Chapter 8

Maxwell Field

All electrodynamic processes are described by the Maxwell equations. Surprisingly they represent the equations of motion of a first-quantized theory, although the Planck constant $\hbar$ does not appear explicitly. This apparent contradiction is resolved by the following consideration. If the quanta of the Maxwell field, i.e. the photons, had a finite rest mass M, then it would appear due to dimensional reasons together with spatio-temporal derivatives as a mass term in the equations of motion in form of the inverse Compton wave length (7.21). Thus, performing the limit of a vanishing rest mass, i.e. $M \to 0$, also the Planck constant $\hbar$ vanishes automatically from the respective equations of motion.

In this chapter we first review the relativistic covariant formulation of this first-quantized Maxwell theory. Afterwards, we invoke the canonical field quantization formalism and work out systematically the second quantization of the Maxwell theory. In particular, we have to deal with the intricate consequences of the underlying local gauge symmetry, which is due to the vanishing rest mass of the quanta of the Maxwell field. In this way we determine step by step the respective properties of a single photon as, for instance, its energy, its momentum, and its spin. Finally, we discuss the photon propagator, which represents an important building block in the Feynman diagrams of quantum electrodynamics describing the interaction between light and matter.

8.1 Maxwell Equations

Forces of an electromagnetic field upon electric charges, which are at rest or move, are mediated by both the electric field strength $\mathbf{E}$ and the magnetic induction $\mathbf{B}$. Physically both vector fields are generated by the charge density ρ and the current density $\mathbf{j}$. Mathematically they are determined by partial differential equations, which were first formulated by James Clerk Maxwell. The general structure of the Maxwell equations is prescribed by the Helmholtz vector decomposition theorem, which states that any vector field is uniquely determined by its respective divergence and rotation in combination with appropriate boundary conditions. With

this the electric field strength $\mathbf{E}$ follows from

$$\begin{aligned} \operatorname{div}\mathbf{E} &= \frac{\rho}{\varepsilon_0}\,, && (8.1)\\ \operatorname{rot}\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t}\,, && (8.2) \end{aligned}$$

whereas the magnetic induction $\mathbf{B}$ is defined by

$$\begin{aligned} \operatorname{div}\mathbf{B} &= 0\,, && (8.3)\\ \operatorname{rot}\mathbf{B} &= \mu_0\,\mathbf{j} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}\,. && (8.4) \end{aligned}$$

Here the vacuum dielectric constant ϵ_0, the vacuum permeability μ_0, and the vacuum light velocity c are related via

$$c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}\,. \tag{8.5}$$

We remark that (8.1), (8.4) and (8.2), (8.3) are denoted as the inhomogeneous and homogeneous Maxwell equations, respectively. Furthermore, we read off from the inhomogeneous Maxwell equations (8.1) and (8.4) the consistency equation that charge density ρ and current density $\mathbf{j}$ are not independent from each other but must fulfill the continuity equation

$$\frac{\partial\rho}{\partial t} + \operatorname{div}\mathbf{j} = 0\,, \tag{8.6}$$

which corresponds to the charge conservation similar to the discussion in (7.39)–(7.41). Note that we formulate the Maxwell equations (8.1)–(8.4) according to the International System of Units, which is abbreviated by SI from the French Système international d'unités. Instead, in quantum field theory quite often the rational Lorentz-Heaviside unit system is used, where one assumes $\varepsilon_0 = \mu_0 = c = 1$ in order to simplify the notation. But we stick consistently to the SI unit system, although this might be considered to be more cumbersome, as this has the advantage that at each stage of the calculation one obtains results, which are, at least in principle, directly accessible in an experiment.

8.2 Local Gauge Symmetry

From the homogeneous Maxwell equations (8.2) and (8.3) we conclude straight-forwardly that both the electric field strength $\mathbf{E}$ and the magnetic induction $\mathbf{B}$ follow from differentiation of a scalar field φ and a vector potential $\mathbf{A}$:

$$\begin{aligned} \mathbf{B} &= \operatorname{rot}\mathbf{A}\,, && (8.7)\\ \mathbf{E} &= -\operatorname{grad}\varphi - \frac{\partial\mathbf{A}}{\partial t}\,. && (8.8) \end{aligned}$$

From the inhomogeneous Maxwell equations (8.1) and (8.4) as well as from (8.7) and (8.8) we then determine coupled partial differential equations for the scalar field φ and the vector potential $\mathbf{A}$:

$$-\Delta\,\varphi - \frac{\partial}{\partial t}\,\mathrm{div}\,\mathbf{A} = \frac{\rho}{\varepsilon_0}\,, \tag{8.9}$$

$$\frac{1}{c^2}\,\frac{\partial^2\mathbf{A}}{\partial t^2} - \Delta\mathbf{A} + \mathrm{grad}\left(\frac{1}{c^2}\,\frac{\partial\varphi}{\partial t} + \mathrm{div}\,\mathbf{A}\right) = \mu_0\,\mathbf{j}\,. \tag{8.10}$$

The equations (8.7)–(8.10) turn out to be invariant with respect to a local gauge transformation with an arbitrary gauge function Λ:

$$\varphi' = \varphi + \frac{\partial\Lambda}{\partial t} \tag{8.11}$$

$$\mathbf{A}' = \mathbf{A} - \mathrm{grad}\,\Lambda\,. \tag{8.12}$$

Thus, a local gauge transformation does not have any physical consequences, but it changes the mathematical description of the electromagnetic field. For instance, choosing a particular gauge allows to decouple the coupled equations of motion (8.9) and (8.10). In the following we briefly discuss the two most prominent gauges.

The *Coulomb gauge* assumes that the longitudinal part of the vector potential $\mathbf{A}$ vanishes, i.e.

$$\mathrm{div}\,\mathbf{A} = 0\,. \tag{8.13}$$

With this (8.9) and (8.10) reduce to

$$\Delta\,\varphi = -\frac{\rho}{\varepsilon_0}\,, \tag{8.14}$$

$$\frac{1}{c^2}\,\frac{\partial^2\mathbf{A}}{\partial t^2} - \Delta\,\mathbf{A} = \mu_0\,\mathbf{j} - \frac{1}{c^2}\,\frac{\partial}{\partial t}\,\mathrm{grad}\varphi\,. \tag{8.15}$$

As the scalar potential $\varphi(\mathbf{x},t)$ obeys the Poisson equation (8.14), it is determined at each time instant t by the corresponding value of the charge density $\rho(\mathbf{x},t)$ according to

$$\varphi(\mathbf{x},t) = \int d^3x'\,\frac{\rho(\mathbf{x}',t)}{4\pi\varepsilon_0|\mathbf{x}-\mathbf{x}'|}\,. \tag{8.16}$$

Due to (8.13) and (8.16) we conclude that from the original four fields φ and $\mathbf{A}$ only two of them represent dynamical degrees of freedom. As a consequence, the quantization of the electromagnetic field thus yields later on two types of photons. The advantage of the Coulomb gauge is that the remaining two dynamical degrees of freedom of the electromagnetic field can be physically identified with the two transversal degrees of freedom of the vector potential $\mathbf{A}$. The disadvantage of the Coulomb gauge is that it is not manifestly Lorentz invariant. Thus, the Coulomb gauge is only valid in a particular inertial system.

The *Lorentz gauge* is defined via

$$\frac{1}{c^2}\,\frac{\partial\varphi}{\partial t} + \mathrm{div}\,\mathbf{A} = 0\,. \tag{8.17}$$

With this the coupled equations of motions (8.9) and (8.10) yield uncoupled wave equations:

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \Delta\,\varphi = \frac{\rho}{\varepsilon_0}\,, \tag{8.18}$$

$$\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \Delta\,\mathbf{A} = \mu_0\,\mathbf{j}\,. \tag{8.19}$$

The advantage is here that the Lorentz gauge (8.17) as well as the decoupled equations of motion (8.18), (8.19) are Lorentz invariant. On the other hand, the quantization of the electromagnetic field on the basis of the Lorentz gauge, as worked out by Suraj Gupta and Konrad Bleuler, turns out to have an essential disadvantage. Namely, apart from the two physical transversal degrees of freedom also an unphysical longitudinal degree of freedom of the electromagnetic field emerges, which has to be eliminated afterwards with some effort.

8.3 Field Strength Tensors

In view of a manifestly Lorentz invariant formulation of the Maxwell theory both the electric field strength $\mathbf{E}$ and the magnetic induction $\mathbf{B}$ are considered as elements of an anti-symmetric 4×4 matrix F, which is called the electromagnetic field strength tensor. Its contravariant components read

$$(F^{\mu\nu}) = \Big(F^{\mu\nu}(\mathbf{E},\mathbf{B})\Big) = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}, \tag{8.20}$$

which fulfill, indeed, the anti-symmetry condition:

$$F^{\mu\nu} = -F^{\nu\mu}\,. \tag{8.21}$$

Its corresponding covariant components

$$F_{\mu\nu} = g_{\mu\lambda}g_{\nu\kappa}\,F^{\lambda\kappa} \tag{8.22}$$

are given by

$$(F_{\mu\nu}) = \Big(F^{\mu\nu}(-\mathbf{E},\mathbf{B})\Big) = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}. \tag{8.23}$$

Furthermore, it turns out to be useful to introduce in addition the dual electromagnetic field strength tensor ${}^*F$ by contracting the electromagnetic field strength tensor F with the totally anti-symmetric unity tensor ϵ, which was already used in (6.140):

$${}^*F^{\mu\nu} = \frac{1}{2}\,\epsilon^{\mu\nu\lambda\kappa}\,F_{\lambda\kappa}\,. \tag{8.24}$$

Thus, its contravariant components turn out to be

$$({}^*F^{\mu\nu}) = \Big(F^{\mu\nu}(c\mathbf{B}, -\mathbf{E}/c)\Big) = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{pmatrix} \tag{8.25}$$

and the covariant components

$${}^*F_{\mu\nu} = g_{\mu\lambda} g_{\nu\kappa}\, {}^*F^{\lambda\kappa} \tag{8.26}$$

result in

$$({}^*F_{\mu\nu}) = \Big(F^{\mu\nu}(-c\mathbf{B}, -\mathbf{E}/c)\Big) = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z/c & -E_y/c \\ -B_y & -E_z/c & 0 & E_x/c \\ -B_z & E_y/c & -E_x/c & 0 \end{pmatrix} . \tag{8.27}$$

With these definitions we can now concisely summarize the homogeneous Maxwell equations (8.2), (8.3) with the help of the dual electromagnetic field strength tensor ${}^*F$

$$\partial_\mu {}^*F^{\mu\nu} = 0\,, \tag{8.28}$$

whereas, correspondingly, the inhomogeneous Maxwell equations (8.1), (8.4) can be united with the help of the electrodynamic field strength tensor F:

$$\partial_\mu F^{\mu\nu} = \mu_0\, j^\nu\,. \tag{8.29}$$

Here the contravariant current density four-vector j^λ consists of both the charge density ρ in the temporal component and the current density $\mathbf{j}$ in the spatial components:

$$\big(j^\lambda\big) = (c\rho, \mathbf{j})\ . \tag{8.30}$$

Indeed, taking into account (6.99), an explicit calculation reproduces the homogeneous Maxwell equations

$$\begin{aligned} (\partial_\mu {}^*F^{\mu\nu}) &= \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{pmatrix} \\ &= \left(\operatorname{div}\mathbf{B}, -\frac{1}{c}\operatorname{rot}\mathbf{E} - \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}\right) = (0, \mathbf{0}) \end{aligned} \tag{8.31}$$

as well as also the inhomogeneous Maxwell equations

$$\begin{aligned} (\partial_\mu F^{\mu\nu}) &= \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} \\ &= \left(\frac{1}{c}\operatorname{div}\mathbf{E}, \operatorname{rot}\mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\right) = \mu_0(c\rho, \mathbf{j})\,. \end{aligned} \tag{8.32}$$

Evaluating the four-divergence of (8.29) one obtains due to the anti-symmetry (8.21) a consistency condition, which is the continuity equation for the contravariant current density

$$\partial_\nu \partial_\mu F^{\mu\nu} = \mu_0 \, \partial_\nu j^\nu \qquad \Longrightarrow \qquad \partial_\nu j^\nu = 0 \, . \tag{8.33}$$

Note that (8.33) represents the manifest Lorentz invariant formulation of (8.6).

