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SIMPLE SINGULARITIES OF REDUCIBLE CURVES IN CHARACTERISTIC $p > 0$

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ABSTRACT. Kolgushkin and Sadykov classified the simple singularities of multigerms of curves over the field of complex numbers [21]. We develop a similar classification over algebraically closed fields of characteristic $p > 0$.

1. Introduction

The study and the classification of singularities have a long history. Very important contributions go back to Zariski (cf. [28], [29]) and Arnold. Arnold introduced and studied the famos $A-D-E$-singularities (cf. [2], [20]). For the classification of curve singularities you can use two approaches. The first approach is using the defining equations for the classification. Here we have contributions by Arnold [2], Guisti [15], Wall [27] and many others ([22], [1],...). Most of the results were obtained over the complex numbers. Greuel and his students started a classification for hypersurface singularities in characteristic $p > 0$ (cf. [6],[7],[12], [14]). The second approach is to use the parametrization of a curve singularity for the classification. There are several contributions in characteristic $p = 0$. Bruce and Gaffney (cf. [8]) resp. Gibson and Hobbs (cf. [11]) classified the simple plane resp. space curve singularities. Arnold [3] gave a classification for simple curves in n-space. Kolgushkin and Sadykov [21] classified multigerms of simple parametrized curves. In characteristic $p > 0$ simple parametrized plane curves and space curves are classified in [26], [19].

The aim of our paper is to give similar to the work of Kolgushkin and Sadykov a classification in characteristic $p > 0$.

Let K be an algebraically closed field of characteristic p. In this paper, we classify the simple parameterized curves with two components, one of them is smooth with respect to $\mathcal{A}$ -equivalence.

Let $f_i : K[[x_1, \ldots, x_n]] \to K[[t]]$ define a germ of a parameterized curve singularity, $i = 1, \ldots, k$. Let $L = Aut_K(K[[x_1, \ldots, x_n]])$, $R = Aut_K(K[[t]])$ and $G = L \times R^k$. Let G act on the set $E = \{(f_1, \ldots, f_k) : f_i : K[[x_1, \ldots, x_n]] \to K[[t]], dim_K(K[[t]]/Im(f_i)) < \infty\}$ by

$$(g, (h_1, \ldots, h_k)) \circ (f_1, \ldots, f_k) = (h_1^{-1} \circ f_1 \circ g, \ldots, h_k^{-1} \circ f_k \circ g).$$

Definition 1.1. Let $(f_1, \ldots, f_k), (g_1, \ldots, g_k) \in E$. They are called $\mathcal{A}$ -equivalent if they are in the same orbit under the action of G. We will write in this case $(f_1, \ldots, f_k) \sim_{\mathcal{A}} (g_1, \ldots, g_k)$.

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Let us consider a special case. Let $f_1 = (t, 0, \dots, 0)$, $g_1 = (t, 0, \dots, 0)$, $f_2 = (x_1(t), \dots, x_n(t))$ and $g_2 = (y_1(t), \dots, y_n(t))$. Then $(f_1, f_2) \sim_{\mathcal{A}} (g_1, g_2)$ if and only if for a suitable $(g, (h_1, h_2)) \in G$

$$(g_1, g_2) = (h_1^{-1} \circ f_1 \circ g, h_2^{-1} \circ f_2 \circ g).$$

Let $g = (H_1(x_1, \dots, x_n), H_2(x_1, \dots, x_n), \dots, H_n(x_1, \dots, x_n))$ then we must have $H_i(x_1, 0, \dots, 0) = 0$, $i = 2, \dots, n$ and $\partial H_1/\partial x_1(0) \neq 0$. This implies that the classification of parameterized curves with two components, one of them smooth, is equivalent to the classification of simple irreducible curves with respect to the action of the following group $G^r = L^r \times R$ (the action is as above for $k = 1$) with

$$L^r = \{\varphi \in Aut_K(K[[x_1, \dots, x_n]]) : \varphi(x_i)(x_1, 0, \dots, 0) = 0, i = 2, \dots, n\}.$$

Definition 1.2. Let $f, g : K[[x_1, \dots, x_n]] \to K[[t]]$ define germs of irreducible curves. They are called restricted $\mathcal{A}$ -equivalent denoted by $f \sim_{\mathcal{A}_r} g$ if they are in the same orbit under the action of G^r.

Definition 1.3. A curve f is called simple (with respect to $\mathcal{A}_r$-equivalence) if there are only finitely many $\mathcal{A}_r$-equivalence classes in a deformation of f.

It is part of the work of Kolgushkin and Sadykov [21] to make a classification of simple irreducible curves under $\mathcal{A}_r$ -equivalence over a field of characteristic 0. The advantage here is that one can use for the classification the results of Mather (cf. [9] and [23]) using complete transversals. This theory is not available in characteristic $p > 0$. The idea of our classification is the following:

(1) Find special classes of non-simple curves [1] such that all non-simple curves have one of them represented in a suitable deformation.
(2) Especially curves with 5 -jet $(t^5, 0, \dots, 0)$ or $(0, 0, \dots, 0)$ are not simple.
(3) Find weak normal forms (see Definition 2.5) based on the semigroup and restricted semigroup (see Definition 2.1). The weak normal form can be obtained by only using the action of the group L^r and is independent of the characteristic.
(4) Use the weak normal form and the action of the group R (which depends on the special characteristic) to find normal forms.

Example 1.4. Curves $(w(t), x(t), y(t), z(t))$ with $ord_t(w(t)) = 3$, $ord_t(x(t)) = 6$, $ord_t(y(t)) = 7$ and $ord_t(z(t)) = 11$ are not simple (see Proposition 3.1).
Let us consider the plane curve defined by $(t^2, t^5 + t^8 + t^9)$. The semigroup is $\Gamma = \langle 2, , 5 \rangle = \{0, 2, 4, 5, \dots\}$ and the restricted semigroup (see Definition 2.1 is $\Gamma_r = \{0, 5, 7, 9, 10, \dots\}$. The curve is not in weak normal form (see Definition 2.6) since $9 \in \Gamma_r$, but it is $\mathcal{A}_r$ -equivalent to $(t^2, t^5 + t^8)$, a weak normal form. If the characteristic $p \neq 5$ we can use the automorphism φ of $K[[t]]$ defined by $\varphi(t) = t - \frac{1}{5}t^4$ to obtain the $\mathcal{A}_r$-equivalent curve (t^2, t^5), the normal form. In characteristic $p = 5$ we cannot do this and obtain as normal form $(t^2, t^5 + t^8)$.

As a the main result of the paper we obtain the following classification:

Theorem 1.5. *Let $(x_1(t), \dots, x_n(t))$ be a simple curve then it is $\mathcal{A}_r$-equivalent to one of the curves in the following tables.*

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[1] Here we use several times the computer algebra system SINGULAR , cf. [10], [13].

