TU KAISERSLAUTERN
Department of Physics

Winter Term 2022/2023
Priv.-Doz. Dr. Axel Pelster

Quantum Mechanics II — Problem Sheet 2

Problem 3: Feynman-Hellmann Theorem (2 points)

For a given system we assume that the Hamilton operator $\hat{H}(\lambda)$, its eigenvalues $E_n(\lambda)$, and the normalized eigenstates $|\psi_n(\lambda)\rangle$ depend on some parameter λ, so we have

$$\hat{H}(\lambda)|\psi_n(\lambda)\rangle = E_n(\lambda)|\psi_n(\lambda)\rangle \,. \tag{1}$$

Proof that then the following relationship holds:

$$\left\langle \psi_n(\lambda) \left| \frac{\partial \hat{H}(\lambda)}{\partial \lambda} \right| \psi_n(\lambda) \right\rangle = \frac{\partial E_n(\lambda)}{\partial \lambda} \,. \tag{2}$$

Problem 4: Expectation Values for Hydrogen Atom (6 points)

Consider the eigenfunctions of the hydrogen atom $\psi_{nlm}(\mathbf{x})$ and determine their expectation values

$$\int d^3x \, \frac{\hbar c \alpha}{|\mathbf{x}|} \, |\psi_{nlm}(\mathbf{x})|^2 = -2E_n \,, \qquad \int d^3x \, \frac{(\hbar c \alpha)^2}{|\mathbf{x}|^2} \, |\psi_{nlm}(\mathbf{x})|^2 = \frac{8nE_n^2}{2l+1} \,, \tag{3}$$

where $E_n = -Mc^2\alpha^2/(2n^2)$ denotes the hydrogen atom eigenenergies with the Sommerfeld fine structure constant α by applying the Feynman-Hellmann Theorem of **Problem 3**. **Hint:** The principal quantum number $n = n_r + l + 1$ decomposes into the radial quantum number n_r and the angular quantum number l.

Problem 5: Relativistic Corrections for Hydrogen Atom (6 points)

Expand the special relativistic kinetic energy $T = \sqrt{\mathbf{p}^2c^2 + M^2c^4}$ in a Taylor series of $|\mathbf{p}|/(Mc) \ll 1$:

$$T = Mc^2 + \frac{\mathbf{p}^2}{2M} - \frac{\mathbf{p}^4}{8M^3c^2} + \ldots \,. \tag{4}$$

Here the first term denotes the rest energy and only leads to a shift of the energy, the second term stands for the usual non-relativistic kinetic energy, and the third term represents the leading relativistic correction.

a) Argue why non-degenerate perturbation theory can be used.

b) Determine the first-order relativistic correction to the energy eigenvalues of the hydrogen atom. **Hint:** Use the results of **Problem 4**.

Problem 6: Fourth-Order Non-Degenerate Perturbation Theory (10 points)

Determine with the notation of the lecture the general formula for the corrections of the energy eigenvalues of a quantum mechanical system up to fourth order in non-degenerate perturbation theory.

Drop the solutions in the post box on the 5th floor of building 46 or, in case of illness/quarantine, send them via email to jkrauss@rhrk.uni-kl.de until November 7 at 11.00.