8.4 Four-Vector Potential

We now combine both the scalar potential φ and the vector potential $\mathbf{A}$ to the contravariant four-vector potential

$$\left(A^\lambda\right) = \left(\frac{\varphi}{c}, \mathbf{A}\right) \, . \tag{8.34}$$

With this the relations (8.7) and (8.8) between the electric field strength $\mathbf{E}$ and the magnetic induction $\mathbf{B}$ as well as the scalar potential φ and the vector potential $\mathbf{A}$ are combined into one single relation between the electromagnetic field strength tensor $F^{\mu\nu}$ and the four-vector potential A^λ:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \, . \tag{8.35}$$

Here the contravariant nabla four-vector is defined via

$$\left(\partial^\mu\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right) \, . \tag{8.36}$$

For instance, we obtain from (8.34)–(8.36):

$$F^{01} = \partial^0 A^1 - \partial^1 A^0 = \frac{1}{c}\frac{\partial A_x}{\partial t} + \frac{1}{c}\frac{\partial \varphi}{\partial x} = -\frac{1}{c} E_x \, , \tag{8.37}$$

$$F^{12} = \partial^1 A^2 - \partial^2 A^1 = \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} = -B_z \, . \tag{8.38}$$

We remark that the definitions (8.24) and (8.35) have the consequence that the homogeneous Maxwell equations (8.28) are automatically fulfilled:

$$\partial_\mu {}^*F^{\mu\nu} = \frac{1}{2}\, \epsilon^{\mu\nu\lambda\kappa}\, \partial_\mu F_{\lambda\kappa} = \frac{1}{2}\, \epsilon^{\mu\nu\lambda\kappa}\, \left(\partial_\mu \partial_\lambda A_\kappa - \partial_\mu \partial_\kappa A_\lambda\right) = 0 \, . \tag{8.39}$$

Note that we have used here the anti-symmetry of the ϵ tensor and that we have assumed that the covariant four-vector potential fulfills the theorem of Schwarz, i.e. partial derivatives commute:

$$\left(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu\right) A_\kappa = 0 \, . \tag{8.40}$$

Furthermore, due to the definition (8.35) the inhomogeneous Maxwell equations (8.29) go over into the manifest Lorentz invariant formulation of the coupled equations of motion (8.9) and (8.10):

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = \mu_0 \, j^\nu \, . \tag{8.41}$$

And, finally, the manifest Lorentz invariant formulation of the local gauge transformation (8.11), (8.12) reads

$$A'^{\mu} = A^{\mu} + \partial^{\mu}\Lambda\,. \tag{8.42}$$

Due to such local gauge transformations (8.42) the electromagnetic field strength tensor F defined via (8.35) does not change

$$F'^{\mu\nu} = \partial^{\mu}A'^{\nu} - \partial^{\nu}A'^{\mu} = \partial^{\mu}A^{\nu} + \partial^{\mu}\partial^{\nu}\Lambda - \partial^{\nu}A^{\mu} - \partial_{\nu}\partial_{\mu}\Lambda = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu} = F^{\mu\nu}\,, \tag{8.43}$$

provided that the gauge function Λ also fulfills the theorem of Schwarz:

$$\left(\partial_{\mu}\partial_{\nu} - \partial_{\nu}\partial_{\mu}\right)\Lambda = 0\,. \tag{8.44}$$

Furthermore, we conclude from (8.24) and (8.43) that then also the dual electromagnetic field strength tensor ${}^{*}F$ is gauge invariant:

$${}^{*}F'^{\mu\nu} = {}^{*}F^{\mu\nu}\,. \tag{8.45}$$

Thus, finally, we conclude that the local gauge transformation (8.42) leaves both the homogeneous and the inhomogeneous Maxwell equations (8.28) and (8.29) invariant.

8.5 Euler-Lagrange Equations

Now we set up a covariant variational principle, whose Euler-Lagrange equations are equivalent to the Maxwell equations. According to (8.35) the electromagnetic field strength tensor is completely determined from the knowledge of the four-vector potential. Therefore, we take here the point of view that the primary dynamical degree of freedom is provided by the four-vector potential. As the homogeneous Maxwell equations (8.28) are already automatically fulfilled by defining (8.35), the covariant variational principle must only reproduce the inhomogeneous Maxwell equations (8.29) or (8.41).

The action $\mathcal{A}$ as a functional of the covariant components A_{ν} of the four-vector potential is defined as an integral of a Lagrange density $\mathcal{L}$ over a volume Ω of the four-dimensional space-time:

$$\mathcal{A}\left[A_{\nu}(\bullet)\right] = \frac{1}{c}\int_{\Omega} d^4x\,\mathcal{L}\,. \tag{8.46}$$

As the inhomogeneous Maxwell equations (8.29) or (8.41) are of second order in the derivatives of the four-vector potential, the Lagrange density can only contain derivatives up to first order:

$$\mathcal{L} = \mathcal{L}\left(A_{\nu}\left(x^{\lambda}\right); \partial_{\mu}A_{\nu}\left(x^{\lambda}\right)\right)\,. \tag{8.47}$$

Then the corresponding Hamilton principle states that the functional derivative of the action with respect to the covariant components of the four-vector potential vanishes:

$$\frac{\delta\mathcal{A}}{\delta A_{\nu}(x^{\lambda})} = 0\,. \tag{8.48}$$

The resulting Euler-Lagrange equations of this classical field theory then read

$$\frac{\partial \mathcal{L}}{\partial A_\nu(x^\lambda)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu\,(x^\lambda))} = 0\,. \tag{8.49}$$

Thus, it remains to find a Lagrange density, whose Euler-Lagrange equations (8.49) coincide with the inhomogeneous Maxwell equations (8.29) or (8.41). As the Maxwell equations are Lorentz invariant, the same must also hold for the Lagrange density. To this end we perform the following covariant ansatz for the Lagrange density of the electrodynamic field:

$$\mathcal{L} = \alpha F^{\lambda\kappa}\, F_{\lambda\kappa} + \beta j^\lambda\, A_\lambda\,. \tag{8.50}$$

Here α and β denote some constants, which are fixed below. Taking into account (8.35) the ansatz (8.50) reduces after some straight-forward algebraic transformations to the expression

$$\mathcal{L} = 2\alpha g^{\lambda\rho} g^{\kappa\sigma}\,(\partial_\rho A_\sigma - \partial_\sigma A_\rho)\,\partial_\lambda A_\kappa + \beta j^\lambda A_\lambda\,. \tag{8.51}$$

With this we obtain the partial derivative

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = \beta j^\nu \tag{8.52}$$

and, correspondingly, due to (8.35) also

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = 4\alpha F^{\mu\nu}\,. \tag{8.53}$$

Thus, with (8.52) and (8.53) the Euler-Lagrange equations (8.49) turn out to be of the form

$$\partial_\mu F^{\mu\nu} = \frac{\beta}{4\alpha}\, j^\nu\,. \tag{8.54}$$

A comparison of (8.54) with the inhomogeneous Maxwell equations (8.29) allows to fix the constant β according to

$$\frac{\beta}{4\alpha} = \mu_0 \qquad \Longrightarrow \qquad \beta = 4\alpha\mu_0\,. \tag{8.55}$$

Due to (8.55) the Lagrange density (8.50) is then given by

$$\mathcal{L} = \alpha F^{\mu\nu} F_{\mu\nu} + 4\alpha\mu_0\, j^\nu A_\nu\,, \tag{8.56}$$

where the constant α is still not yet determined.

8.6 Hamilton Function

We consider now the free electrodynamic field, where neither electric charges nor currents are present:

$$\rho(\mathbf{x}, t) = 0\,, \qquad \mathbf{j}(\mathbf{x}, t) = \mathbf{0}\,. \tag{8.57}$$

Furthermore, we restrict ourselves from now on to the Coulomb gauge (8.13) as it represents the basis of the standard formulation for the second quantization of the Maxwell theory and is commonly used in quantum optics. From (8.13), (8.16), and (8.57) we then conclude that the scalar potential vanishes:

$$\varphi(\mathbf{x},t) = 0\,. \tag{8.58}$$

Note that (8.13) and (8.58) together is also known as the radiation gauge. From (8.14), (8.57), and (8.58) we then read off that the vector potential obeys the wave equation:

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}(\mathbf{x},t)}{\partial t^2} - \Delta \mathbf{A}(\mathbf{x},t) = \mathbf{0}\,. \tag{8.59}$$

Thus, in radiation gauge the vector potential $\mathbf{A}(\mathbf{x},t)$ is determined from solving the wave equation (8.59) by taking into account the Coulomb gauge (8.13). Once the vector potential is known, one obtains from (8.7) the magnetic induction, whereas the electric field (8.8) reduces due to the radiation gauge (8.13) and (8.58) to

$$\mathbf{E}(\mathbf{x},t) = -\frac{\partial \mathbf{A}(\mathbf{x},t)}{\partial t}\,. \tag{8.60}$$

Furthermore, the Lagrange density of the free electrodynamic field reads due to (8.20), (8.23), (8.56), and (8.57)

$$\mathcal{L} = 2\alpha\left(\mathbf{B}^2 - \frac{\mathbf{E}^2}{c^2}\right)\,. \tag{8.61}$$

Due to (8.7) and (8.60) the Lagrange density (8.61) can be expressed in terms of the vector potential:

$$\mathcal{L} = 2\alpha\left\{\Big[\boldsymbol{\nabla}\times\mathbf{A}(\mathbf{x},t)\Big]^2 - \frac{1}{c^2}\left[\frac{\partial \mathbf{A}(\mathbf{x},t)}{\partial t}\right]^2\right\}\,. \tag{8.62}$$