Table 1 $\Gamma =< 1 >$

Normal form	Conditions
$(0, t)$	
(t, t^a)	$a > 1$

Table 2 Characteristic $p \neq 2$, $\Gamma =< 2, a >$, $a > 2$, odd

Normal form	Conditions
$(0, t^2, t^a)$	
$(t^a, t^2, 0)$	
$(t^2, t^a + t^b, t^c)$	$b = \infty$ or b is even and $2 < b < 2a$ $c = \infty$ or $c \equiv \lfloor \frac{b}{a} \rfloor + 1 \bmod 2$ if $c < \infty$ and $b < \infty$ then $max(a, b) \leq c < a + min(a, b)$ $a < p + 6$ or $b < 6$ or $c < p + 10$

Table 3 Characteristic $p = 2$, $\Gamma =< 2, a >$, $a > 2$, odd

Normal form	Conditions
$(0, t^2 + t^b, t^a)$	$b = \infty$ or b odd and $2 < b < a$
$(t^a, t^2 + t^b, 0)$	$b = \infty$ or b odd and $2 < b < a + 2$
(t^2, t^3)	
$t^2 + \alpha t^3, t^4, t^5)$	$\alpha \in \{0, 1\}$

Table 4 Characteristic $p \neq 3$, $\Gamma =< 3, k, r >$, 3-jet $(0, t^3, 0 \dots, 0)$

Normal form	Conditions
$(0, t^3, t^k + t^b, t^r)$ $(t^k + t^b, t^3, t^r)$ $(t^r, t^3, t^k + t^b)$	$(r = \infty$ and $3 \nmid k)$ or $(k < r < 2k - 2$ and $k.r \equiv 2 \bmod 3)$ $b = \infty$ or $(k < b < r$ and $k.b \equiv 2 \bmod 3)$ $k < p + 9$ or $k < 2p + 9$ and $(b < p + k$ or $r \leq p + k$ in the first two cases) or $k < 2p + 9$ and ($b < p + k$ or $r + 3 \leq p + k$ in the last case) note that the conditions imply for $p = 2$ that $k \leq 10$

Table 5 Characteristic $p = 3$, $\Gamma =< 3, k, r >$, 3-jet $(0, t^3, 0 \dots, 0)$

Normal form	Conditions
$(0, t^3 + \alpha t^4 + \beta t^5, t^k, t^r)$ $(t^k, t^3 + \alpha t^4 + \beta t^5, t^r)$ $(t^r, t^3 + \alpha t^4, t^5)$	$k = 5$, $r \in \{7, \infty\}$, $\beta = 0$, $\alpha \in \{0, 1\}$ or $k = 7$, $r = 8$, $\alpha\beta = 0$, $\alpha, \beta \in \{0, 1\}$

Table 6 Characteristic $p \neq 2, 3$, $\Gamma =< 3, k, r >$, 3-jet $(t^3, 0 \ldots, 0)$

Normal form	Conditions
(t^3, t^4) $(t^3, t^4 + t^6)$ (t^3, t^4, t^6) (t^3, t^4, t^9) $(t^3, t^4 + t^6, t^9)$	$k = 4$, $r = \infty$
(t^3, t^4, t^5) (t^3, t^4, t^5, t^6)	$k = 4$, $r = 5$
(t^3, t^5, t^6, t^7)	$k = 5$, $r = 7$
(t^3, t^5) $(t^3, t^5 + t^6)$ $(t^3, t^5 + t^9)$ (t^3, t^5, t^6) $(t^3, t^5, t^6 + t^7)$ (t^3, t^5, t^9) $(t^3, t^5 + t^6, t^9)$ (t^3, t^5, t^{12}) $(t^3, t^5 + t^6, t^{12})$ $(t^3, t^5 + t^9, t^{12})$	$p \neq 5$ $k = 5$, $r = \infty$
(t^3, t^5, t^7) $(t^3, t^5 + t^6, t^7)$	$p \neq 5, 7$ $k = 5$, $r = 7$
(t^3, t^5, t^7, t^9) $(t^3, t^5 + t^6, t^7, t^9)$	$p \neq 5$, $k = 5$, $r = 7$
(t^3, t^6, t^7, t^8)	$p \neq 5$, $k = 7$, $r = 8$

Table 7 Characteristic $p = 3$, $\Gamma =< 3, k, r >$, 3-jet $(t^3, 0 \ldots, 0)$

Normal form	Conditions
$(t^3 + \alpha t^5, t^4, t^s)$	$k = 4$, $r = \infty$ $\alpha \in \{0, 1\}$, $s \in \{6, 9, \infty\}$
(t^3, t^4, t^5) (t^3, t^4, t^5, t^6)	$k = 4$, $r = 5$
$(t^3 + \alpha t^4 + \beta t^7, t^5, t^s)$	$k = 5$, $r = \infty$ $\alpha, \beta \in \{0, 1\}$, $\alpha\beta = 0$, $s \in \{6, 9, 12, \infty\}$
$(t^3 + \alpha t^4, t^5, t^7, t^s)$	$k = 5$, $r = 7$ $\alpha \in \{0, 1\}$, $s \in \{6, 9, \infty\}$
$(t^3 + \alpha t^4 + \beta t^5, t^6, t^7, t^8)$	$k = 7$, $r = 8$, $\alpha, \beta \in \{0, 1\}$, $\alpha\beta = 0$

Table 8 Characteristic $p = 2$, $\Gamma =< 3, k, r >$, 3-jet $(t^3, 0 \ldots, 0)$

Normal form	Conditions
(t^3, t^4, t^5) $(t^3, t^4 + t^6, t^5)$	$k = 4$, $r = 5$

Table 9 Characteristic $p \neq 2$, 4-jet $(t^4, 0 \ldots, 0)$

Normal form	Semigroup
(t^4, t^5, t^6, t^7)	$< 4, 5, 6, 7 >$
$(t^4, t^5, t^6, t^7, t^8)$	
additionally if $p = 3$	
$(t^4, t^5, t^6 + t^8, t^7)$	
additionally if $p = 5$	
$(t^4, t^5 + t^8, t^6, t^7)$	
additionally if $p = 7$	
$(t^4, t^5, t^6, t^7 + t^8)$	

Table 10 Characteristic $p \neq 2$, 4-jet $(0, t^4, 0 \ldots, 0)$

Normal form	Semigroup
$(0, t^4, t^5, t^6)$	$< 4, 5, 6 >$
additionally if $p = 3$	
$(0, t^4, t^5, t^6 + t^7)$	
additionally if $p = 5$	
$(0, t^4, t^5 + t^7, t^6)$	
$(0, t^4, t^5, t^7)$	$< 4, 5, 7 >$
additionally if $p = 5$	
$(0, t^4, t^5 + t^6, t^7)$	
$(0, t^4, t^6, t^7)$	$< 4, 6, 7 >$
additionally if $p = 3$	
$(0, t^4, t^6 + t^9, t^7)$	
additionally if $p = 7$	
$(0, t^4, t^6, t^7 + t^9)$	
$(0, t^4, t^5, t^6, t^7)$	$< 4, 5, 6, 7 >$
(t^5, t^4, t^6, t^7)	
(t^6, t^4, t^5, t^7)	
(t^7, t^4, t^5, t^6)	
additionally if $p = 3$	
$(t^7, t^4, t^5, t^6 + t^7)$	
additionally if $p = 5$	
$(t^6, t^4, t^5 + t^6, t^7)$	
$(t^7, t^4, t^5 + t^7, t^6)$	
$(0, t^4, t^6, t^7, t^9)$	$< 4, 6, 7, 9 >$
(t^9, t^4, t^6, t^7)	
additionally if $p = 3$	
$(t^9, t^4, t^6 + t^9, t^7)$	
additionally if $p = 7$	
$(t^9, t^4, t^6, t^7 + t^9)$	

