adjoint matrix of H and $G := NG'$
$h = Y - y' - d^e G(y')T,\ T = (T_1, \dots, T_n)$
Write $f(Y) - f(y') = \sum_j d^e \partial f/\partial Y_j(y') G_j(y')T + d^{2e}Q$
Write $f(y') = d^{e+1}a$
for $i = 1$ to r **do**
  $g_i = a_i + T_i + d^{e-1}Q_i$
end for
$E := A[Y, T]/(I, g, h)$
Compute s the $r \times r$ minor defined by the first r columns of $(\partial g/\partial T)$
Write $P(y' + d^e G(y')T) = ds'$
return $E_{ss'}$.

Next we present the algorithm for uniform desingularization for the higher dimensional case.

Algorithm 2 UniformNeronDesingularization

Input: A, B, v, k, f, N given by the following data. $A = K[x]_{(x)}/J$, $J = (h_1, \dots, h_{p'})$, $h_i \in K[x]$, $x = (x_1, \dots, x_t)$, $\dim(A) = m$, K a field. $B = A[Y]/I$, $I = (g_1, \dots, g_l)$, $g_i \in K[x, Y]$, $Y = (Y_1, \dots, Y_n)$, integer k, c, $f = (f_1, \dots, f_r), f_i \in I$, $v : B \to A/\mathfrak{m}_A^c$ defined by $\bar{y} \in K[x]^n$, for $i \in [m]$, $f^{(i)} = (f_1^{(i)}, \dots, f_{r_i}^{(i)})$, $r_i \le n$, $f_j^{(i)} \in I$, $N_i \in ((f^{(i)}) : I)$, M_i an $r_i \times r_i$ minor of $(\partial f^{(i)}/\partial Y)$.

Output: D, π given by the following data. $D = (A[Z]/(g))_{hM}$ a standard smooth, $Z = (Z_1, \dots, Z_p)$, $g = (g_1, \dots, g_q)$, $q \le p$, $h \in A[Z]$, M a $q \times q$ minor of $(\partial g/\partial Z)$, $\pi : B \to D$ given by $\pi(Y) = y'$ and factorizing v or the message "$y', f^{(i)}, N_i$ are not well chosen."

```
for i ∈ [m] do
    P_i := M_i N_i; d_i := P_i(ȳ)
end for
if I(ȳ) ⊈ (x)^{3k} + J or (x)^k ⊈ (d_1, …, d_m) + J then
    return "ȳ, f^{(i)}, N_i are not well chosen."
end if
if m = 1 then
    return UniformNeronDesingularizationDim1(A, B, v, k, f^{(1)}, N_1)
end if
Ā := A/(d_1^3, …, d_{m-1}^3), v̄ := Ā ⊗_A v, B̄ := Ā ⊗_A B
(D̄, π̄) := UniformNeronDesingularization (Ā, B̄, v̄, k, f^{(m)}, N_m, M_m)
    D̄ = (Ā[Z]/(ḡ))_{h̄M̄}, Z = (Z_1, …, Z_p), g = (g_1, …, g_q), g_i ∈ K[x, Z]
    ḡ = g mod (d_1^3, …, d_{m-1}^3), h ∈ k[x, Z], h̄ = h mod (d_1^3, …, d_{m-1}^3), M a q × q
    minor of (∂g/∂Z), M̄ = M mod (d_1^3, …, d_{m-1}^3), π̄(Y) = y'
D := (A[Z]/(g))_{hM}
Complete the Jacobian matrix associated to f^{(m)} by rows of 0 and 1 to obtain
    a square matrix H_m with det(H_m) = M_m
Compute G'_m the adjoint matrix of H_m and G_m := N_m G'_m
P := M_m N_m, d := P(y'), f = f^{(m)}, r = r_m
Write P(y') = ds for some s ∈ D, s ≡ 1 mod d
h := s(Y − y') − d ∑_{j=1}^{r} G_j(y')T_j, T_j = (T_1, …, T_r, T_{j,r+1}, …, T_{j,n})
p = max{deg(f_i^{(m)})}
Write s^p(f(Y) − f(y')) ≡ ∑_{j'} s^{p−1} d(∂f/∂Y_{j'})(y') ∑_{j=1}^{q} G_{jj'}(y')T_{jj'} + d^2 Q mod h.
Write f(y') = d^2 b, b ∈ dD^r.
for i ∈ [r] do
    g_i := s^p b_i + s^p T_i + Q_i.
end for
E := D[Y, T]/(I, g, h).
Compute s' the r × r-minor of (∂g/∂T) given by the first r variables of T.
Compute s'' such that P(y' + s^{−1} d ∑_{j=1}^{q} G_j(y')T_j) = ds''.
Define π : B → (D[Y, T]/(I, g, h)_{ss's''} by π(Y_i) = y'_i.
return ((D[Y, T]/(I, g, h)_{ss's''}, π).
```

Algorithm 3 NeronDesingularization

Input: A, B, v, N given by the following data: $A = k[x]_{(x)}/J$, $J = (h_1, \dots, h_q) \subset k[x]$, $x = (x_1, \dots, x_t)$, k a field, $k' = Q(k[U]/\bar{J})$, $\bar{J} = (a_1, \dots, a_r) \subset k[U]$, $U = (U_1, \dots, U_{t'})$ separable over k, $B = A[Y]/I$, $I = (g_1, \dots, g_l) \subset k[x, Y]$, $Y = (Y_1, \dots, Y_n)$, $v : B \to A' \subset K[[x]]/JK[[x]]$ an A-morphism, given by $\bar{y} = (\bar{y}_1, \dots, \bar{y}_n) \in k[x, U]^n$ approximations mod $(x)^N$ of $v(Y_i)$, $K \supset k'$ a field.