With this the momentum field $\boldsymbol{\pi}$, which is canonically conjugated to the vector potential $\mathbf{A}$, follows as

$$\boldsymbol{\pi}(\mathbf{x},t) = \frac{\delta\mathcal{A}\left[\mathbf{A}(\bullet,\bullet)\right]}{\delta\frac{\partial \mathbf{A}(\mathbf{x},t)}{\partial t}} = \frac{\partial\mathcal{L}}{\partial\frac{\partial \mathbf{A}(\mathbf{x},t)}{\partial t}} = -\frac{4\alpha}{c^2}\,\frac{\partial \mathbf{A}(\mathbf{x},t)}{\partial t}\,. \tag{8.63}$$

A subsequent Legendre transformation

$$\mathcal{H} = \boldsymbol{\pi}(\mathbf{x},\mathbf{t})\,\frac{\partial \mathbf{A}(\mathbf{x},t)}{\partial t} - \mathcal{L} \tag{8.64}$$

converts then the Lagrange density (8.62) to the Hamilton density

$$\mathcal{H} = -\frac{c^2}{8\alpha}\,\boldsymbol{\pi}(\mathbf{x},t)^2 - 2\alpha\Big[\boldsymbol{\nabla}\times\mathbf{A}(\mathbf{x},t)\Big]^2\,, \tag{8.65}$$

which should coincide with the well-known energy density of the free electromagnetic field in SI units

$$\mathcal{H} = \frac{\epsilon_0}{2}\left[\frac{\partial \mathbf{A}(\mathbf{x},t)}{\partial t}\right]^2 + \frac{1}{2\mu_0}\Big[\boldsymbol{\nabla}\times\mathbf{A}(\mathbf{x},t)\Big]^2\,. \tag{8.66}$$

By taking into account (8.5) and (8.63) this fixes the parameter α according to

$$\alpha = -\frac{1}{4\mu_0}\,. \tag{8.67}$$

Thus, we obtain from (8.5), (8.63), and (8.67) the following result for the momentum field:

$$\boldsymbol{\pi}(\mathbf{x},t) = \epsilon_0\,\frac{\partial \mathbf{A}(\mathbf{x},t)}{\partial t}\,. \tag{8.68}$$

This corresponds to the classical expression for the momentum $\mathbf{p} = m\dot{\mathbf{x}}$, provided we identify the coordinate $\mathbf{x}$ with the vector potential $\mathbf{A}$ and the mass m with the vacuum dielectric constant ε_0. Furthermore, a spatial integral over the Hamilton density yields the Hamilton function

$$H = \int d^3x\,\mathcal{H}\,, \tag{8.69}$$

which follows from (8.66) to be

$$H = \frac{1}{2}\int d^3x\left\{\frac{1}{\epsilon_0}\,\boldsymbol{\pi}(\mathbf{x},t)^2 + \frac{1}{\mu_0}\left[\boldsymbol{\nabla}\times\mathbf{A}(\mathbf{x},t)\right]^2\right\}\,. \tag{8.70}$$

Note that the first (second) term represents the kinetic (potential) energy of the electromagnetic field. With an additional calculation the Hamilton function (8.70) can be simplified. To this end we consider

$$(\boldsymbol{\nabla}\times\mathbf{A})^2 = \epsilon_{jkl}\,\partial_k A_l\,\epsilon_{jmn}\,\partial_m A_n\,, \tag{8.71}$$

which reduces with the help of (6.56) to

$$(\boldsymbol{\nabla}\times\mathbf{A})^2 = \partial_k A_l \partial_k A_l - \partial_k\left(A_l\partial_l A_k\right) + A_l\partial_l\partial_k A_k\,. \tag{8.72}$$

Inserting (8.72) into (8.70), the second term vanishes due to applying the theorem of Gauß and the third term is zero in the Coulomb gauge (8.13), so we end up with

$$H = \frac{1}{2}\int d^3x\left\{\frac{1}{\epsilon_0}\,\pi_k(\mathbf{x},t)\pi_k(\mathbf{x},t) + \frac{1}{\mu_0}\,\partial_k A_l(\mathbf{x},t)\partial_k A_l(\mathbf{x},t)\right\}\,. \tag{8.73}$$

8.7 Canonical Field Quantization

The electrodynamic field is now quantized by exchanging the fields $A_j(\mathbf{x},t)$ and $\pi_j(\mathbf{x},t)$ with their corresponding field operators $\hat{A}_j(\mathbf{x},t)$ and $\hat{\pi}_j(\mathbf{x},t)$. To this end we perform a bosonic field quantization and demand equal-time commutation relations. At first, we demand that the field operators $\hat{A}_j(\mathbf{x},t)$ and $\hat{\pi}_j(\mathbf{x},t)$ commute, as usual, among themselves, respectively:

$$\left[\hat{A}_k(\mathbf{x},t),\hat{A}_l(\mathbf{x}',t)\right]_- = 0, \tag{8.74}$$

$$\left[\hat{\pi}_k(\mathbf{x},t),\hat{\pi}_l(\mathbf{x}',t)\right]_- = 0. \tag{8.75}$$

But when it comes to the equal-time commutation relations between the field operators $\hat{A}_j(\mathbf{x},t)$ and $\hat{\pi}_j(\mathbf{x},t)$, the situation turns out to be more intriguing. Let us investigate whether naive equal-time commutation relations of the form

$$\left[\hat{A}_k(\mathbf{x},t),\hat{\pi}_l(\mathbf{x}',t)\right]_- = i\hbar\,\delta_{kl}\delta(\mathbf{x}-\mathbf{x}') \tag{8.76}$$

are possible. On the one hand, a derivative with respect to x_k then yields at the left-hand side of (8.76) to

$$\partial_k\left[\hat{A}_k(\mathbf{x},t),\hat{\pi}_l(\mathbf{x}',t)\right]_- = \left[\partial_k\hat{A}_k(\mathbf{x},t),\hat{\pi}_l(\mathbf{x}',t)\right]_- = 0\,, \tag{8.77}$$

as we have to demand the quantized version of the Coulomb gauge (8.13):

$$\partial_j\hat{A}_j(\mathbf{x},t) = 0\,. \tag{8.78}$$

On the other, a derivative with respect to x_k at the right-hand side of (8.76) leads to

$$i\hbar\,\delta_{kl}\partial_k\delta(\mathbf{x}-\mathbf{x}') = i\hbar\,\partial_l\delta(\mathbf{x}-\mathbf{x}') \neq 0\,, \tag{8.79}$$

i.e. to an expression, which is non-zero in obvious contradiction to (8.77). Therefore, we are forced to modify the naive equal-time commutation relations (8.76) in such a way that it becomes compatible with the quantized version of the Coulomb gauge (8.78). To this end we consider the Fourier transformed of the right-hand side of (8.76)

$$i\hbar\,\delta_{kl}\delta(\mathbf{x}-\mathbf{x}') = i\hbar\int\frac{d^3k}{(2\pi)^3}\,\delta_{kl}\,e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')} \tag{8.80}$$

and substitute this expression by a yet to be determined transversal delta function

$$i\hbar\,\delta^T_{kl}(\mathbf{x}-\mathbf{x}') = i\hbar\int\frac{d^3k}{(2\pi)^3}\,\delta^T_{kl}(\mathbf{k})\,e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}\,. \tag{8.81}$$

The Fourier transformed of the transversal delta function is then fixed from demanding that the derivative of (8.81) with respect to x_k vanishes, i.e.

$$i\hbar\,\partial_k\delta^T_{kl}(\mathbf{x}-\mathbf{x}') = i\hbar\int\frac{d^3k}{(2\pi)^3}\,ik_k\,\delta^T_{kl}(\mathbf{k})\,e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')} = 0\,. \tag{8.82}$$

For this to be valid it is sufficient that the transversality condition

$$k_k\,\delta^T_{kl}(\mathbf{k}) = 0 \tag{8.83}$$

is fulfilled. By comparing (8.80) and (8.81) a suitable ansatz for the Fourier transformed of the transversal delta function reads

$$\delta^T_{kl}(\mathbf{k}) = \delta_{kl} + k_k k_l\,f(\mathbf{k})\,. \tag{8.84}$$

The yet unknown function $f(\mathbf{k})$ follows then from inserting (8.84) into (8.83):

$$f(\mathbf{k}) = -\frac{1}{\mathbf{k}^2}\,. \tag{8.85}$$

Thus, from (8.81), (8.84), and (8.85) we then conclude for the transversal delta function

$$\delta^T_{kl}(\mathbf{x}-\mathbf{x}') = \delta_{kl}\delta(\mathbf{x}-\mathbf{x}') + \partial'_k\partial'_l \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{\mathbf{k}^2}\,e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}\,. \tag{8.86}$$

The remaining integral is known, for instance, within the realm of electrostatics from determining the Green function of the Poisson equation and yields the Coulomb potential. Thus, we obtain for the transversal delta function

$$\delta^T_{kl}(\mathbf{x}-\mathbf{x}') = \delta_{kl}\,\delta(\mathbf{x}-\mathbf{x}') + \frac{1}{4\pi}\,\partial'_k\partial'_l\,\frac{1}{|\mathbf{x}-\mathbf{x}'|}\,. \tag{8.87}$$

And, finally, we summarize our derivation by stating that the naive equal-time commutation relations (8.76) have to be modified by

$$\left[\hat{A}_k(\mathbf{x},t),\hat{\pi}_l(\mathbf{x}',t)\right]_- \;=\; i\hbar\,\delta^T_{kl}(\mathbf{x}-\mathbf{x}') \tag{8.88}$$

in order to be compatible with the quantized version of the Coulomb gauge (8.78).

However, one should be aware that a derivation of commutation relations always has an essential caveat. As hard as one tries to consistently determine such basic principles, they are always attached with heuristic elements. Whether commutation relations are at the end correct or not can only be verified by checking any prediction following from them against experimental measurements. In this spirit we will show later on that demanding the bosonic equal-time commutation relations (8.74), (8.75), and (8.88) leads, indeed, to a consistent description of the electromagnetic field with the help of usual annihilation and creation operators for photons, i.e. the quanta of light.