Table 11 Characteristic $p \neq 2$, 5-jet $(0, t^5, 0 \ldots, 0)$

Normal form	Semigroup
$(0, t^5, t^6, t^7, t^8)$ additionally if $p = 3$ $(0, t^5, t^6 + t^9, t^7, t^8)$ additionally if $p = 5$ $(0, t^5 + t^9, t^6, t^7, t^8)$ additionally if $p = 7$ $(0, t^5, t^6, t^7 + t^9, t^8)$	$< 5, 6, 7, 8 >$
$(0, t^5, t^6, t^7, t^9)$ additionally if $p = 3$ $(0, t^5, t^6 + t^8, t^7, t^9)$ additionally if $p = 5$ $(0, t^5 + t^8, t^6, t^7, t^9)$ additionally if $p = 7$ $(0, t^5, t^6, t^7 + t^8, t^9)$	$< 5, 6, 7, 9 >$
$(0, t^5, t^6, t^7, t^8, t^9)$ $(t^9, t^5, t^6, t^7, t^8)$ $(t^8, t^5, t^6, t^7, t^9)$ additionally if $p = 3$ $(t^8, t^5, t^6 + t^8, t^7, t^9)$ $(t^9, t^5, t^6 + t^9, t^7, t^8)$ additionally if $p = 5$ $(t^9, t^5 + t^9, t^6, t^7, t^8)$ $(t^8, t^5 + t^8, t^6, t^7, t^9)$ additionally if $p = 7$ $(t^9, t^5, t^6, t^7 + t^9, t^8)$ $(t^8, t^5, t^6, t^7 + t^8, t^9)$	$< 5, 6, 7, 8, 9 >$

2. Weak Normal Form

In this section we will define a weak normal form of a parametrization with respect to $\mathcal{A}_r$ -equivalence and compute it for multiplicity smaller or equal to five. This will be the basis of our classification, since it is independent of the characteristic p.

Definition 2.1. Let $(x_1(t), \ldots, x_n(t))$ be a curve. Define the semigroup

$$\Gamma = \Gamma(K[[x_1(t), \ldots, x_n(t)]]) = \{ord_t(f) : f \in K[[x_1(t), \ldots, x_n(t)]], f \neq 0\}$$

and the restricted semigroup

$$\Gamma_r = \Gamma(\langle x_2(t), \ldots, x_n(t)\rangle K[[x_1(t), \ldots, x_n(t)]])$$
$$= \{ord_t(f) : f \in \langle x_2(t), \ldots, x_n(t)\rangle K[[x_1(t), \ldots, x_n(t)]], f \neq 0\} \cup \{0\}.$$

Remark 2.2. If $\{g_1(t), \ldots, g_s(t)\}$ is a Sagbi basis of $K[[x_1(t), \ldots, x_n(t)]]$ (resp. a standard basis [2] of $\langle x_2(t), \ldots, x_n(t)\rangle K[[x_1(t), \ldots, x_n(t)]]$) then $\Gamma = \langle ord_t(g_1(t)), \ldots, ord_t(g_s(t))\rangle_{\mathbb{Z}}$ (resp. $\Gamma_r = \langle ord_t(g_1(t)), \ldots, ord_t(g_s(t))\rangle_{\Gamma}$).

[2]For a definition and properties see [16].

Example 2.3. (1) Given (t^4, t^5, t^4+t^6, t^8) then $\Gamma = \langle 4,5,6\rangle = \{0,4,5,6,8,\dots\}$ and $\Gamma_r = \{0,4,5,8,\dots\}$.
(2) Given $(0, t^4, t^6+t^7)$ then $\Gamma = \Gamma_r = \langle 4,5,13\rangle$.

First of all, we state a generalization of Zariski's theorem (cf. [28], [29]) to $\mathcal{A}_r$ -equivalence. The proof is an easy extension of Zariski's proof.

Theorem 2.4. *Given a parametrization $(x_1(t),\dots,x_n(t))$ with semigroup Γ and restricted semigroup Γ_r. Then there exists a parametrization $(y_1(t),\dots,y_k(t))$ with the following properties:*

1. $(x_1(t),\dots,x_n(t)) \sim_{L^r} (y_1(t),\dots,y_k(t))$.[3]
2. $y_1 = t^{m_1} + \sum\limits_{\substack{j>m_1\\ j\notin\Gamma}} a_j^{(1)} t^j$ *and* $y_i = t^{m_i} + \sum\limits_{\substack{j>m_i\\ j\notin\Gamma_r}} a_j^{(i)} t^j, i\geq 2$
3. $m_1 \notin \Gamma_r$, *($m_1 = \infty$ is included)*
4. $m_2 < \dots < m_k$, $m_i \notin \Gamma(\langle y_2(t),\dots,y_{i-1}(t)\rangle K[[x_1(t),\dots,x_{i-1}(t)]])$.

Example 2.5. (1) $(t^4,t^5,t^4+t^6,t^8) \sim_{\mathcal{A}_r} (t^6, t^4+t^6, t^5, t^8) \sim_{\mathcal{A}_r} (t^6, t^4+t^6, t^5)$ therefore $\Gamma = \langle 4,5,6\rangle = \{0,4,5,6,8,\dots\}$ and $\Gamma_r = \{0,4,5,8,\dots\}$.
(2) $(0,t^4,t^6+t^7,t^{13}) \sim_{\mathcal{A}_r} (0,t^4,t^6+t^7,t^{15})$ since $13 \in \Gamma(\langle t^4, t^6+t^7\rangle K[[t^4, t^6+t^7]]$.
(3) $(t^2,t^4,t^6+t^7) \sim_{\mathcal{A}_r} (t^2,t^4,t^7)$ since $6 \in \Gamma(\langle t^4, t^6+t^7\rangle K[[t^2,t^4,t^6+t^7]]$.

Definition 2.6. $(y_1(t),\dots,y_k(t))$ is called a weak normal form of $(x_1(t),\dots,x_n(t))$ if satisfies the properties of Theorem 2.4 and we call $(ord_t(y_1(t)),\dots,ord_t(y_k(t)))$, the order sequence of the weak normal form.

We will give now the weak normal forms for special parametrizations of multiplicities ≤ 5.

Proposition 2.7. *Let $(x_1(t),\dots,x_n(t))$ be a curve of multiplicity 2 with semigroup $\langle 2,a\rangle$, where $a>2$ and a is odd. Then $(x_1(t),\dots,x_n(t))$ is $\mathcal{A}_r$-equivalent to one of the following weak normal forms:*

