Quantum Mechanics II — Problem Sheet 12

Problem 33: Klein Paradox for Dirac Equation (24 points)

The Dirac equation describing a spin 1/2-particle in a static potential $V(\mathbf{x})$ can be written in the form of a Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}\psi(\mathbf{x},t) = \Big[- ic\hbar\, \boldsymbol{\alpha}\boldsymbol{\nabla} + Mc^2\beta + V(\mathbf{x})\Big]\psi(\mathbf{x},t)\,, \tag{1}$$

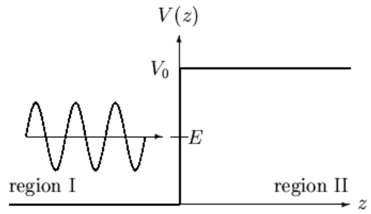
which determines the 4-component Dirac spinor $\psi(\mathbf{x},t) = \Big[\psi_1(\mathbf{x},t), \psi_2(\mathbf{x},t), \psi_3(\mathbf{x},t), \psi_4(\mathbf{x},t)\Big]^T$. Here we have introduced the 4×4-matrices

$$\alpha^k = \begin{pmatrix} O & \sigma^k \\ \sigma^k & O \end{pmatrix}, \qquad \beta = \begin{pmatrix} I & O \\ O & -I \end{pmatrix}, \tag{2}$$

where σ^k stands for the Pauli matrices as well as O and I represent the 2×2 zero and unit matrix, respectively :

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3}$$

In the following you investigate the situation that the spin 1/2-particle propagates in z-direction and hits a potential step of strength V_0:



a) Assume that the incoming wave in region I is given by $E > 0$ and $p > 0$ as well as

$$\psi_{\rm i} = u_{\rm i} e^{i(pz-Et)/\hbar}\,. \tag{4}$$

Which equation fulfills the spinor $u_{\rm i}$? Conclude from the requirement $u_{\rm i} \neq 0$ that E and p satisfy the relativistic energy-momentum dispersion.

b) The reflected wave in region I must have a momentum $-p$, whereas the transmitted wave in region II has a momentum $\overline{p} > 0$. With this the respective wave functions read

$$\psi_{\rm r} = u_{\rm r} e^{i(-pz-Et)/\hbar}\,, \qquad \psi_{\rm t} = u_{\rm t} e^{i(\overline{p}z-Et)/\hbar}\,. \tag{5}$$

Write down the resulting equations for the spinors $u_{\rm r}$ and $u_{\rm t}$. Conclude from the requirement $u_{\rm t} \neq 0$ the dispersion relation between E and $\overline{p}$.

c) The total wave function must be continuous at the boundary, i.e. for $z = 0$. Show that this leads to the condition $u_{\rm i} + u_{\rm r} = u_{\rm t}$.

d) Conclude from **a)**–**c)** that the amplitudes u_r and u_i are proportional to each other according to

$$u_\mathrm{r} = \frac{2V_0}{c} \frac{-E/c + \alpha_z p}{V_0^2/c^2 - (p+\overline{p})^2}\, u_\mathrm{i}\,. \tag{6}$$

e) Show that the fraction R of spin 1/2-particles, which are reflected, follows to be given by

$$u_\mathrm{r}^\dagger u_\mathrm{r} = R u_\mathrm{i}^\dagger u_\mathrm{i}\,, \qquad R = \left[\frac{2V_0 M}{V_0^2/c^2 - (p+\overline{p})^2}\right]^2 . \tag{7}$$

Which result do you get for R in the respective cases $V_0 = 0$ and $V_0 = E - Mc^2$?

f) If V_0 increases still further, i.e. $V_0 > E - Mc^2$, then $\overline{p}$ becomes imaginary and we set

$$\psi_\mathrm{t} = u_\mathrm{t} e^{-\mu z - iEt/\hbar}\,, \tag{8}$$

where μ is now real. Why must μ be greater than zero? Show that $u_\mathrm{r}^\dagger u_\mathrm{r} = u_\mathrm{i}^\dagger u_\mathrm{i}$ holds, i.e. the reflected current is equal to the incoming one. Discuss μ for increasing V_0 more and more. At which value V_0 does the maximal value of μ occur and at which V_0 does μ vanish again for further increasing V_0?

g) Consider now the case $V_0 > E + Mc^2$. Then $\overline{p}$ becomes real again, so that the above result (7) holds. Which values do you now get for R? What is the Klein paradox?

Drop the solutions in the post box on the 5th floor of building 46 or, in case of illness/quarantine, send them via email to jkrauss@rhrk.uni-kl.de until February 6 at 12.00.