8.8 Heisenberg Equations

Furthermore, proceeding with the second-quantized formalism, we obtain from the Hamilton function (8.73) the Hamilton operator

$$\hat{H} = \frac{1}{2}\int d^3x'\left\{\frac{1}{\epsilon_0}\,\hat{\pi}_k(\mathbf{x}',t)\hat{\pi}_k(\mathbf{x}',t) + \frac{1}{\mu_0}\,\partial'_k\hat{A}_l(\mathbf{x}',t)\partial'_k\hat{A}_l(\mathbf{x}',t)\right\}\,. \tag{8.89}$$

Note that the order of the operators in (8.89) does not play a role due to the commutation relations (8.74) and (8.75). Let us now evaluate the Heisenberg equation (3.62) for the field operator

$$i\hbar\,\frac{\partial\hat{A}_j(\mathbf{x},t)}{\partial t} = \left[\hat{A}_j(\mathbf{x},t),\hat{H}\right]_- \tag{8.90}$$

by inserting therein the Hamilton operator (8.89). After applying (3.10) as well as the equal-time commutation relations (8.74), (8.75), and (8.88) we get at first

$$i\hbar\,\frac{\partial\hat{A}_j(\mathbf{x},t)}{\partial t} = \frac{i\hbar}{\epsilon_0}\int d^3x'\,\delta^T_{jk}(\mathbf{x}-\mathbf{x}')\,\hat{\pi}_k(\mathbf{x}',t)\,. \tag{8.91}$$

Taking into account the transversal delta function (8.87), a partial integration yields

$$i\hbar\,\frac{\partial \hat{A}_j(\mathbf{x},t)}{\partial t} = \frac{i\hbar}{\epsilon_0}\left\{\hat{\pi}_j(\mathbf{x},t) - \frac{1}{4\pi}\int d^3x'\,\left(\partial'_j\,\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\,\partial'_k\hat{\pi}_k(\mathbf{x}',t)\right\}\,. \tag{8.92}$$

With this we reproduce the quantized version of (8.68), as the last term in (8.92) vanishes due to the quantized version of the Coulomb gauge (8.78):

$$\frac{\partial \hat{A}_j(\mathbf{x},t)}{\partial t} = \frac{1}{\epsilon_0}\,\hat{\pi}_j(\mathbf{x},t)\,. \tag{8.93}$$

Correspondingly, the Heisenberg equation (3.62) for the momentum field operator reads

$$i\hbar\frac{\partial \hat{\pi}_j(\mathbf{x},t)}{\partial t} = \left[\hat{\pi}_j(\mathbf{x},t),\hat{H}\right]_-\,. \tag{8.94}$$

Using (3.10) as well as the equal-time commutation relations (8.74), (8.75), and (8.88) we get at first

$$i\hbar\,\frac{\partial \hat{\pi}_j(\mathbf{x},t)}{\partial t} = \frac{-i\hbar}{\mu_0}\int d^3x'\,\partial'_k\delta^T_{jl}(\mathbf{x}-\mathbf{x}')\,\partial'_k\hat{A}_l(\mathbf{x}',t)\,, \tag{8.95}$$

so a partial integration yields

$$i\hbar\,\frac{\partial \hat{\pi}_j(\mathbf{x},t)}{\partial t} = \frac{i\hbar}{\mu_0}\int d^3x'\,\delta^T_{jl}(\mathbf{x}-\mathbf{x}')\,\Delta'\hat{A}_l(\mathbf{x}',t)\,, \tag{8.96}$$

Due to the explicit form of the transversal delta function (8.87) and a partial integration we then get

$$i\hbar\,\frac{\partial \hat{\pi}_j(\mathbf{x},t)}{\partial t} = \frac{i\hbar}{\mu_0}\,\left\{\partial_k\partial_k\hat{A}_j(\mathbf{x},t) - \frac{1}{4\pi}\int d^3x'\,\left(\partial'_j\,\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\,\Delta'\,\partial'_l\hat{A}_l(\mathbf{x}',t)\right\}\,. \tag{8.97}$$

With the quantized version of the Coulomb gauge (8.78) this reduces finally to

$$\frac{\partial \hat{\pi}_j(\mathbf{x},t)}{\partial t} = \frac{1}{\mu_0}\,\Delta\hat{A}_j(\mathbf{x},t)\,. \tag{8.98}$$

Thus, we conclude from (8.5), (8.93), and (8.98) that the field operator $\hat{\mathbf{A}}(\mathbf{x},t)$ obeys like the classical field $\mathbf{A}(\mathbf{x},t)$ in (8.59) the wave equation:

$$\frac{1}{c^2}\,\frac{\partial^2\hat{\mathbf{A}}(\mathbf{x},t)}{\partial t^2} - \Delta\hat{\mathbf{A}}(\mathbf{x},t) = \mathbf{0}\,. \tag{8.99}$$

8.9 Decomposition in Plane Waves

The wave equation (8.99) can be solved with a Fourier decomposition into plane waves:

$$\hat{\mathbf{A}}(\mathbf{x},t) = \int d^3k\,\hat{\mathbf{A}}(\mathbf{k},t)\,e^{i\mathbf{kx}}\,. \tag{8.100}$$

Inserting (8.100) into (8.99) one obtains for the expansion operators $\hat{\mathbf{A}}(\mathbf{k},t)$ the differential equation of a harmonic oscillator:

$$\frac{\partial^2 \hat{\mathbf{A}}(\mathbf{k},t)}{\partial t^2} + \omega_{\mathbf{k}}^2\, \hat{\mathbf{A}}(\mathbf{k},t) = \mathbf{0}\,, \tag{8.101}$$

where the dispersion relation is given by

$$\omega_{\mathbf{k}} = c|\mathbf{k}|\,. \tag{8.102}$$

The general solution of (8.101) reads

$$\hat{\mathbf{A}}(\mathbf{k},t) = \hat{\mathbf{A}}^{(1)}(\mathbf{k})\, e^{-i\omega_{\mathbf{k}}t} + \hat{\mathbf{A}}^{(2)}(\mathbf{k})\, e^{+i\omega_{\mathbf{k}}t}\,. \tag{8.103}$$

so that the field operator (8.100) results in

$$\hat{\mathbf{A}}(\mathbf{x},t) = \int d^3k \left\{ \hat{\mathbf{A}}^{(1)}(\mathbf{k}) e^{i(\mathbf{kx}-\omega_{\mathbf{k}}t)} + \hat{\mathbf{A}}^{(2)}(\mathbf{k}) e^{i(\mathbf{kx}+\omega_{\mathbf{k}}t)} \right\}. \tag{8.104}$$

Performing in the second integral the substitution $\mathbf{k} \to -\mathbf{k}$ and taking into account the symmetry of the dispersion relation (8.102), i.e.

$$\omega_{\mathbf{k}} = \omega_{-\mathbf{k}}\,, \tag{8.105}$$

the Fourier decomposition (8.104) is converted into

$$\hat{\mathbf{A}}(\mathbf{x},t) = \int d^3k \left\{ \hat{\mathbf{A}}^{(1)}(\mathbf{k}) e^{i(\mathbf{kx}-\omega_{\mathbf{k}}t)} + \hat{\mathbf{A}}^{(2)}(-\mathbf{k}) e^{-i(\mathbf{kx}-\omega_{\mathbf{k}}t)} \right\}. \tag{8.106}$$

Thus, the adjoint field operator reads

$$\hat{\mathbf{A}}^{\dagger}(\mathbf{x},t) = \int d^3k \left\{ \hat{\mathbf{A}}^{(1)\dagger}(\mathbf{k}) e^{-i(\mathbf{kx}-\omega_{\mathbf{k}}t)} + \hat{\mathbf{A}}^{(2)\dagger}(-\mathbf{k}) e^{i(\mathbf{kx}-\omega_{\mathbf{k}}t)} \right\} \tag{8.107}$$

As the vector potential of electrodynamics is real, we demand that the field operator as its second-quantized counterpart is self-adjoint, i.e.

$$\hat{\mathbf{A}}(\mathbf{x},t) = \hat{\mathbf{A}}^{\dagger}(\mathbf{x},t)\,, \tag{8.108}$$

and conclude from (8.106) and (8.107):

$$\hat{\mathbf{A}}(\mathbf{k}) = \hat{\mathbf{A}}^{(1)}(\mathbf{k}) \quad , \quad \hat{\mathbf{A}}^{\dagger}(\mathbf{k}) = \hat{\mathbf{A}}^{(2)}(-\mathbf{k})\,, \tag{8.109}$$

Inserting the finding (8.109) into the Fourier decomposition (8.106), we finally obtain

$$\hat{\mathbf{A}}(\mathbf{x},t) = \int d^3k \left\{ \hat{\mathbf{A}}(\mathbf{k}) e^{i(\mathbf{kx}-\omega_{\mathbf{k}}t)} + \hat{\mathbf{A}}^{\dagger}(\mathbf{k}) e^{-i(\mathbf{kx}-\omega_{\mathbf{k}}t)} \right\}. \tag{8.110}$$

8.10 Construction of Polarization Vectors

Before we can continue with working out the second quantization of the Maxwell theory we have to acquire beforehand a more detailed understanding of the description of plane waves. To this end we define two linearly polarized plane waves with the wave vector $\mathbf{k}$ and the dispersion (8.102) via

$$\mathbf{A}_1(\mathbf{x},t) = A_1 \boldsymbol{\epsilon}_1 e^{i(\mathbf{kx}-\omega_\mathbf{k} t)}\,, \qquad \mathbf{A}_2(\mathbf{x},t) = A_2 \boldsymbol{\epsilon}_2 e^{i(\mathbf{kx}-\omega_\mathbf{k} t)}\,. \tag{8.111}$$

Here A_1, A_2 represent the respective complex-valued amplitudes and $\boldsymbol{\epsilon}_1$, $\boldsymbol{\epsilon}_2$ denote two complex-valued polarization vectors, which are normalized according to

$$\boldsymbol{\epsilon}_1 \boldsymbol{\epsilon}_1^* = \boldsymbol{\epsilon}_2 \boldsymbol{\epsilon}_2^* = 1\,. \tag{8.112}$$

Let us consider now the sum of those two linearly polarized plane waves:

$$\mathbf{A}(\mathbf{x},t) = \mathbf{A}_1(\mathbf{x},t) + \mathbf{A}_2(\mathbf{x},t) = (A_1 \boldsymbol{\epsilon}_1 + A_2 \boldsymbol{\epsilon}_2)\, e^{i(\mathbf{kx}-\omega_\mathbf{k} t)}\,. \tag{8.113}$$

Provided that both complex amplitudes $A_1 = |A_1| e^{i\varphi}$ and $A_2 = |A_2| e^{i\varphi}$ have the same phase φ, also their sum (8.113) is linearly polarized and we get

$$\mathbf{A}(\mathbf{x},t) = A \boldsymbol{\epsilon} e^{i(\mathbf{kx}-\omega_\mathbf{k} t)}\,. \tag{8.114}$$

Here the resulting amplitude A is given by

$$A = \sqrt{|A_1|^2 + |A_2|^2}\, e^{i\varphi} \tag{8.115}$$

and the resulting polarization vector $\boldsymbol{\epsilon}$ has the angle

$$\vartheta = \arctan \frac{|A_2|}{|A_1|} \tag{8.116}$$

with respect to $\boldsymbol{\epsilon}_1$, see Fig. 8.1. However, in the more general case that both complex amplitudes $A_1 = |A_1| e^{i\varphi_1}$ and $A_2 = |A_2| e^{i\varphi_2}$ have different phases $\varphi_1 \neq \varphi_2$, the sum (8.113) represents an ellipticly polarized plane wave. Let us illustrate this for the simpler situation of a circularly polarized plane wave, which occurs provided that both complex amplitudes A_1 and A_2 have the same absolute value and their phases differ by 90°:

$$A_1 = \frac{A_0}{\sqrt{2}}\,, \qquad A_2 = \pm i \frac{A_0}{\sqrt{2}}\,. \tag{8.117}$$

Inserting (8.117) into (8.113) we obtain for the sum of the two linearly polarized plane waves

$$\mathbf{A}(\mathbf{x},t) = \frac{A_0}{\sqrt{2}}\, (\boldsymbol{\epsilon}_1 \pm i \boldsymbol{\epsilon}_2)\, e^{i(\mathbf{kx}-\omega_\mathbf{k} t)}\,. \tag{8.118}$$

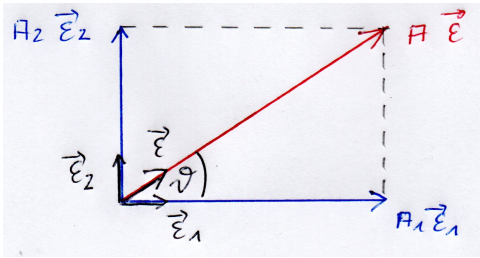

Figure 8.1: Adding two linearly polarized plane waves according to (8.113) with complex amplitudes A_1 and A_2, which have the same phase.