1. $(0, t^2 + \sum\limits_{\substack{i>2\\ i\notin\Gamma}} a_i t^i, t^a)$ *with* $\Gamma = \Gamma_r$,
2. $(t^a, t^2 + \sum\limits_{\substack{i>2\\ i\notin\Gamma_r}} a_i t^i)$ *with* $\Gamma_r = \{0,2,4,\dots,a+1,\dots\}$,
3. $(t^2 + \sum\limits_{\substack{i>2\\ i\notin\Gamma}} a_i t^i, t^a + \sum\limits_{\substack{i>a\\ i\notin\Gamma_r}} b_i t^i)$ *with* $\Gamma_r = \{0,a,a+2,\dots,2a-1,\dots\}$,
4. $(t^2 + \sum\limits_{\substack{i>2\\ i\notin\Gamma}} a_i t^i, t^a + \sum\limits_{\substack{i>a\\ i\notin\Gamma_r}} b_i t^i, t^c)$ *with* $\Gamma_r = \{0,a,a+2,\dots,c-1,\dots\}$, *$c$ is even, $a<c<2a$,*
5. $(t^2 + \sum\limits_{\substack{i>2\\ i\notin\Gamma}} a_i t^i, t^c, t^a)$ *with* $\Gamma_r = \{0,c,c+2,\dots,a-1,\dots\}$, *$c$ is even, $c<a$,*
6. $(t^2 + \sum\limits_{\substack{i>2\\ i\notin\Gamma}} a_i t^i, t^b + t^a + \sum\limits_{\substack{i>a\\ i\notin\Gamma_r}} b_i t^i)$ *with* $\Gamma_r = \{0,b,b+2,\dots,a+b-1,\dots\}$, *$b$ is even, $2<b<a$,*
7. $(t^2 + \sum\limits_{\substack{i>2\\ i\notin\Gamma}} a_i t^i, t^b + t^a + \sum\limits_{\substack{i>a\\ i\notin\Gamma_r}} b_i t^i, t^c)$ *with* $\Gamma_r = \{0,b,b+2,\dots,c-1,\dots\}$, *$c$ is odd, $c>a$, b is even, $2<b<a$.*

[3] This means that $(x_1(t),\dots,x_n(t)) \sim_{\mathcal{A}_r} (y_1(t),\dots,y_k(t))$ using only the group L^r.

Proof. We may assume that $(x_1(t),\dots,x_n(t))$ is a weak normal form. If $\Gamma=\Gamma_r$ then we obtain (1) by using Theorem 2.4. If $\Gamma\neq\Gamma_r$ and $2\in\Gamma_r$ then we obtain (2), since $a\notin\Gamma_r$. If $2\notin\Gamma_r$ then $x_1(t)=t^2+\sum_{\substack{i>2\\ i\notin\Gamma}}a_it^i$. If $ord_t(x_2(t))=a$ then $\Gamma_r=\{0,a,a+2,\dots,b-1,\dots\}$, b is even, $a<b\leq 2a$ and $x_2(t)=t^a+\sum_{\substack{i>a\\ i\notin\Gamma_r}}b_it^i$. If $b=2a$ then $x_i(t)=0$ for all $i\geq 3$ and if $b<2a$ then $x_3(t)=t^b$ and $x_i(t)=0$ for all $i\geq 4$. We obtained the cases (3) and (4).
If $ord_t(x_2(t))=b\neq 0$ then b is even. If $a\in\Gamma_r$ then we obtain $\Gamma_r=\{0,b,b+2,\dots,a-1,\dots\}$, $x_2(t)=t^b+\sum_{\substack{i>b\\ i\notin\Gamma_r}}b_it^i$, $x_3(t)=t^a$ and $x_i(t)=0$ for all $i\geq 4$. This is the case (5). If $a\notin\Gamma_r$ then $\Gamma_r=\{0,b,b+2,\dots,c-1,\dots\}$, c is odd, $c\geq a$ and we obtain the cases (6) and (7). □

Corollary 2.8. *Assume that the characteristic $p\neq 2$. Let $(x_1(t),\dots,x_n(t))$ be a curve of multiplicity 2 with semigroup $\Gamma=\langle 2,a\rangle$ and 2-jet $(t^2,0,\dots,0)$. Then $(x_1(t),\dots,x_n(t))$ is equivalent to the weak normal form called W_{abc}*

$$(t^2,t^a+t^b+\sum_{\substack{i>max\{a,b\}\\ i\notin\Gamma_r}}a_it^i,t^c),$$

$b=\infty$ or b is even and $2<b<2a$. $c=\infty$ or $c\equiv\left\lfloor\frac{b}{a}\right\rfloor+1$ mod 2. If $c<\infty$ and $b<\infty$ then $max(a,b)\leq c<a+min(a,b)$.

Proposition 2.9. *Let $(x_1(t),\dots,x_n(t))$ be a curve in weak normal form with 3-jet $(0,t^3,0,\dots,0)$ and semigroup $\Gamma=\langle 3,k,r\rangle$, $r=\infty$ and $3\nmid k$ or $k<r<2k-2$ and $k.r\equiv 2$ mod 3. Then $(x_1(t),\dots,x_n(t))$ is $\mathcal{A}_r$-equivalent to one of the following weak normal forms:*

1. $(0,t^3+\sum_{\substack{i>3\\ i\notin\Gamma_r}}a_it^i,t^k+\sum_{\substack{i>k\\ i\notin\Gamma_r}}b_it^i,t^r)$ *with* $\Gamma=\Gamma_r$,
2. $(t^k+\sum_{\substack{i>k\\ i\notin\Gamma}}a_it^i,t^3+\sum_{\substack{i>3\\ i\notin\Gamma_r}}b_it^i,t^r)$ *with* $\Gamma_r=\langle 3,k+3,r\rangle$,
3. $(t^r,t^3+\sum_{\substack{i>3\\ i\notin\Gamma_r}}a_it^i,t^k+\sum_{\substack{i>k\\ i\notin\Gamma_r}}b_it^i)$ *with* $\Gamma_r=\langle 3,k,r+3\rangle$.

Proof. If $\Gamma=\Gamma_r$ then $x_1(t)=0$ and we obtain (1). If $\Gamma\neq\Gamma_r$ then $x_1(t)\neq 0$ and $a=ord_t(x_1(t))$ is not divisible by 3 and $a+3\in\Gamma_r$. Since $a\in\Gamma$, we obtain $a=k$ or $a=r$. These are the cases (2) and (3). □

Proposition 2.10. *Let $(x_1(t),\dots,x_n(t))$ be a curve in weak normal form with 3-jet $(t^3,0,\dots,0)$ and semigroup $\Gamma=\langle 3,k,r\rangle$, $r=\infty$ and $3\nmid k$ or $k<r<2k-2$ and $k.r\equiv 2$ mod 3. Let $x_i(t)=t^{m_i}+\sum_{\substack{j>m_i\\ j\notin\Gamma_r}}a_j^{(i)}t^j$ then $(m_1,\dots,m_n)$ is one of the following:*

1. $(3,k)$,
2. $(3,k,r)$,
3. $(3,k,r,s)$ *with* $3\mid s$ *and* $r<s<k+r$,
4. $(3,k,s)$ *with* $3\mid s$ *and* $k<s<3k-2$,
5. $(3,k,s,r)$ *with* $3\mid s$ *and* $k<s<r$,

(6) $(3, s)$ *with* $3 \mid s$ *and* $3 < s < k$,
(7) $(3, s, k)$ *with* $3 \mid s$ *and* $3 < s < k$,
(8) $(3, s, k, r)$ *with* $3 \mid s$ *and* $3 < s < k$.

Proof. We have $m_1 = 3$. If $3 \nmid m_2$ then $m_2 = k$. If $n = 2$ then we obtain (1). If $3 \nmid m_3$ then $m_3 = r$. If $n = 3$ then we obtain (2). If $n > 3$ then $n = 4$ and $3 \mid m_4$ and we obtain (3). If $3 \mid m_3$ and $n = 3$ then we obtain (4). If $n > 3$ then $n = 4$ and $m_4 = r$ and we obtain (5). If $3 \mid m_2$ and $n = 2$ then we obtain (6). If $n = 3$ then $m_3 = k$ and we obtain (7). If $n > 3$ then $n = 4$, $m_4 = r$ and we obtain (8). □

Remark 2.11. For later use, we give here some weak normal forms $(x_1(t), \dots, x_n(t))$ explicitly. Let $m_i = ord_t(x_i(t))$ and $m = (m_1, \dots, m_n)$.

