TU KAISERSLAUTERN — Winter Term 2022/2023

Department of Physics — Priv.-Doz. Dr. Axel Pelster

Quantum Mechanics II — Problem Sheet 9

Problem 25: Path Integral for Harmonic Oscillator on Finite Time Lattice (24 points)

The path integral for a one-dimensional harmonic oscillator of mass M and frequency ω is defined in the continuum limit of a vanishing time lattice spacing according to

$$G(x_b,t_b;x_a,t_a) = \lim_{\substack{\varepsilon \to 0 \\ N \to \infty \\ \varepsilon N = t_b - t_a}} G_N^{(\varepsilon)}(x_b,t_b;x_a,t_a)\,, \tag{1}$$

where the discretized path integral on an equidistant time lattice with spacing $\varepsilon = (t_b - t_a)/N$ reads

$$G_N^{(\varepsilon)}(x_b,t_b;x_a,t_a) = \int d^{N-1}x \left(\frac{M}{2\pi i\hbar\varepsilon}\right)^{N/2} \exp\left\{\frac{i}{\hbar}\,\varepsilon \sum_{j=0}^{N-1}\left[\frac{M}{2}\left(\frac{x_{j+1}-x_j}{\varepsilon}\right)^2 - \frac{M}{2}\,\omega^2 x_j^2\right]\right\}. \tag{2}$$

a) Rewrite (2) in the form of the $(N-1)$-dimensional Fresnel integral

$$G_N^{(\varepsilon)}(x_b,t_b;x_a,t_a) = \int d^{N-1}x \left(\frac{M}{2\pi i\hbar\varepsilon}\right)^{N/2} \exp\left[-\frac{M}{2i\hbar\varepsilon}\,\left(\mathbf{x}^{\mathrm{T}}A\mathbf{x} - 2\mathbf{b}^{\mathrm{T}}\mathbf{x} + c\right)\right] \tag{3}$$

and show that A is the $(N-1)\times(N-1)$ tridiagonal band matrix

$$A = \begin{pmatrix} a & b & 0 & \dots & 0 & 0 & 0 \\ b & a & b & \dots & 0 & 0 & 0 \\ 0 & b & a & & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b & 0 \\ 0 & 0 & 0 & \cdots & b & a & b \\ 0 & 0 & 0 & \cdots & 0 & b & a \end{pmatrix}. \tag{4}$$

Determine the coefficients a and b as well as the vector $\mathbf{b}$ and the constant c.

b) Calculate the D-dimensional Gauß integral

$$I = \int d^D x \,\exp\left(-\frac{1}{2}\,\mathbf{x}^{\mathrm{T}}A\mathbf{x} + \mathbf{b}^{\mathrm{T}}\mathbf{x}\right) \tag{5}$$

for a positive definite matrix A with the symmetry $A = A^{\mathrm{T}}$ and show that the result is given by

$$I = \sqrt{\frac{(2\pi)^D}{\mathrm{Det}\,A}}\,\exp\left(\frac{1}{2}\,\mathbf{b}^{\mathrm{T}}A^{-1}\mathbf{b}\right). \tag{6}$$

c) Calculate the determinant $D_{N-1} = \mathrm{Det}\,A$ along the following way.

i) Show that the determinant fulfills a recursion relation of the form

$$D_{N-1} = c_1(a,b)D_{N-2} + c_2(a,b)D_{N-3} \tag{7}$$

and determine the coefficients $c_1(a,b)$ and $c_2(a,b)$.

ii) Such a homogeneous difference equation of second order with constant coefficients is solved in close analogy to a homogeneous differential equation of second order with constant coefficients. Solve (7) therefore with the ansatz $D_{N-1} = \lambda^{N-1}$ and derive the characteristic equation for λ. Show that λ can have the two values $\lambda_\pm = e^{\pm i\varepsilon\tilde{\omega}}$. How does the lattice frequency $\tilde{\omega}$ depend on both ω and ε?

iii) The general solution of the difference equation (7) is given by

$$D_{N-1} = c_+ \lambda_+^{N-1} + c_- \lambda_-^{N-1} \,. \tag{8}$$

Determine the coefficients c_+ and c_- from the initial conditions D_1 and D_2.

iv) Show that the determinant follows as $D_{N-1} = \sin(\varepsilon\tilde{\omega}N)/\sin(\varepsilon\tilde{\omega})$.

d) Calculate the inverse of the $(N-1)\times(N-1)$ tridiagonal band matrix (4), which is defined as

$$\sum_{l=1}^{N-1} A_{kl} A^{-1}_{lm} = \delta_{k,m} \tag{9}$$

according to the following steps.

i) Consider first the case $k = 2, \ldots, m-1$ and solve (9) with the ansatz $A^{-1}_{lm} = \mu^l$. Which characteristic equation do you get for μ and what are its solutions?

ii) Solve (9) now for $k = 1, \ldots, m-1$ and show that the result reads $A^{-1}_{lm} = c^<_m \sin(\varepsilon\tilde{\omega}l)$.

iii) Proceed analogously to i), ii) and show that (9) for $k = m+1, \ldots, N-1$ is solved by $A^{-1}_{lm} = c^>_m \sin[\varepsilon\tilde{\omega}(N-l)]$.

iv) Having determined A^{-1}_{lm} both for $l < m$ in i), ii) and $l > m$ in iii), we now demand continuity for $l = m$. Which relation between $c^<_m$ and $c^>_m$ does this imply?

v) Determine the remaining unknown constant from (9) in case of $k = m$ and show that the inverse matrix of (4) is, finally, given by

$$A^{-1}_{lm} = \begin{cases} \dfrac{\sin[\varepsilon\tilde{\omega}(N-m)]\sin(\varepsilon\tilde{\omega}l)}{\sin(\varepsilon\tilde{\omega})\sin(\varepsilon\tilde{\omega}N)}\,; & l = 1, \ldots, m \\[2ex] \dfrac{\sin[\varepsilon\tilde{\omega}(N-l)]\sin(\varepsilon\tilde{\omega}m)}{\sin(\varepsilon\tilde{\omega})\sin(\varepsilon\tilde{\omega}N)}\,; & l = m, \ldots, N-1 \end{cases} \tag{10}$$

e) Combine all previous results and write down the propagator of the harmonic oscillator on a finite time lattice. Does $G^{(\varepsilon)}_{N=1}(x_b, t_b; x_a, t_a)$ reproduce the proper short-time propagator? Which result do you get in the continuum limit (1) for the long-time propagator?

Drop the solutions in the post box on the 5th floor of building 46 or, in case of illness/quarantine, send them via email to jkrauss@rhrk.uni-kl.de until January 9 at 12.00.