Output: (E, π) given by the following data: $E = (A[Z]/L)_{hM}$ standard smooth, $Z = (Z_1, \dots, Z_p)$, $L = (b_1, \dots, b_{q'}) \subset k[x, Z]$, $h \in k[x, Z]$, M $q \times q$-minor of $(\partial b_i/\partial Z_j)$, $\pi : B \to E$ an A-morphism given by $\pi(Y_1) = y'_1, \dots, \pi(Y_n) = y'_n$ factorizing v, i.e. there exists $\omega : E \to A'$ with $\omega\pi = v$.

Compute $w := (a_{i_1}, \dots, a_{i_p})$, ρ a $p \times p$-minor of $(\partial a_{i_\nu}/\partial U_j)$ such that $\rho \notin \bar{J}$. Compute $\tau \in (w) : \bar{J}$ such that $k[U]_{\rho\tau}/(a_1, \dots, a_r) = k[U]_{\rho\tau}/(w)$, $D := A[U]_{\rho\tau}/(w)$.
if If $\dim(A) = 0$ **then**
    compute $f = (f_1, \dots, f_r)$ in I, $N \in (f) : I$ and M an $r \times r$-minor of $(\partial f_i/\partial Y_j)$ such that B_{NM} is standard smooth, return $((D[Y]/I)_{NM}, Y)$.
end if
Compute $H_{B/A} = (b_1, .., b_q)$ and $\gamma_1, .., \gamma_m \in H_{B/A}(\bar{y})$, system of parameters in A.
for $j \in [m]$ **do**
    write $\gamma_j = \sum_{i=1}^q b_i(\bar{y})\bar{y}_{n+i+(j-1)q}$ mod $(\gamma_1^2, ..., \gamma_m^2)$, $\bar{y}_j \in k[x, U]$.
end for
for $j \in [m]$ **do**
    $g_{l+j} := -\gamma_j + \sum_{i=1}^q b_i Y_{n+i+(j-1)q}$;
end for
$Y := (Y_1, \dots, Y_{n+mq})$; $\bar{y} = (\bar{y}_1 \dots, \bar{y}_{n+mq})$; $I := (g_1, \dots, g_{l+m})$; $l := l+m$; $n := n+mq$, $B := A[Y]/I$, $\gamma := \gamma_m$
$B = S_B(I/I^2)$, v trivially extended. Write $B := A[Y]/I$, $Y = (Y_1, \dots, Y_n)$; $Y := (Y, Z)$; $I := (I, Z)$, $B := A[Y]/I$, $Z = (Z_1, \dots, Z_n)$, v trivially extended.
Compute $f = (f_1, \dots, f_r)$ such that a power d of γ is in $((f) : I)\Delta_f$.
Compute e such that $(0 : d^e) = (0 : d^{e+1})$ and $p := \max_i\{\deg f_i\}$.
Choose $r \times r$-minors M_i of $(\partial f/\partial Y)$ and $N_i \in ((f) : I)$ such that for $P := \sum M_i N_i$ we have $d \equiv P$ mod I.
Complete the Jacobian submatrices of $(\partial f/\partial Y)$ corresponding to M_i by $n - r$ rows of 0 and 1 to obtain square matrices H_i with $\det H_i = M_i$.
for $j \in [m]$ **do**
    compute G'_j the adjoint matrix of H_j and $G_j := N_j G'_j$
end for
$\bar{A} := A/(d^{2e+1})$; $\bar{B} := \bar{A} \otimes_A B$, $\bar{v} := \bar{A} \otimes v$.
$(\bar{E}, \bar{\pi})$ =NeronDesingularization$(\bar{A}, \bar{B}, \bar{v}, N)$, $\bar{E} = (\bar{A}[Z_1, \dots, Z_p]/\bar{L})_{hM}$ standard smooth, $L = (b_1, \dots, b_q) \subset k[x, Z]$, $\bar{L} = L$ mod d^{2e+1}, $h \in k[x, Z]$, M $q \times q$-minor of $(\partial b/\partial Z)$, $\bar{\pi} : \bar{B} \to \bar{E}$ factorization of $\bar{v}$ given by y' from $k[x, Z]^n$.
$D := A[Z]_{hM}/L$, write $P(y') = ds$, $f(y') = d^{e+1}b$, $b \in d^e D^r$, $s \in D$, $d | s - 1$.
$h := s(Y - y') - d^e \sum_{i=1}^q G_j(y')T_j$, $T_j = (T_1, \dots, T_r, T_{j,r+1}, \dots, T_{j,n})$.
Write $s^p(f(Y) - f(y')) \equiv \sum_{j'} s^{p-1} d^e (\partial f/\partial Y_{j'})(y') \sum_j G_{jj'}(y')T_{jj'} + d^e Q$ mod h.
for $i \in [r]$ **do**
    $g_i := s^p b_i + s^p T_i + d^{e-1} Q_i$.
end for
$E := D[Y, T]/(I, g, h)$.
Compute s' the $r \times r$-minor of $(\partial g/\partial T)$ given by the first r variables of T.
Compute s'' such that $P(y' + s^{-1} d^e \sum_{j=1}^q G_j(y')T_j) = ds''$.
Define $\pi : B \to (D[Y, T]/(I, g, h)_{ss's''}$ by $\pi(Y_i) = y'_i$.
return $((D[Y, T]/(I, g, h)_{ss's''}, \pi)$.

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