(1) $\Gamma = \langle 3, 4, r\rangle$, $r \in \{5, \infty\}$,
  (a) $m = (3, 4)$, $\Gamma_r = \{0, 4, 7, 10, \dots\}$, $(t^3 + a_5t^5, t^4 + b_5t^5 + b_6t^6 + b_9t^9)$,
  (b) $m = (3, 4, 5)$, $\Gamma_r = \{0, 5, 7, \dots\}$, $(t^3, t^4 + b_6t^6, t^5 + c_6t^6)$,
  (c) $m = (3, 4, 5, 6)$, $\Gamma_r = \{0, 4, \dots\}$, (t^3, t^4, t^5, t^6),
  (d) $m = (3, 4, 6)$, $\Gamma_r = \{0, 4, 6, \dots\}$, $(t^3 + a_5t^5, t^4 + b_5t^5, t^6)$,
  (e) $m = (3, 4, 9)$, $\Gamma_r = \{0, 4, 7, \dots\}$, $(t^3 + a_5t^5, t^4 + b_5t^5 + b_6t^6, t^9)$,

(2) $\Gamma = \langle 3, 5, r\rangle$, $r \in \{7, \infty\}$,
  (a) $m = (3, 5)$, $\Gamma_r = \{0, 5, 8, 10, 11, 13 \dots\}$, $(t^3 + a_4t^4 + a_7t^7, t^5 + b_6t^6 + b_7t^7 + b_9t^9 + b_{12}t^{12})$,
  (b) $m = (3, 5, 6)$, $\Gamma_r = \{0, 5, 6, 8, \dots\}$, $(t^3 + a_4t^4 + a_7t^7, t^5 + b_7t^7, t^6 + c_7t^7)$,
  (c) $m = (3, 5, 6, 7)$, $\Gamma_r = \{0, 5, \dots\}$, $(t^3 + a_4t^4, t^5, t^6, t^7)$,
  (d) $m = (3, 5, 7)$, $\Gamma_r = \{0, 5, 7, 8, 10, \dots\}$, $(t^3 + a_4t^4, t^5 + b_6t^6 + b_9t^9, t^7 + c_9t^9)$,
  (e) $m = (3, 5, 7, 9)$, $\Gamma_r = \{0, 5, 7, \dots\}$, $(t^3 + a_4t^4, t^5 + b_6t^6, t^7, t^9)$,
  (f) $m = (3, 5, 9)$, $\Gamma_r = \{0, 5, 8, \dots\}$, $(t^3 + a_4t^4 + a_7t^7, t^5 + b_6t^6 + b_7t^7, t^9)$,
  (g) $m = (3, 5, 12)$, $\Gamma_r = \{0, 5, 8, 10, \dots\}$, $(t^3 + a_4t^4 + a_7t^7, t^5 + b_6t^6 + b_7t^7 + b_9t^9, t^{12})$,

(3) $\Gamma = \langle 3, 7, 8\rangle$, $m = (3, 6, 7, 8)$, $\Gamma_r = \{0, 6, \dots\}$, $(t^3 + a_4t^4 + a_5t^5, t^6, t^7, t^8)$.

Proposition 2.12. *Let* $\Gamma = \langle 4, 5, 6, 7\rangle$ *be the semigroup of a curve* $(x_1(t), \dots, x_n(t))$ *in weak normal form and assume that* 4*-jet of the curve is* $(t^4, 0, \dots, 0)$. *Then* $n = 4$ *or* 5 *with the normal forms*

(1) $(t^4, t^5 + a_8t^8, t^6 + a_8t^8, t^7 + c_8t^8)$,
(2) $(t^4, t^5, t^6, t^7, t^8)$.

Proof. Note that $5, 6, 7, 9, \dots \in \Gamma_r$. If $8 \notin \Gamma_r$, we obtain (1) and if $8 \in \Gamma_r$, we obtain (2). □

Proposition 2.13. *Let* $(x_1(t), \dots, x_n(t))$ *be a curve in weak normal form with* 4*-jet* $(0, t^4, 0, \dots, 0)$. *Let* $m_i = ord_t(x_i(t))$ *and* $m = (m_1, \dots, m_n)$. *Let* Γ *and* Γ_r *are semigroup and restricted semigroup of the curve. We obtain for the following choices of* Γ, Γ_r *and* m *as weak normal forms:*

(1) $\Gamma = \Gamma_r = \langle 4, 5, 6\rangle$, $(0, t^4 + a_7t^7, t^5 + b_7t^7, t^6 + c_7t^7)$,
(2) $\Gamma = \Gamma_r = \langle 4, 5, 7\rangle$, $(0, t^4 + a_6t^6, t^5 + b_6t^6, t^7)$,
(3) $\Gamma = \Gamma_r = \langle 4, 6, 7\rangle$, $(0, t^4 + a_5t^5 + a_9t^9, t^6 + b_9t^9, t^7 + c_9t^9)$,
(4) $\Gamma = \langle 4, 5, 6, 7\rangle$,
  (a) $m = (\infty, 4, 5, 6, 7)$, $\Gamma = \Gamma_r$, $(0, t^4, t^5, t^6, t^7)$,
  (b) $m = (5, 4, 6, 7)$, $\Gamma_r = \{0, 4, 6, \dots\}$, $(t^5, t^4 + a_5t^5, t^6, t^7)$,
  (c) $m = (6, 4, 5, 7)$, $\Gamma_r = \{0, 4, 5, 7, \dots\}$, $(t^6, t^4 + a_6t^6, t^5 + b_6t^6, t^7)$,

(d) $m = (7, 4, 5, 6)$, $\Gamma_r = \{0, 4, 5, 6, 8, \dots\}$, $(t^7, t^4+a_7t^7, t^5+b_7t^7, t^6+c_7t^7)$,

(5) $\Gamma = \langle 4, 6, 7, 9\rangle$,

(a) $m = (\infty, 4, 6, 7, 9)$, $\Gamma = \Gamma_r$, $(0, t^4 + a_5t^5, t^6, t^7, t^9)$,

(b) $m = (9, 4, 6, 7)$, $\Gamma_r = \{0, 4, 6, 7, 8, 10, \dots\}$, $(t^9, t^4 + a_5t^5 + a_9t^9, t^6 + b_9t^9, t^7 + c_9t^9)$.

