Quantum Mechanics II — Problem Sheet 8

Problem 23: Yukawa Potential (12 points)

We consider the scattering of a particle with mass μ at a weak Yukawa potential

$$V(\mathbf{x}) = V_0 \, \frac{a}{|\mathbf{x}|} \, e^{-|\mathbf{x}|/a} \, . \tag{1}$$

a) Show that the scattering amplitude reads in first Born approximation:

$$f^{(1)}(\vartheta) = -\frac{2\mu V_0 a^3}{\hbar^2} \, \frac{1}{2(ka)^2(1-\cos\vartheta)+1} \, . \tag{2}$$

b) Express the phase shifts δ_l for a weak Yukawa potential in terms of the Legendre functions of the second kind

$$Q_l(x) = \frac{1}{2} \int_{-1}^{1} dx' \, \frac{P_l(x')}{x-x'} \, . \tag{3}$$

Hint: In case of a rotationally invariant scattering potential the scattering amplitude $f(\vartheta)$ has the partial wave decomposition

$$f(\vartheta) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) e^{i\delta_l} \sin\delta_l P_l(\cos\vartheta) \, , \tag{4}$$

where $P_l(x)$ stands for the Legendre functions of the first kind, which fulfill the orthonormality relation

$$\int_{-1}^{1} dx \, P_l(x) P_{l'}(x) = \frac{2}{2l+1} \, \delta_{l,l'} \, . \tag{5}$$

Furthermore, δ_l denotes the scattering phases and a weak scattering potential implies $\delta_l \ll 1$.

c) Show that the scattering phases for an attractive (repulsive) Yukawa potential is positive (negative) and determine the scattering phases δ_l in the limit of low energies, i.e. $ka \ll 1$. Explain heuristically why in this limit δ_l falls off rapidly with l, so that the dominant contribution stems from s-wave scattering.
Hint: The Legendre functions of the second kind have the following expansion

$$Q_l(x) = \sum_{n=0}^{\infty} \frac{(l+2n)!}{(2n)!!(2l+2n+1)!!} \, \frac{1}{x^{l+2n+1}} \, , \qquad |x| > 1 \, . \tag{6}$$

d) Use the partial wave decomposition (4) in order to express the total cross-section $\sigma = \int d\Omega |f(\vartheta)|^2$ in terms of the scattering phases δ_l. Check explicitly that the optical theorem

$$\sigma = \frac{4\pi}{k} \, \mathrm{Im} f(\vartheta = 0) \tag{7}$$

yields the same result. Define the scattering length via $\sigma = 4\pi a^2$ and show that the scattering length is given by

$$a = -\lim_{k\to 0} \frac{\delta_0}{k} \, . \tag{8}$$

Determine with this a in case of the weak Yukawa potential by taking the result from **c)** into account.

Problem 24: Eikonal Approximation (12 points)

The Born approximation of the scattering amplitude assumes that the scattering potential $V(\mathbf{x})$ is weak enough so that it can be treated as a perturbation. Here we consider instead the scattering in the semiclassical Wentzel-Kramers-Brillouin (WKB) approximation, i.e. we assume that $V(\mathbf{x})$ varies slowly on the scale of the de Broglie wave length:

$$\frac{|\nabla V(\mathbf{x})|}{E - V(\mathbf{x})} \ll \frac{p(\mathbf{x})}{\hbar}, \qquad p(\mathbf{x}) = \sqrt{2\mu[E - V(\mathbf{x})]}. \tag{9}$$

a) Solve the time-independent Schrödinger equation

$$\left[-\frac{\hbar^2}{2\mu}\Delta + V(\mathbf{x})\right]\psi(\mathbf{x}) = E\psi(\mathbf{x}) \tag{10}$$

with the ansatz $\psi(\mathbf{x}) = e^{iS(\mathbf{x})/\hbar}$ and determine the resulting differential equation for the eikonal $S(\mathbf{x})$. Expand then $S(\mathbf{x})$ with respect to $\hbar$, i.e. $S(\mathbf{x}) = S_0(\mathbf{x}) - i\hbar S_1(\mathbf{x}) + \ldots$ and show that the classical action $S_0(\mathbf{x})$ obeys the Hamilton-Jakobi equation $[\nabla S_0(\mathbf{x})]^2 = p(\mathbf{x})^2$.

b) Solve the Hamilton-Jakobi equation for the case that the energy $E = \hbar^2 k^2/2\mu$ is much larger than the strength $|V|$ of the scattering potential $V(\mathbf{x})$. Then you can assume that the classical trajectory is a straight line $\mathbf{x} = \mathbf{b} + z\mathbf{e}_z$ with the impact parameter b and $\mathbf{b} \perp \mathbf{e}_z$. Why is it justified to choose the integration constant such that $S_0(\mathbf{x})/\hbar \to kz$ in the limit of a vanishing potential $V(\mathbf{x})$? Show that the wave function turns out to have the form

$$\psi(\mathbf{x}) \approx \exp\left[ikz - i\frac{\mu}{\hbar^2 k}\int_{-\infty}^{z} dz'\, V(\mathbf{b} + z'\mathbf{e}_z)\right]. \tag{11}$$

c) The scattering amplitude for a rotationally symmetric potential $V(\mathbf{x}) = V(|\mathbf{x}|)$ reads

$$f(\vartheta) = -\frac{\mu}{2\pi\hbar^2}\int d^3x'\, e^{-i\mathbf{k}'\mathbf{x}'}V(\mathbf{x}')\psi(\mathbf{x}), \qquad \mathbf{k}' = k\,\frac{\mathbf{x}'}{|\mathbf{x}|}. \tag{12}$$

Evaluate the integral in (12) by using cylinder coordinates and show that the scattering amplitude reduces in the eikonal approximation (11) to the expression

$$f(\vartheta) = -ik\int_0^\infty db\, b\, J_0(kb\vartheta)\left[e^{2i\Delta(b)} - 1\right], \qquad \Delta(b) = -\frac{\mu}{2\hbar^2 k}\int_{-\infty}^{\infty} dz\, V\left(\sqrt{b^2 + z^2}\right). \tag{13}$$

Hint: The Bessel function has the integral representation $J_0(x) = \int_0^{2\pi} d\varphi/(2\pi)\, e^{-ix\cos\varphi}$.

d) Evaluate the partial wave decomposition (4) in the classical limit of large energies and, i.e. large wave vectors k. Why does then hold $l \approx bk$? Thus, in that limit the discrete angular quantum number l gets continuous so that the sum over l in (4) can approximately be evaluated by an integral with respect to b. Use furthermore that $P_l(\cos\vartheta) \approx J_0(l\vartheta)$ holds for large l and small ϑ and derive in comparison with (13) the eikonal approximation for the scattering phases

$$\delta_l = \left|\Delta(b)\right|_{b=l/k}. \tag{14}$$

e) Evaluate (14) for the Yukawa potential (1) and compare your result with the corresponding one of **Problem 23 b)**.

Drop the solutions in the post box on the 5th floor of building 46 or, in case of illness/quarantine, send them via email to jkrauss@rhrk.uni-kl.de until January 2 at 12.00.