In order to be concrete we choose now the coordinate axes in such a way that the plane wave propagates in z-direction, whereas the two polarization vectors $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$, which are normalized according to (8.112), point in x- and y-direction:

$$\mathbf{k} = k \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} , \qquad \boldsymbol{\epsilon}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \qquad \boldsymbol{\epsilon}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \tag{8.119}$$

With this Eq. (8.118) reduces to

$$\mathbf{A}(\mathbf{x}, t) = \frac{A_0}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix} e^{i(k\mathbf{e}_z\mathbf{x} - \omega_{k\mathbf{e}_z} t)} . \tag{8.120}$$

Considering the real part of the vector potential $\mathbf{A}(\mathbf{x}, t)$ at a fixed space point $\mathbf{x}$, it represents a vector in the xy-plane with constant absolute value A_0, which rotates on a circle with the frequency $\omega_{k\mathbf{e}_z}$:

$$\mathrm{Re}\, A_x(\mathbf{x}, t) = \frac{A_0}{\sqrt{2}} \cos\left(kz - \omega_{k\mathbf{e}_z} t\right) , \mathrm{Re}\, A_y(\mathbf{x}, t) = \mp \frac{A_0}{\sqrt{2}} \sin\left(kz - \omega_{k\mathbf{e}_z} t\right) , \mathrm{Re}\, A_z(\mathbf{x}, t) = 0 . \tag{8.121}$$

For the upper (lower) sign the rotation is performed anti-clockwise (clockwise) for an observer looking in the direction of the oncoming light beam. Such a plane wave is called in optics left-(right-) circularly polarized light, whereas in elementary particle physics one says that such a plane wave has positive (negative) helicity, see Fig. 8.2.

In view of a more detailed discussion of the helicity we remind us upon its definition in Eq. (6.185). Here the spin vector (6.163) of the electromagnetic field is given by the representation matrices $N^{\alpha\beta}$ of the Lorentz algebra in the space of the four-vector potential, which coincide with the representation matrices $L^{\alpha\beta}$ of the Lorentz algebra in the Minkowskian space-time according to (6.108) and (6.113). Thus, taking into account (6.53) and restricting us upon the spatial components, the helicity operator

$$\hat{h}(\mathbf{k}) = \frac{\mathbf{k}}{k} \mathbf{L} \tag{8.122}$$

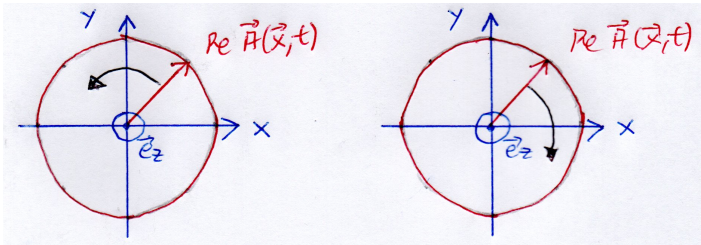

Figure 8.2: Adding two linearly polarized plane waves according to (8.113) with complex amplitudes A_1 and A_2 with the same absolute value and phases, which differ by 90°.

turns out to be defined by

$$\hat{h}(\mathbf{k}) = \frac{i}{k} \begin{pmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{pmatrix} . \tag{8.123}$$

Now we introduce the polariation vectors $\boldsymbol{\epsilon}(\mathbf{k}, \lambda)$ for plane waves which propagate with the wave vector $\mathbf{k}$ and the helicity $\lambda = \pm 1$:

$$\mathbf{A}(\mathbf{x}, t) = A\boldsymbol{\epsilon}(\mathbf{k}, \lambda) e^{i(\mathbf{kx} - \omega_{\mathbf{k}} t)} . \tag{8.124}$$

Here the polarization vectors $\boldsymbol{\epsilon}(\mathbf{k}, \lambda)$ represent the eigenvectors of the helicity operator (8.122) with the eigenvalues λ:

$$\hat{h}(\mathbf{k})\boldsymbol{\epsilon}(\mathbf{k}, \lambda) = \lambda\boldsymbol{\epsilon}(\mathbf{k}, \lambda) . \tag{8.125}$$

From (8.120) and (8.124) we read off the polarization vectors $\boldsymbol{\epsilon}(k\mathbf{e}_z, \lambda)$ for a propagation in z-direction:

$$\boldsymbol{\epsilon}(k\mathbf{e}_z, \lambda) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \lambda i \\ 0 \end{pmatrix} . \tag{8.126}$$

Indeed, the polarization vectors (8.126) fulfill due to (8.123) the eigenvalue problem

$$\hat{h}(k\mathbf{e}_z)\boldsymbol{\epsilon}(k\mathbf{e}_z, \lambda) = \lambda\boldsymbol{\epsilon}(k\mathbf{e}_z, \lambda) . \tag{8.127}$$

Now we construct the polarization vectors $\boldsymbol{\epsilon}(\mathbf{k}, \lambda)$ with a general wave vector $\mathbf{k}$ by rotating the polarization vectors $\boldsymbol{\epsilon}(k\mathbf{e}_z, \lambda)$ in the same way as the original wave vector $k\mathbf{e}_z$. To this end we need the rotation matrix $R(\theta, \phi)$, which rotates the original wave vector $k\mathbf{e}_z$ to the general wave vector $\mathbf{k}$, where the latter is described in terms of spherical coordinates k, θ, and ϕ:

$$\mathbf{k} = k \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\phi \end{pmatrix} . \tag{8.128}$$

Here the rotation matrix $R(\theta,\phi)$ is constructed such that first the rotation $R_y(\theta)$ around the y-axis with angle θ and then the rotation $R_z(\phi)$ around the z-axis with angle ϕ is applied:

$$R(\theta,\phi) = R_z(\phi)\, R_y(\theta)\,. \tag{8.129}$$

The individual rotation matrices follow from evaluating matrix exponential functions

$$R_z(\phi) \;=\; e^{-iL_3\phi} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{8.130}$$

$$R_y(\theta) \;=\; e^{-iL_2\theta} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \tag{8.131}$$

where the respective generators stem from (6.53). As a result we obtain for the rotation (8.129)

$$R(\theta,\phi) = \begin{pmatrix} \cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi \\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}. \tag{8.132}$$

Indeed, the rotation matrix $R(\theta,\phi)$ maps the original wave vector $k\mathbf{e}_z$ to the general wave vector (8.128) as follows from the third column of (8.132):

$$R(\theta,\phi)k\mathbf{e}_z = \mathbf{k}\,. \tag{8.133}$$

Transforming correspondingly also the polarization vectors $\boldsymbol{\epsilon}(k\mathbf{e}_z,\lambda)$ from (8.126) with the rotation matrix $R(\theta,\phi)$, i.e.

$$\boldsymbol{\epsilon}(\mathbf{k},\lambda) = R(\theta,\phi)\boldsymbol{\epsilon}(k\mathbf{e}_z,\lambda)\,, \tag{8.134}$$

we obtain the explicit result

$$\boldsymbol{\epsilon}(\mathbf{k},\lambda) = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\theta\cos\phi - \lambda i\sin\phi \\ \cos\theta\sin\phi + \lambda i\cos\phi \\ -\sin\theta \end{pmatrix}. \tag{8.135}$$

Indeed, taking into account (8.123) and (8.128) one can show that the polarization vectors (8.135) fulfill the eigenvalue problem of the helicity operator (8.125). Furthermore, as expected, the polarization vectors (8.135) reduce for the special case $\theta = \phi = 0$ to the original polarization vectors (8.126).

8.11 Properties of Polarization Vectors

Due to the second-quantized formulation of the Coulomb gauge (8.78) the Fourier operators $\hat{\mathbf{A}}(\mathbf{k})$ in the decomposition (8.110) must obey the transversality condition

$$\mathbf{k}\hat{\mathbf{A}}(\mathbf{k}) = 0\,. \tag{8.136}$$

This means that the Fourier operators $\hat{\mathbf{A}}(\mathbf{k})$ have two transversal dynamical degrees of freedom. Performing the ansatz

$$\hat{\mathbf{A}}(\mathbf{k}) = N_{\mathbf{k}} \sum_{\lambda=\pm 1} \boldsymbol{\epsilon}(\mathbf{k},\lambda)\hat{a}_{\mathbf{k},\lambda} \tag{8.137}$$

with some normalization constants $N_{\mathbf{k}}$ the transversality condition (8.136) is fulfilled provided that the polarization vectors $\boldsymbol{\epsilon}(\mathbf{k},\lambda)$ are perpendicular to the propagation direction, which is defined by the wave vector $\mathbf{k}$:

$$\mathbf{k}\boldsymbol{\epsilon}(\mathbf{k},\lambda) = 0\,. \tag{8.138}$$

Due to (8.128) it is straight-forward to show that the polarization vectors determined in (8.135) obey (8.138).