Proof. The proof is an immediate consequence of Theorem 2.4. □

Proposition 2.14. *Let $(x_1(t), \dots, x_n(t))$ be a curve in weak normal form with 5-jet $(0, t^5, 0, \dots, 0)$. Let $m_i = ord_t(x_i(t))$ and $m = (m_1, \dots, m_n)$. Let Γ and Γ_r are semigroup and restricted semigroup of the curve. We obtain for the following choices of Γ, Γ_r and m as weak normal forms.*

(1) $\Gamma = \langle 5, 6, 7, 8\rangle$,

(a) $m = (\infty, 5, 6, 7, 8)$, $\Gamma = \Gamma_r$, $(0, t^5 + a_9t^9, t^6 + b_9t^9, t^7 + c_9t^9, t^8 + d_9t^9)$,

(b) $m = (9, 5, 6, 7, 8)$, $\Gamma = \Gamma_r$, $(t^9, t^5 + a_9t^9, t^6 + b_9t^9, t^7 + c_9t^9, t^8 + d_9t^9)$,

(2) $\Gamma = \langle 5, 6, 7, 8, 9\rangle$, $m = (\infty, 5, 6, 7, 8, 9)$, $\Gamma = \Gamma_r$, $(0, t^5, t^6, t^7, t^8, t^9)$,

(3) $\Gamma = \langle 5, 6, 7, 9\rangle$,

(a) $m = (\infty, 5, 6, 7, 9)$, $\Gamma = \Gamma_r$, $(0, t^5 + a_8t^8, t^6 + b_8t^8, t^7 + c_8t^8, t^9)$,

(b) $m = (8, 5, 6, 7, 9)$, $\Gamma = \Gamma_r$, $(t^8, t^5 + a_8t^8, t^6 + b_8t^8, t^7 + c_8t^8, t^9)$.

Proof. The proof is an immediate consequence of Theorem 2.4. □

3. Minimal Non-simple Curves

In this section, we will give families of non-simple curves with the property that most of the non-simple curves have a member of one of these families in a suitable deformation.

Proposition 3.1. *The following curves $(w(t), x(t), y(t), z(t))$ are not simple:*

(1) $ord_t(w(t)) = 3$, $ord_t(x(t)) = 6$, $ord_t(y(t)) = 7$ *and* $ord_t(z(t)) = 11$,

(2) $ord_t(w(t)) = 6$, $ord_t(x(t)) = 4$, $ord_t(y(t)) = 5$ *and* $z(t) = 0$,

(3) $ord_t(w(t)) = 8$, $ord_t(x(t)) = 5$, $ord_t(y(t)) = 6$ *and* $ord_t(z(t)) = 7$.

Proof. In case (1), we have $\Gamma = \langle 3, 7, 11\rangle$ and $\Gamma_r = \{0, 6, 7, 9, \dots\}$, in case (2), we have $\Gamma = \langle 4, 5, 6\rangle$ and $\Gamma_r = \{0, 4, 5, 8, \dots\}$ and in case (3), we have $\Gamma = \langle 5, 6, 7, 8\rangle$ and $\Gamma_r = \{0, 5, 6, 7, 10, \dots\}$. It is a consequence of the weak normal form (Theorem 2.4) that the curve $(w(t), x(t), y(t), z(t))$ is equivalent to one of the following parametrization:

(1) $(t^3 + a_4t^4 + a_5t^5 + a_8t^8, t^6 + b_8t^8, t^7 + c_8t^8, t^{11})$,

(2) $(t^6 + a_7t^7, t^4 + b_6t^6 + b_7t^7, t^5 + c_6t^6 + c_7t^7)$,

(3) $(t^8 + a_9t^9, t^5 + b_8t^8 + b_9t^9, t^6 + c_8t^8 + c_9t^9, t^7 + d_8t^8 + d_9t^9)$.

Using Singular, we can show that in characteristic $p \neq 3$, $(t^3 + a_4t^4 + a_5t^5 + a_8t^8, t^6 + b_8t^8, t^7 + ut^8, t^{11}) \sim_{\mathcal{A}_r} (t^3 + a_4t^4 + a_5t^5 + a_8t^8, t^6 + b_8t^8, t^7 + vt^8, t^{11})$ implies $u = v$ or $u = -v$. Since the field is algebraically closed, we have infinitely many equivalence classes in that family. Here are the computations.

```
int ch=0;
ring R=(ch,a,b,c,d,e,f,g,h,k,l,m,n,H1,H2,H3,H4,H5,H6,H7,H8,H9,H10,H11
,H12,H13,H14,H15,H16,H17,H18,H19,L1,L2,L3,L4,L5,L6,L7,L8,L9,L10,L11,
L12,L13,L14,L15,L16,L17,L18,L19,M1,M2,M3,M4,M5,M6,M7,M8,M9,M10,M11,
M12,M13,M14,M15,M16,M17,M18,M19,N2,N3,N5,N6,N7,N8,N9,N11,N12,N13,N14,
N15,N16,N17,N18,N19,u,v,o),(x,y,z,w,t),ds;
```

```
poly p=n*t+a*t2+b*t3+c*t4+d*t5+e*t6+f*t7+g*t8+h*t9+k*t10+l*t11+m*t12;
poly H=H1*x+H2*y+H3*z+H4*w+H5*x2+H6*xy+H7*xz+H8*xw+H9*y2+H10*yz+H11*yw
       +H12*z2+H13*zw+H14*w2+H15*x3+H16*x2y+H17*x2z+H18*x2w+H19*y3;
poly L=L2*y+L3*z+L5*w+L6*xy+L7*xz+L8*xw+L9*y2+L11*yz+L12*yw+L13*z2+
       L14*zw+L15*w2+L16*x2y+L17*x2z+L18*x2w+L19*y2;
poly M=M2*y+M3*z+M5*w+M6*xy+M7*xz+M8*xw+M9*y2+M11*yz+M12*yw+M13*z2+
       M14*zw+M15*w2+M16*x2y+M17*x2z+M18*x2w+M19*y2;
poly N=N2*y+N3*z+N5*w+N6*xy+N7*xz+N8*xw+N9*y2+N11*yz+N12*yw+N13*z2+
       N14*zw+N15*w2+N16*x2y+N17*x2z+N18*x2w+N19*y2;

poly q1=jet(p^3-subst(H,x,t3,y,t6+o*t8,z,t7+v*t8,w,t11),8);
poly q2=jet(p^6+o*p^8-subst(L,x,t3,y,t6+o*t8,z,t7+v*t8,w,t11),8);
poly q3=jet(p^7+u*p^8-subst(M,x,t3,y,t6+o*t8,z,t7+v*t8,w,t11),8);
poly q4=jet(p^11-subst(N,x,t3,y,t6+o*t8,z,t7+v*t8,w,t11),8);
matrix MM1=coef(q1,t);
matrix MM2=coef(q2,t);
matrix MM3=coef(q3,t);
matrix MM4=coef(q4,t);
ideal I;
int ii;
for(ii=1;ii<=ncols(MM1);ii++){I[size(I)+1]=MM1[2,ii];}
for(ii=1;ii<=ncols(MM2);ii++){I[size(I)+1]=MM2[2,ii];}
for(ii=1;ii<=ncols(MM3);ii++){I[size(I)+1]=MM3[2,ii];}
for(ii=1;ii<=ncols(MM4);ii++){I[size(I)+1]=MM4[2,ii];}
ring S=integer,(a,b,c,d,e,f,g,h,k,l,m,H1,H2,H3,H4,H5,H6,H7,H8,H9,H10,H11,
       H12,H13,H14,H15,H16,H17,H18,H19,L1,L2,L3,L4,L5,L6,L7,L8,L9,L10,L11,
       L12,L13,L14,L15,L16,L17,L18,L19,M1,M2,M3,M4,M5,M6,M7,M8,M9,M10,M11,
       M12,M13,M14,M15,M16,M17,M18,M19,N2,N3,N5,N6,N7,N8,N9,N11,N12,N13,
       N14,N15,N16,N17,N18,N19,u,v,n,o),lp;
ideal I=imap(R,I);
ideal J=std(I);
J;
// display the first two elements in the Groebner basis
J[1]=n^8*o-n^6*o
J[2]=3*u*n^6*o-3*v*n^7*o
// The first 2 elements of the Groebner  basis.
```

