Quantum Mechanics II — Problem Sheet 6

Problem 15: Central Potential (6 points)

Investigate the s-wave scattering at the central potential $V(r) = -\hbar^2\lambda^2/M\cosh^2(\lambda r)$. Determine both the scattering shift δ_0 and the total cross-section for small particle energies. **Hint:** The general solution of the radial Schrödinger equation for $l = 0$ reads

$$R_0(r) = \frac{A\,[\lambda\tanh(\lambda r) - ik]\,e^{ikr} + B\,[\lambda\tanh(\lambda r) + ik]\,e^{-ikr}}{r}\,, \tag{1}$$

where A, B denote constants.

Problem 16: Delta Function (8 points)

Work out the s-wave scattering at the potential $V(r) = \alpha\,\delta(r-a)$ for small energies. Express both the scattering phase and the cross-section via the dimensionless parameter $\beta = 2Ma\alpha/\hbar^2$. **Hint:** Use for the solution of the corresponding Schrödinger equation for $l = 0$ the ansatz $u_0(r) = rR_0(r)$.

Problem 17: Fermi Pseudopotential (10 points)

Consider at first the scattering of a particle of mass μ by the Fermi pseudopotential $U(\mathbf{x})$, which acts on the wave function $\psi(\mathbf{x})$ according to

$$U(\mathbf{x})\psi(\mathbf{x}) = \frac{2\pi\hbar^2 a_\mathrm{s}}{\mu}\,\delta(\mathbf{x})\,\frac{\partial}{\partial r}\Big[r\,\psi(\mathbf{x})\Big] \tag{2}$$

with $r = |\mathbf{x}|$. Here the parameter a_s denotes the s-wave scattering length, which characterizes the scattering properties of the Fermi pseudopotential $U(\mathbf{x})$.

a) Write down the stationary Schrödinger equation for the corresponding scattering problem and obtain its exact solution. **Hint:** Define the quantity

$$A = \frac{\partial}{\partial r}\Big[r\,\psi(\mathbf{x})\Big]\Big|_{\mathbf{x}=\mathbf{0}} \tag{3}$$

and determine it self-consistently.

b) Which result do you read off for the scattering amplitude? Determine both the differential and the total cross-section as a function of the incident energy $E = \hbar^2k^2/2\mu$.

Look now for the bound state of the Fermi pseudopotential $U(\mathbf{x})$.

c) Solve now exactly the stationary Schrödinger equation for the bound state along similar lines as **a)**. Determine for the bound state both the normalized wave function and the binding energy. **Hint:** The incident energy $E = \hbar^2k^2/2\mu$ of the above scattering problem is related to the bound-state energy $E = -\hbar^2\kappa^2/2\mu$ via the analytic continuation $k = i\kappa$.

Drop the solutions in the post box on the 5th floor of building 46 or, in case of illness/quarantine, send them via email to jkrauss@rhrk.uni-kl.de until December 5 at 12.00.