As another property of the polarization vectors (8.135) we investigate whether they obey orthonormality relations. Showing separately

$$\boldsymbol{\epsilon}(\mathbf{k},\lambda)\boldsymbol{\epsilon}(\mathbf{k},\lambda)^* = 1\,, \tag{8.139}$$

$$\boldsymbol{\epsilon}(\mathbf{k},\lambda)\boldsymbol{\epsilon}(\mathbf{k},-\lambda)^* = 0\,, \tag{8.140}$$

we arrive, indeed, due to $\lambda = \pm 1$ at the orthonormality relations

$$\boldsymbol{\epsilon}(\mathbf{k},\lambda)\boldsymbol{\epsilon}(\mathbf{k},\lambda')^* = \delta_{\lambda,\lambda'}\,. \tag{8.141}$$

Another property of the polarization vectors (8.135), which will turn out to be quite useful for later calculations, is their behaviour concerning the inversion $\mathbf{k} \to -\mathbf{k}$. Obviously, such an inversion is obtained in spherical coordinates (8.126) via

$$\phi \to \phi + \pi : \qquad \sin\phi \to -\sin\phi\,, \qquad \cos\phi \to -\cos\phi\,, \tag{8.142}$$

$$\theta \to \theta - \pi : \qquad \sin\theta \to \sin\theta\,, \qquad \cos\theta \to -\cos\theta\,. \tag{8.143}$$

With this we then conclude from (8.135

$$\boldsymbol{\epsilon}(-\mathbf{k},\lambda) = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\theta\cos\phi + \lambda i \sin\phi \\ \cos\theta\sin\phi - \lambda i\cos\phi \\ -\sin\theta \end{pmatrix}\,. \tag{8.144}$$

Thus, from (8.135) and (8.144) we read off

$$\boldsymbol{\epsilon}(-\mathbf{k},\lambda) = \boldsymbol{\epsilon}(\mathbf{k},-\lambda) = \boldsymbol{\epsilon}(\mathbf{k},\lambda)^*\,. \tag{8.145}$$

And, inserting the decomposition (8.137) into (8.110) by taking into account (8.145), we finally get

$$\hat{\mathbf{A}}(\mathbf{x},t) = \sum_{\lambda=\pm 1} \int d^3k\, N_{\mathbf{k}} \left\{ \boldsymbol{\epsilon}(\mathbf{k},\lambda) e^{i(\mathbf{kx}-\omega_{\mathbf{k}}t)} \hat{a}_{\mathbf{k},\lambda} + \boldsymbol{\epsilon}(\mathbf{k},\lambda)^* e^{-i(\mathbf{kx}-\omega_{\mathbf{k}}t)} \hat{a}^\dagger_{\mathbf{k},\lambda} \right\}\,. \tag{8.146}$$

Note that this plane wave decomposition fulfills, indeed, the Coulomb gauge (8.78) due to the transversality condition (8.138). In the following we aim at unravelling the physical interpretation of the Fourier operators $\hat{a}_{\mathbf{k},\lambda}$ and $\hat{a}^\dagger_{\mathbf{k},\lambda}$ in the plane wave decomposition (8.146), which leads to straight-forward but quite lengthy calculations. Therefore, we relegate the respective technical details to the exercises and restrict ourselves in the subsequent four sections to present a concise summary of the corresponding derivations.

8.12 Fourier Operators

We start with noting the plane wave decomposition for the momentum field operator, which follows from (8.93) and (8.146):

$$\hat{\boldsymbol{\pi}}(\mathbf{x},t) = \sum_{\lambda=\pm 1} \int d^3k\, \epsilon_0\, N_{\mathbf{k}} \Big\{ -i\omega_{\mathbf{k}} \boldsymbol{\epsilon}(\mathbf{k},\lambda) e^{i(\mathbf{kx}-\omega_{\mathbf{k}}t)} \hat{a}_{\mathbf{k},\lambda} + i\omega_{\mathbf{k}} \boldsymbol{\epsilon}(\mathbf{k},\lambda)^* e^{-i(\mathbf{kx}-\omega_{\mathbf{k}}t)} \hat{a}^\dagger_{\mathbf{k},\lambda} \Big\} . \quad (8.147)$$

The plane wave decompositions (8.146) and (8.147) for both the field operator $\hat{\mathbf{A}}(\mathbf{x},t)$ and the momentum field operator $\hat{\boldsymbol{\pi}}(\mathbf{x},t)$ can now be solved for the Fourier operators $\hat{a}_{\mathbf{k},\lambda}$ and $\hat{a}^\dagger_{\mathbf{k},\lambda}$:

$$\hat{a}_{\mathbf{k},\lambda} = \frac{1}{2(2\pi)^3 N_{\mathbf{k}}} \int d^3x\, \boldsymbol{\epsilon}(\mathbf{k},\lambda)^* e^{-i(\mathbf{kx}-\omega_{\mathbf{k}}t)} \left\{ \hat{\mathbf{A}}(\mathbf{x},t) + i\, \frac{\hat{\boldsymbol{\pi}}(\mathbf{x},t)}{\epsilon_0 \omega_{\mathbf{k}}} \right\} , \quad (8.148)$$

$$\hat{a}^\dagger_{\mathbf{k},\lambda} = \frac{1}{2(2\pi)^3 N_{\mathbf{k}}} \int d^3x\, \boldsymbol{\epsilon}(\mathbf{k},\lambda) e^{i(\mathbf{kx}-\omega_{\mathbf{k}}t)} \left\{ \hat{\mathbf{A}}(\mathbf{x},t) - i\, \frac{\hat{\boldsymbol{\pi}}(\mathbf{x},t)}{\epsilon_0 \omega_{\mathbf{k}}} \right\} . \quad (8.149)$$

This allows us now to determine the commutator relations between the Fourier operators $\hat{a}_{\mathbf{k},\lambda}$ and $\hat{a}^\dagger_{\mathbf{k},\lambda}$ from the equal-time commutator relations (8.74), (8.75), and (8.88) for the field operator $\hat{\mathbf{A}}(\mathbf{x},t)$ and the momentum field operator $\hat{\boldsymbol{\pi}}(\mathbf{x},t)$:

$$\Big[\hat{a}_{\mathbf{k},\lambda}, \hat{a}_{\mathbf{k}',\lambda'}\Big]_- = 0 , \quad (8.150)$$

$$\Big[\hat{a}^\dagger_{\mathbf{k},\lambda}, \hat{a}^\dagger_{\mathbf{k}',\lambda'}\Big]_- = 0 , \quad (8.151)$$

$$\Big[\hat{a}_{\mathbf{k},\lambda}, \hat{a}^\dagger_{\mathbf{k}',\lambda'}\Big]_- = \frac{\hbar}{2(2\pi)^3 \epsilon_0 \omega_{\mathbf{k}} N_{\mathbf{k}}^2}\, \delta_{\lambda\lambda'}\, \delta(\mathbf{k}-\mathbf{k}') . \quad (8.152)$$

Thus, fixing the yet undetermined normalization constant according to

$$N_{\mathbf{k}} = \sqrt{\frac{\hbar}{2(2\pi)^3 \epsilon_0 \omega_{\mathbf{k}}}} , \quad (8.153)$$

we end up with the bosonic canonical commutation relation

$$\Big[\hat{a}_{\mathbf{k},\lambda}, \hat{a}^\dagger_{\mathbf{k}',\lambda'}\Big]_- = \delta_{\lambda\lambda'}\, \delta(\mathbf{k}-\mathbf{k}') . \quad (8.154)$$

This means that the Fourier operators $\hat{a}_{\mathbf{k},\lambda}$ and $\hat{a}^\dagger_{\mathbf{k},\lambda}$ can be interpreted as the annihilation and creation operators of bosonic particles, which are characterized by the wave vector $\mathbf{k}$ and the polarization λ. In order to determine the respective properties of these particles we investigate in the subsequent three sections their contribution to the energy, the momentum, and the spin angular momentum of the electromagnetic field in second quantization.

8.13 Energy

Taking into account the normalization constant (8.153) in the plane wave decompositions (8.146) and (8.147) for both the field operator $\hat{\mathbf{A}}(\mathbf{x},t)$ and the momentum field operator $\hat{\boldsymbol{\pi}}(\mathbf{x},t)$ we get

$$\hat{\mathbf{A}}(\mathbf{x},t) = \sum_{\lambda=\pm1}\int d^3k\sqrt{\frac{\hbar}{2(2\pi)^3\epsilon_0\omega_{\mathbf{k}}}}\left\{\boldsymbol{\epsilon}(\mathbf{k},\lambda)e^{i(\mathbf{kx}-\omega_{\mathbf{k}}t)}\hat{a}_{\mathbf{k},\lambda}+\boldsymbol{\epsilon}(\mathbf{k},\lambda)^*e^{-i(\mathbf{kx}-\omega_{\mathbf{k}}t)}\hat{a}^\dagger_{\mathbf{k},\lambda}\right\}, \quad (8.155)$$

$$\hat{\boldsymbol{\pi}}(\mathbf{x},t) = \sum_{\lambda=\pm1}\int d^3k\sqrt{\frac{\hbar\epsilon_0\omega_{\mathbf{k}}}{2(2\pi)^3}}\left\{-i\boldsymbol{\epsilon}(\mathbf{k},\lambda)e^{i(\mathbf{kx}-\omega_{\mathbf{k}}t)}\hat{a}_{\mathbf{k},\lambda}+i\boldsymbol{\epsilon}_k(\mathbf{k},\lambda)^*e^{-i(\mathbf{kx}-\omega_{\mathbf{k}}t)}\hat{a}^\dagger_{\mathbf{k},\lambda}\right\}. \quad (8.156)$$

Inserting (8.155) and (8.156) in the expression for the Hamilton operator (8.89) and using (8.5), (8.102), and (8.105) we yield

$$\hat{H} = \frac{1}{2}\sum_{\lambda=\pm1}\int d^3k\,\hbar\omega_{\mathbf{k}}\left(\hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda}+\hat{a}_{\mathbf{k},\lambda}\hat{a}^\dagger_{\mathbf{k},\lambda}\right). \quad (8.157)$$

Thus, comparing (3.6) with (8.157) we recognize that the second quantized electromagnetic field consists of independent harmonic oscillators, where each energy $\hbar\omega_{\mathbf{k}}$ is doubly degenerate due to the polarization degree of freedome λ. Defining the vacuum state as usual

$$\hat{a}_{\mathbf{k},\lambda}|0\rangle = 0 \qquad \Longleftrightarrow \qquad \langle 0|\hat{a}^\dagger_{\mathbf{k},\lambda} = 0\,, \quad (8.158)$$

we find that the vacuum energy of the electrodynamic field is given by a sum of the zero-point energy of all independent harmonic oscillators

$$\langle 0|\hat{H}|0\rangle = \int d^3k\,\hbar\omega_{\mathbf{k}}\,, \quad (8.159)$$

which turns out to be divergent due to the linear dispersion (8.102). Therefore, using the commutator relation (8.154) we obtain for the renormalized Hamilton operator

$$\hat{H} = \hat{H} - \langle 0|\hat{H}|0\rangle \quad (8.160)$$

the normal ordered result

$$\hat{H} = \sum_{\lambda=\pm1}\int d^3k\,\hbar\omega_{\mathbf{k}}\,\hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda}\,. \quad (8.161)$$

Here $\hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda}$ represents the occupation number operator, which counts the number of photons with wave vector $\mathbf{k}$ and polarization λ once it is applied to a photon state.