This proves that $n^2 = 1$ and $u = nv$.
In characteristic 3, we show that $(t^3 + a_4t^4 + ut^5 + a_8t^8, t^6 + b_8t^8, t^7 + c_8t^8, t^{11}) \sim_{\mathcal{A}_r}$ $(t^3 + a_4t^4 + vt^5 + a_8t^8, t^6 + b_8t^8, t^7 + c_8t^8, t^{11})$ implies $u = v$.

```
int ch=3;
ring R=(ch,a,b,c,d,e,f,g,h,k,l,m,n,H1,H2,H3,H4,H5,H6,H7,H8,H9,H10,H11,H12,H13
       ,H14,H15,H16,H17,H18,H19,L1,L2,L3,L4,L5,L6,L7,L8,L9,L10,L11,L12,L13,L14
       ,L15,L16,L17,L18,L19,M1,M2,M3,M4,M5,M6,M7,M8,M9,M10,M11,M12,M13,M14,M15,
       M16,M17,M18,M19,N2,N3,N5,N6,N7,N8,N9,N11,N12,N13,N14,N15,N16,N17,N18,
       N19,u,v,r,o,s),(x,y,z,w,t),ds;
poly p=n*t+a*t2+b*t3+c*t4+d*t5+e*t6+f*t7+g*t8+h*t9+k*t10+l*t11+m*t12;
poly H=H1*x+H2*y+H3*z+H4*w+H5*x2+H6*xy+H7*xz+H8*xw+H9*y2+H10*yz+H11*yw+
```

```
       H12*z2+H13*zw+H14*w2+H15*x3+H16*x2y+H17*x2z+H18*x2w+H19*y3;
poly L=L2*y+L3*z+L5*w+L6*xy+L7*xz+L8*xw+L9*y2+L11*yz+L12*yw+L13*z2+
       L14*zw+L15*w2+L16*x2y+L17*x2z+L18*x2w+L19*y2;
poly M=M2*y+M3*z+M5*w+M6*xy+M7*xz+M8*xw+M9*y2+M11*yz+M12*yw+M13*z2+
       M14*zw+M15*w2+M16*x2y+M17*x2z+M18*x2w+M19*y2;
poly N=N2*y+N3*z+N5*w+N6*xy+N7*xz+N8*xw+N9*y2+N11*yz+N12*yw+N13*z2+
       N14*zw+N15*w2+N16*x2y+N17*x2z+N18*x2w+N19*y2;

poly q1=jet(p^3+r*p^4+u*p^5+o*p^8-subst(H,x,t3+r*t4+v*t5+o*t8,y,
        t6+s*t8,z,t7,w,t11),8);
poly q2=jet(p^6+s*p^8-subst(L,x,t3+r*t4+v*t5+o*t8,y,t6+s*t8,z,t7,w,t11),8);
poly q3=jet(p^7-subst(M,x,t3+r*t4+v*t5+o*t8,y,t6+s*t8,z,t7,w,t11),8);
poly q4=jet(p^11-subst(N,x,t3+r*t4+v*t5+o*t8,y,t6+s*t8,z,t7,w,t11),8);

matrix MM1=coef(q1,t);
matrix MM2=coef(q2,t);
matrix MM3=coef(q3,t);
matrix MM4=coef(q4,t);
ideal I;
int ii;
for(ii=1;ii<=ncols(MM1);ii++){I[size(I)+1]=MM1[2,ii];}
for(ii=1;ii<=ncols(MM2);ii++){I[size(I)+1]=MM2[2,ii];}
for(ii=1;ii<=ncols(MM3);ii++){I[size(I)+1]=MM3[2,ii];}
for(ii=1;ii<=ncols(MM4);ii++){I[size(I)+1]=MM4[2,ii];}
ring S=ch,(a,b,c,d,e,f,g,h,k,l,m,H1,H2,H3,H4,H5,H6,H7,H8,H9,H10,H11,H12,H13,H14,
       H15,H16,H17,H18,H19,L1,L2,L3,L4,L5,L6,L7,L8,L9,L10,L11,L12,L13,L14,L15,
       L16,L17,L18,L19,M1,M2,M3,M4,M5,M6,M7,M8,M9,M10,M11,M12,M13,M14,M15,M16,
       M17,M18,M19,N2,N3,N5,N6,N7,N8,N9,N11,N12,N13,N14,N15,N16,N17,N18,N19,
       u,v,r,o,s,n),lp;
ideal I=imap(R,I);
ideal J=std(I);
J;
//display the first 6 elements in the Groebner basis
J[1]=n^4*r-n^3*r
J[2]=n^8*s-n^6*s
J[3]=v*n^6*s-v*n^4*s
J[4]=v^2*n^5*s-v^2*n^3*s
J[5]=u*n^3*r-v*n^3*r
J[6]=u*n^5-v*n^3
```

This proves that for generic a_4 we obtain $n = 1$ and therefore $u = v$.
Similarly we treat the case (2) with $c_7 = u$ (resp. v) in characteristic $p \neq 2$ and $b_7 = u$ (resp. v) in characteristic 2. Since the code is the same as above and only the definition of the integer ch and the polynomials $q1$, $q2$, $q3$, $q4$ change we will only give those data and the result.

```
int ch=0;
poly q1=jet(p^6 -subst(H,x,t6,y,t4,z,t5+r*t6+v*t7,w,0),7);
poly q2=jet(p^4-subst(L,x,t6,y,t4,z,t5+r*t6+v*t7,w,0),7);
```

```
poly q3=jet(p^5+r*p^6+u*p^7-subst(M,x,t6,y,t4,z,t5+r*t6+v*t7,w,0),7);

J[1]=n^7*r-n^6*r
J[2]=u*n^6*r-v*n^6*r
J[3]=u*n^8-v*n^6
```

We obtain in characteristic $p \neq 2$ for generic a_6 that $n = 1$ and $u = v$, Now let us consider the case $p = 2$.

```
int ch=2;
poly q1=jet(p^6+r*p^7 -subst(H,x,t6+r*t7,y,t4+o*t6+v*t7,z,t5,w,0),8);
poly q2=jet(p^4+o*p^6+u*p^7-subst(L,x,t6+r*t7,y,t4+o*t6+v*t7,z,t5,w,0),8);
poly q3=jet(p^5-subst(M,x,t6+r*t7,y,t4+o*t6+v*t7,z,t5,w,0),8);

J[1]=n^5*r*o+n^4*r*o
J[2]=n^6*o+n^4*o
J[3]=n^7*r+n^6*r
J[4]=v*n^5*r+v*n^4*r
```

We obtain for generic a_7 that $n = 1$ and $u = v$.
In case (3), we use in characteristic $p \neq 2, 3$, the family with $d_9 = u$ (resp. v) and in characteristic $2, 3$ with $c_9 = u$ (resp. v).

```
int ch=0;
poly q1=jet(p^8 -subst(H,x,t8,y,t5+r*t8,z,t6,w,t7+o*t8+v*t9),9);
poly q2=jet(p^5+r*p^8-subst(L,x,t8,y,t5+r*t8,z,t6,w,t7+o*t8+v*t9),9);
poly q3=jet(p^6-subst(M,x,t8,y,t5+r*t8,z,t6,w,t7+o*t8+v*t9),9);
poly q4=jet(p^7+o*p^8+u*p^9-subst(N,x,t8,y,t5+r*t8,z,t6,w,t7+o*t8+v*t9),9);

J=quotient(J,,n^8*o4);
J[1]=8*n-8
J[14]=12*u-12*v+2*n^2*o^2-2*o^2
```