8.14 Momentum

Due to classical electrodynamics the momentum of the electromagnetic field is defined by

$$\mathbf{P} = \int d^3x\,\frac{\mathbf{S}(\mathbf{x},t)}{c^2} \quad (8.162)$$

with the Poynting vector

$$\mathbf{S}(\mathbf{x},t) = \frac{1}{\mu_0}\,\mathbf{E}(\mathbf{x},t)\times\mathbf{B}(\mathbf{x},t)\,. \tag{8.163}$$

Taking into account (8.5), (8.7), and (8.60), the momentum (8.163) is expressed in terms of the vector potential and the canonically conjugated momentum field via

$$\mathbf{P} = \int d^3x\,\left[\boldsymbol{\nabla}\times\mathbf{A}(\mathbf{x},t)\right]\times\boldsymbol{\pi}(\mathbf{x},t)\,. \tag{8.164}$$

Thus, in second quantization, the momentum operator of the electromagnetic field reads

$$\hat{\mathbf{P}} = \int d^3x\,\Big(\boldsymbol{\nabla}\times\hat{\mathbf{A}}(\mathbf{x},t)\Big)\times\hat{\boldsymbol{\pi}}(\mathbf{x},t)\,. \tag{8.165}$$

The further evaluation is based on taking into account the plane wave decompositions (8.155) and (8.156) for both the field operator $\hat{\mathbf{A}}(\mathbf{x},t)$ and the momentum field operator $\hat{\boldsymbol{\pi}}(\mathbf{x},t)$. Furthermore, the symmetry of the dispersion relation (8.105), the vector identity

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{ac})\mathbf{b} - (\mathbf{bc})\mathbf{a}\,, \tag{8.166}$$

the transversality condition (8.138), the orthonormality relation (8.141), and (8.145) are needed. Subsequently, performing the substitution $\mathbf{k}\to-\mathbf{k}$ and applying (8.105), (8.150), (8.151), we get the expression

$$\mathbf{P} = \sum_{\lambda=\pm1}\int d^3k\,\frac{\hbar\mathbf{k}}{2}\,\left(\hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda} + \hat{a}_{\mathbf{k},\lambda}\hat{a}^\dagger_{\mathbf{k},\lambda}\right)\,. \tag{8.167}$$

Note that the vacuum state has a vanishing momentum

$$\langle 0|\hat{\mathbf{P}}|0\rangle = \int d^3k\,\hbar\mathbf{k} = \mathbf{0} \tag{8.168}$$

due to the odd symmetry of the integrand. Thus, taking into account the commutator relation (8.154) we recognize that (8.167) coincides with the renormalized momentum operator

$$\hat{\mathbf{P}} = \hat{\mathbf{P}} - \langle 0|\hat{\mathbf{P}}|0\rangle\ , \tag{8.169}$$

which finally yields the normal ordered result

$$\hat{\mathbf{P}} = \sum_{\lambda=\pm1}\int d^3k\,\hbar\mathbf{k}\,\hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda}\,. \tag{8.170}$$

8.15 Spin Angular Momentum

According to the Noether theorem, which is explored and applied to the electromagnetic field in the Appendix, the spin angular momentum of the electromagnetic is given by

$$\mathbf{S} = \int d^3x\,\mathbf{A}(\mathbf{x},t)\times\boldsymbol{\pi}(\mathbf{x},t)\,. \tag{8.171}$$

Thus, the corresponding second quantized spin angular momentum operator reads

$$\hat{\mathbf{S}} = \int d^3x \, \hat{\mathbf{A}}(\mathbf{x},t) \times \hat{\boldsymbol{\pi}}(\mathbf{x},t) \, . \tag{8.172}$$

Inserting the plane wave decompositions (8.155) and (8.156) for both the field operator $\hat{\mathbf{A}}(\mathbf{x},t)$ and the momentum field operator $\hat{\boldsymbol{\pi}}(\mathbf{x},t)$ and performing the substitution $\mathbf{k} \to -\mathbf{k}$ then yields the intermediate result

$$\hat{\mathbf{S}} = \sum_{\lambda=\pm1} \sum_{\lambda'=\pm1} \int d^3k \, \frac{i\hbar}{2} \left[\boldsymbol{\epsilon}(\mathbf{k},\lambda) \times \boldsymbol{\epsilon}(\mathbf{k},\lambda')^* \hat{a}_{\mathbf{k},\lambda} \hat{a}^\dagger_{\mathbf{k},\lambda'} + \boldsymbol{\epsilon}(\mathbf{k},\lambda') \times \boldsymbol{\epsilon}(\mathbf{k},\lambda)^* \hat{a}^\dagger_{\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda'} \right] \, . \tag{8.173}$$

Now we evaluate the vector product between two polarization vectors. At first we obtain from (8.145)

$$\boldsymbol{\epsilon}(\mathbf{k},\lambda) \times \boldsymbol{\epsilon}(\mathbf{k},-\lambda)^* = \mathbf{0} \, , \tag{8.174}$$

whereas we get from (8.128) and (8.135)

$$\boldsymbol{\epsilon}(\mathbf{k},\lambda) \times \epsilon(\mathbf{k},\lambda) = -i\lambda \, \frac{\mathbf{k}}{k} \, . \tag{8.175}$$

Thus, both (8.174) and (8.175) can be summarized by

$$\boldsymbol{\epsilon}(\mathbf{k},\lambda) \times \boldsymbol{\epsilon}(\mathbf{k},\lambda')^* = -i\lambda \, \frac{\mathbf{k}}{k} \, \delta_{\lambda\lambda'} \, . \tag{8.176}$$

With this the intermediate result (8.173) for the spin angular momentum operator of the electromagnetic field reduces to

$$\hat{\mathbf{S}} = \sum_{\lambda=\pm1} \int d^3k \, \lambda \, \frac{\hbar}{2} \, \frac{\mathbf{k}}{k} \left(\hat{a}^\dagger_{\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda} + \hat{a}_{\mathbf{k},\lambda} \hat{a}^\dagger_{\mathbf{k},\lambda} \right) \, . \tag{8.177}$$

Thus, the vacuum state has a vanishing spin angular momentum

$$\langle 0 | \hat{\mathbf{S}} | 0 \rangle = \hbar \left(\sum_{\lambda=\pm1} \lambda \right) \left(\int d^3k \, \frac{\mathbf{k}}{k} \right) = \mathbf{0} \tag{8.178}$$

due to the odd symmetry in both the summand and the integrand. Using the commutator relation (8.154) we read off that (8.177) coincides with the renormalized spin angular momentum operator

$$\hat{\mathbf{S}} = \hat{\mathbf{S}} - \langle 0 | \hat{\mathbf{S}} | 0 \rangle \, , \tag{8.179}$$

leading to the normal ordered result

$$\hat{\mathbf{S}} = \sum_{\lambda=\pm1} \int d^3k \, \lambda\hbar \, \frac{\mathbf{k}}{k} \, \hat{a}^\dagger_{\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda} \, . \tag{8.180}$$

We observe that the decompositions of the second quantized expressions for the energy (8.161), the momentum (8.170), and the spin angular momentum (8.180) of the electromagnetic field turn out to be time independent and, thus, represent conserved quantities. Together with the commutator relations (8.150), (8.151), and (8.154) we furthermore conclude that the Fourier operators $\hat{a}_{\mathbf{k},\lambda}$ and $\hat{a}^\dagger_{\mathbf{k},\lambda}$ represent the annihilation and creation operators of photons with the energy $\hbar\omega_{\mathbf{k}}$, the momentum $\hbar\mathbf{k}$, and the spin angular momentum $\lambda\hbar\mathbf{k}/k$, where the latter amounts to the helicity $\lambda\hbar$.

8.16 Definition of Maxwell Propagator

In close analogy to the Klein-Gordon propagator (7.123) we now define also the Maxwell propagator as the vacuum expectation value of the time-ordered product of two field operators $\hat{A}^\mu(\mathbf{x},t)$ and $\hat{A}^\nu(\mathbf{x}',t')$:

$$D^{\mu\nu}(\mathbf{x},t;\mathbf{x}',t') = \left\langle 0 \left| \hat{T}\left(\hat{A}^\mu(\mathbf{x},t)\hat{A}^\nu(\mathbf{x}',t')\right) \right| 0 \right\rangle . \tag{8.181}$$

Taking into account the definition of the time ordering operator (7.124) the Maxwell propagator reads explicitly

$$D^{\mu\nu}(\mathbf{x},t;\mathbf{x}',t') = \Theta(t-t')\langle 0|\hat{A}^\mu(\mathbf{x},t)\hat{A}^\nu(\mathbf{x}',t')|0\rangle + \Theta(t'-t)\langle 0|\hat{A}^\nu(\mathbf{x}',t')\hat{A}^\mu(\mathbf{x},t)|0\rangle . \tag{8.182}$$

Due to the radiation gauge (8.13) and (8.58) the zeroth component of the field operator $\hat{A}^\mu(\mathbf{x},t)$ vanishes, so only the spatial components of the Maxwell propagator can be non-zero:

$$D^{\mu\nu}(\mathbf{x},t;\mathbf{x}',t') = 0 \qquad \text{if either } \mu = 0 \text{ or } \nu = 0 . \tag{8.183}$$

In order to determine the equation of motion for the spatial components of the Maxwell propagator we evaluate initially their first temporal partial derivative. To this end we take into account (7.125) as well as (8.74) and get from (8.182)

$$\begin{aligned} \frac{\partial D^{jk}(\mathbf{x},t;\mathbf{x}',t')}{\partial t} &= \Theta(t-t')\left\langle 0 \left| \frac{\partial \hat{A}_j(\mathbf{x},t)}{\partial t}\hat{A}_k(\mathbf{x}',t') \right| 0 \right\rangle \\ &\quad + \Theta(t'-t)\left\langle 0 \left| \hat{A}_k(\mathbf{x}',t')\frac{\partial \hat{A}_j(\mathbf{x},t)}{\partial t} \right| 0 \right\rangle . \end{aligned} \tag{8.184}$$

A subsequent time derivative then yields by applying (7.125), (8.88), (8.93), and (8.99)

$$\begin{aligned} \frac{\partial^2 D^{jk}(\mathbf{x},t;\mathbf{x}',t')}{\partial t^2} &= \frac{-i\hbar}{\epsilon_0}\,\delta(t-t')\,\delta^T_{jk}(\mathbf{x}-\mathbf{x}') \\ &\quad + c^2\Delta\Big[\Theta(t-t')\,\langle 0|\hat{A}_j(\mathbf{x},t)\hat{A}_k(\mathbf{x}',t')|0\rangle + \Theta(t'-t)\,\langle 0|\hat{A}_k(\mathbf{x}',t')\hat{A}_j(\mathbf{x},t)|0\rangle\Big] . \end{aligned} \tag{8.185}$$

From (8.182) and (8.185) we then obtain the result that the Maxwell propagator represents the Green function of the wave equation

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right) D^{jk}(\mathbf{x},t;\mathbf{x}',t') = -i\hbar\mu_0\,\delta^T_{jk}(\mathbf{x}-\mathbf{x}')\delta(t-t') . \tag{8.186}$$

We remark that not the delta function but the transversal delta function appears at the right-hand side of the inhomogeneous wave equation (8.186) due to the chosen Coulomb gauge. Therefore, one calls $D^{jk}(\mathbf{x},t;\mathbf{x}',t')$ more specifically to be the transveral Maxwell propagator.

8.17 Calculation of Maxwell Propagator

In order to further evaluate the spatial components of the Maxwell propagator (8.182), we insert the plane wave decomposition (8.155) for the field operator $\hat{\mathbf{A}}(\mathbf{x},t)$ and use the commutation relation (8.154):

$$\begin{aligned} D^{jk}(\mathbf{x},t;\mathbf{x}',t') &= \sum_{\lambda=\pm 1}\int d^3k\,\frac{\hbar}{2(2\pi)^3\epsilon_0\omega_{\mathbf{k}}}\,\Big\{\Theta(t-t')\epsilon_j(\mathbf{k},\lambda)\epsilon_k(\mathbf{k},\lambda)^*\,e^{i[\mathbf{k}(\mathbf{x}-\mathbf{x}')-\omega_{\mathbf{k}}(t-t')]} \\ &\quad +\Theta(t'-t)\epsilon_k(\mathbf{k},\lambda)\epsilon_j(\mathbf{k},\lambda)^*\,e^{-i[\mathbf{k}(\mathbf{x}-\mathbf{x}')-\omega_{\mathbf{k}}(t-t')]}\Big\}\,. \end{aligned} \tag{8.187}$$

Performing in the second term the substitution $\lambda\to-\lambda$, this reduces due to (8.145) to

$$\begin{aligned} D^{jk}(\mathbf{x},t;\mathbf{x}',t') = \int d^3k\,\frac{\hbar}{2(2\pi)^3\epsilon_0\omega_{\mathbf{k}}}\,P^{jk}(\mathbf{k})\,\Big\{&\Theta(t-t')e^{i[\mathbf{k}(\mathbf{x}-\mathbf{x}')-\omega_{\mathbf{k}}(t-t')]} \\ &+\Theta(t'-t)e^{-i[\mathbf{k}(\mathbf{x}-\mathbf{x}')-\omega_{\mathbf{k}}(t-t')]}\Big\}\,. \end{aligned} \tag{8.188}$$

Here we have introduced the polarization sum

$$P^{jk}(\mathbf{k}) = \sum_{\lambda=\pm 1}\epsilon_j(\mathbf{k},\lambda)\epsilon_k(\mathbf{k},\lambda)^*\,, \tag{8.189}$$

which is symmetric with respect to the wave vector according to (8.145)

$$P^{jk}(-\mathbf{k}) = P^{jk}(\mathbf{k})\,. \tag{8.190}$$

The latter symmetry property allows to simplify (8.188) further by performing the substitution $\mathbf{k}\to-\mathbf{k}$ in the second term, yielding

$$D^{jk}(\mathbf{x},t;\mathbf{x}',t') = \int d^3k\,\frac{\hbar P^{jk}(\mathbf{k})}{2(2\pi)^3\epsilon_0\omega_{\mathbf{k}}}\,e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}\,\Big\{\Theta(t-t')e^{-i\omega_{\mathbf{k}}(t-t')}+\Theta(t'-t)e^{i\omega_{\mathbf{k}}(t-t')}\Big\}\,. \tag{8.191}$$

Now we evaluate the polarization sum (8.189) explicitly by taking into account the polar coordinate representations for both the wave vector in (8.128) and the polarization vectors in (8.135). This yields

$$\big(P^{jk}(\mathbf{k})\big) = \begin{pmatrix} 1-k_x^2/\mathbf{k}^2 & -k_xk_y/\mathbf{k}^2 & -k_xk_z/\mathbf{k}^2 \\ -k_xk_y/\mathbf{k}^2 & 1-k_y^2/\mathbf{k}^2 & -k_yk_z/\mathbf{k}^2 \\ -k_xk_z/\mathbf{k}^2 & k_yk_z/\mathbf{k}^2 & 1-k_z^2/\mathbf{k}^2 \end{pmatrix}\,, \tag{8.192}$$

which is concisely summarized by

$$P^{jk}(\mathbf{k}) = \delta_{jk} - \frac{k_jk_k}{|\mathbf{k}|^2}\,. \tag{8.193}$$

With this we read off the transversality property of the polarization sum

$$k_jP^{jk}(\mathbf{k}) = 0\,, \tag{8.194}$$

which implies the corresponding transversality property of the Maxwell propagator (8.191)

$$\partial_j D^{jk}(\mathbf{x}, t; \mathbf{x}', t') = 0\,. \tag{8.195}$$

Due to this transversality property, which originally stems from having chosen the Coulomb gauge, the transversal Maxwell propagator (8.191) is not Lorentz invariant. Therefore, we aim now for decomposing the transversal Maxwell propagator into a Lorentz invariant and a Lorentz non-invariant contribution.

8.18 Four-Dimensional Fourier Representation

To this end we rewrite at first the three-dimensional Fourier representation of the Maxwell propagator in (8.191) in terms of a four-dimensional Fourier representation by using an integral identity, which is analogous to one obtained in (7.163) and (7.167):

$$\lim_{\eta\downarrow 0}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\frac{e^{-i\omega(t-t')}}{\omega^2-\omega_{\mathbf{k}}^2+i\eta} = \frac{-i}{2\omega_{\mathbf{k}}}\left[\Theta(t-t')e^{-i\omega_{\mathbf{k}}(t-t')}+\Theta(t'-t)e^{i\omega_{\mathbf{k}}(t-t')}\right]. \tag{8.196}$$

With this we obtain

$$D^{jk}(\mathbf{x}, t; \mathbf{x}', t') = \lim_{\eta\downarrow 0}\int\frac{d^3k}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\frac{i\hbar}{\epsilon_0}\,P^{jk}(\mathbf{k})\,\frac{e^{i[\mathbf{k}(\mathbf{x}-\mathbf{x}')-\omega(t-t')]}}{\omega^2-\omega_{\mathbf{k}}^2+i\eta}\,. \tag{8.197}$$

Note that the four-dimensional Fourier representation of the Maxwell propagator (8.197) solves evidently the equation of motion (8.186) by taking into account (8.5), (8.87), and (8.193). Introducing the contravariant four-wave vector

$$\left(k^\lambda\right) = \left(k^0, \mathbf{k}\right) = (\omega/c, \mathbf{k}) \tag{8.198}$$

and edging the spatial components of the Maxwell propagator with zeros, we deduce from (8.197) by taking into account the dispersion (8.102)

$$D^{\mu\nu}(x^\lambda; x'^\lambda) = \lim_{\eta\downarrow 0}\int\frac{d^4k}{(2\pi)^4}\,\frac{i\hbar}{c\epsilon_0}\,\frac{e^{-ik_\lambda(x^\lambda-x'^\lambda)}}{k_\lambda k^\lambda+i\eta}\,P^{\mu\nu}(k^\lambda)\,, \tag{8.199}$$

where the polarization sum does not explicitly depend on k^0:

$$P^{\mu\nu}(k^\lambda) = -g^{\mu\nu} + \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & -k_jk_k/\mathbf{k}^2 \end{pmatrix}^{\mu\nu}\,. \tag{8.200}$$

This polarization sum projects due to the transversality property (8.194) into the two-dimensional subspace perpendicular to $(0, \mathbf{k})$. But this projection is not covariant as the zeroth component of the four-vector potential vanishes due to the radiation gauge (8.13) and (8.58). In order to investigate the non-covariance of the polarization sum and, thus, of the transversal Maxwell propagator, in more detail we introduce the time-like vector

$$(\xi^\lambda) = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix} \tag{8.201}$$

and a space-like vector perpendicular to it

$$(\bar{k}^\lambda) = \begin{pmatrix} 0 \\ \mathbf{k}/|\mathbf{k}| \end{pmatrix} . \tag{8.202}$$

An explicit calculation then yields the following decomposition:

$$\bar{k}^\lambda = \frac{k^\lambda - (k\xi)\xi^\lambda}{\sqrt{(k\xi)^2 - k^2}} . \tag{8.203}$$

From (8.200)–(8.203) we then obtain for the polarization sum

$$P^{\mu\nu}(k^\lambda) = -g^{\mu\nu} - k^2 \frac{\xi^\mu \xi^\nu}{(k\xi)^2 - k^2} - \frac{k^\mu k^\nu - (k\xi)(k^\mu \xi^\nu + k^\nu \xi^\mu)}{(k\xi)^2 - k^2} . \tag{8.204}$$

All terms, which contain the time-like vector ξ, are not covariant. Inserting the polarization sum (8.204) into (8.199) the transversal Maxwell propagator decomposes into three terms:

$$D^{\mu\nu}(x;x') = D_F^{\mu\nu}(x;x') - D_C^{\mu\nu}(x;x') - D_R^{\mu\nu}(x;x') . \tag{8.205}$$

The first term is the covariant Maxwell propagator of Feynman

$$D_F^{\mu\nu}(x;x') = \lim_{\eta \downarrow 0} \int \frac{d^4k}{(2\pi)^4} \frac{\hbar}{c\epsilon_0} \frac{ig^{\mu\nu}}{k^2 + i\eta} e^{-ik(x-x')} , \tag{8.206}$$

which also follows from the Gupta-Bleuler quantization of the electromagnetic field. Later on, when we discuss the perturbative calculation of quantum electrodynamic processes, it turns out that the Maxwell propagator in Feynman diagrams can be identified without loss of generality with (8.206). The other two non-covariant terms in the transversal Maxwell propagator (8.205) turn out to not contribute to any physical result. The second term reads

$$D_C^{\mu\nu}(\mathbf{x},t;\mathbf{x}',t') = \lim_{\eta \downarrow 0} \int \frac{d^4k}{(2\pi)^4} \frac{i\hbar}{c\epsilon_0} \frac{k^2 \xi^\mu \xi^\nu}{(k\xi)^2 - k^2} \frac{e^{-ik(x-x')}}{k^2 + i\eta} . \tag{8.207}$$

Note that (8.207) reduces due to (8.6), (8.198), and (8.201) to

$$D_C^{\mu\nu}(\mathbf{x},t;\mathbf{x}',t') = \frac{i\hbar\mu_0}{4\pi} \delta^{\mu 0}\delta^{\nu 0} \frac{\delta(t-t')}{|\mathbf{x}-\mathbf{x}'|} . \tag{8.208}$$

With this we conclude that this contribution of the transversal Maxwell propagator is instantaneous and couples exclusively to the zeroth component of the four-current density, i.e. the charge density. And the third residual term in (8.205) reads

$$D_R^{\mu\nu}(x;x') = \lim_{\eta \downarrow 0} \int \frac{d^4k}{(2\pi)^4} \frac{i\hbar}{c\epsilon_0} \frac{k^\mu k^\nu - (k\xi)(k^\mu \xi^\nu + \xi^\mu k^\nu)}{(k\xi)^2 - k^2} \frac{e^{-ikx}}{k^2 + i\eta} . \tag{8.209}$$

It contains contributions, which are proportional to either k^μ or k^ν. As the electromagnetic field couples to four-current densities, which fulfill the continuity equation (8.33), we have in Fourier space

$$j_\mu(k) k^\mu = 0 . \tag{8.210}$$

Therefore the integral over $D_R^{\mu\nu}(x;x')$ contracted with conserved currents $j_\mu^{(1)}(x)$ and $j_\nu^{(2)}(x')$ produces a vanishing result:

$$\int d^4x \int d^4x' \, j_\mu^{(1)}(x) D_R^{\mu\nu}(x^\lambda; x'^\lambda) j_\nu^{(2)}(x') = \int \frac{d^4k}{(2\pi)^4} \, j_\mu(-k) D_R^{\mu\nu}(k) j_\nu(k) = 0 \,. \tag{8.211}$$

Later on we demonstrate explicitly by discussing the concrete example of a scattering process that both contributions (8.207) and (8.209) of the transversal Maxwell propagator do, indeed, not contribute to any observable quantity like the cross section.