This implies that in characteristic $p \neq 2, 3$ we obtain $n = 1$ and $u = v$.
In characteristic 2 or 3 we use the following computation.

```
int ch=2; //or int ch=3;
poly q1=jet(p^8+r*p^9 -subst(H,x,t8+r*t9,y,t5,z,t6+o*t8+v*t9,w,t7),9);
poly q2=jet(p^5-subst(L,x,t8+r*t9,y,t5,z,t6+o*t8+v*t9,w,t7),9);
poly q3=jet(p^6+o*p^8+u*p^9-subst(M,x,t8+r*t9,y,t5,z,t6+o*t8+v*t9,w,t7),9);
poly q4=jet(p^7-subst(N,x,t8+r*t9,y,t5,z,t6+o*t8+v*t9,w,t7),9);

J[1]=n^9*r+n^8*r
J[2]=n^10*o+n^8*o
J[3]=v*n^7*r+v*n^6*r
J[4]=v*n^8*o+v*n^6*o
J[5]=u*n^8*o+v*n^7*o
```

This implies that for generic a_9 and c_8 we obtain $n = 1$ and $u = v$. □

Corollary 3.2. *The following curves $(w(t), x(t), y(t), z(t))$ are not simple:*

(1) $ord_t(w(t)) = 4$, $ord_t(x(t)) = 5$, $ord_t(y(t)) = 6$ *and* $ord_t(z(t)) = 8$,

(2) *$ord_t(w(t))=5$, $ord_t(x(t))=4$, $ord_t(y(t))=10$ and $z(t)=0$,*
(3) *$ord_t(w(t))=5$, $ord_t(x(t))=6$, $ord_t(y(t))=7$ and $ord_t(z(t))=8$,*
(4) *$ord_t(w(t))=7$, $ord_t(x(t))=4$, $ord_t(y(t))=6$ and $ord_t(z(t))=9$,*
(5) *the 5-jet is $(t^5,0,0,0)$ or $(0,0,0,0)$.*

Proof. In case (1), we consider the deformation $(w(t),\alpha t^4+x(t),\beta t^5+y(t),z(t))$. For generic α and β this family is equivalent to $(\overline{w}(t),\overline{x}(t),\overline{y}(t))$ with $\overline{w}(t)=t^6+$ terms of higher order, $\overline{x}(t)=\alpha t^4+x(t)$ and $\overline{y}(t)=\beta t^5+y(t)$, since $\Gamma_r=\{0,4,5,8,\dots\}$ for this family. The result is a consequence of Proposition 3.1 (2).
In case of (2), we consider the deformation $(\alpha t^6+w(t),x(t),\beta t^5+y(t))$. For generic α and β this family is equivalent to $(\overline{w}(t),x(t),\overline{y}(t))$ with $ord_t(\overline{w}(t))=6$, $ord_t(x(t))=4$, $ord_t(\overline{y}(t))=5$. The result is a consequence of Proposition 3.1 (2).
In case (3), we consider the deformation $(\alpha t^4+w(t),\beta t^5+x(t),\gamma t^6+y(t),z(t))$ and the result is a consequence of (1).
In case (4), we consider the deformation $(\alpha t^6+w(t),x(t),\beta t^5+y(t),z(t))$ and the result follows similarly to (2).
In case (5), we consider the deformation $(\alpha t^4+w(t),\beta t^5+x(t),\gamma t^6+y(t),\delta t^8+z(t))$, which is not simple for generic $\alpha,\beta,\gamma,\delta$ because of (1). □

Proposition 3.3. *Assume that the characteristic $p>2$. The following curves are not simple:*
(1) $(t^2,t^6+t^{p+6}+\sum_{i>p+7}a_it^i,t^{p+10})$,
(2) $(t^2,t^{p+6}+t^{2p+6}+\sum_{i>2p+7}a_it^i,t^{2p+10})$.

Proof. To prove (1), we will show that $(t^2,t^6+t^{p+6}+\sum_{i>p+7}a_it^i,t^{p+10})\sim_{\mathcal{A}_r}(t^2,t^6+t^{p+6}+\sum_{i>p+7}b_it^i,t^{p+10})$ implies $a_{p+8}=b_{p+8}$. Assume the equivalence above is true, then there exist an automorphism $\varphi:K[[t]]\to K[[t]]$ such that $\varphi(t)=\sum_{i=1}^{\infty}\varphi_it^i$ and an automorphism $\phi:K[[x_1,x_2,x_3]]\to K[[x_1,x_2,x_3]]$ defined by $\phi(x_i)=H_i(x_1,x_2,x_3)=\sum_{k+l+m\geq 1}H^{(i)}_{k,l,m}x_1^kx_2^lx_3^m$, where φ_i, $H^{(i)}_{k,l,m}\in K$, $H^{(i)}_{k,0,0}=0$ for all k if $i\geq 2$ such that
(1) $\varphi(t)^2=H_1(t^2,t^6+t^{p+6}+\sum b_it^i,t^{p+10})$,
(2) $\varphi(t)^6+\varphi(t)^{p+6}+\sum a_i\varphi(t)^i=H_2(t^2,t^6+t^{p+6}+\sum b_it^i,t^{p+10})$,
(3) $\varphi(t)^{p+10}=H_3(t^2,t^6+t^{p+6}+\sum b_it^i,t^{p+10})$.

(1) implies that $\varphi_i=0$ for i even and $i\leq p+3$. As a consequence we obtain that $\varphi(t)^6=\sum_{i=1}^{\infty}\tilde{\varphi}_it^i$ with $\tilde{\varphi}_i=0$ for i odd and $i\leq p+8$. Using (2) we obtain $\varphi_1^6t^6=H^{(2)}_{0,1,0}t^6$ and $\varphi_1^6\varphi_1^pt^{p+6}=H^{(2)}_{0,1,0}t^{p+6}$. This implies $\varphi_1=H^{(2)}_{0,1,0}=1$. Using (2) again we obtain $\varphi_1^{p+8}a_{p+8}t^{p+8}=H^{(2)}_{0,1,0}b_{p+8}t^{p+8}$ and therefore $a_{p+8}=b_{p+8}$.
(2) can be proved similarly. □

Proposition 3.4. *Assume that the characteristic $p=2$. The following curves are not simple:*

$$(t^2+a_3t^3,b_4t^4+t^5+b_8t^8,t^6)$$

Proof. We will show if $b_4 \neq 0$ then $(t^2 + a_3t^3, b_4t^4 + t^5 + \sum_{i>5} b_it^i, t^6) \sim_{\mathcal{A}_r} (t^2 + \tilde{a}_3t^3, b_4t^4 + t^5 + \sum_{i>5} \tilde{b}_it^i, t^6)$ implies $a_3 = \tilde{a}_3$. Assume the equivalence above is true, then there exist an automorphism $\varphi : K[[t]] \to K[[t]]$ such that $\varphi(t) = \sum_{i=1}^{\infty} \varphi_it^i$ and an automorphism $\phi : K[[x_1, x_2, x_3]] \to K[[x_1, x_2, x_3]]$ defined by $\phi(x_i) = H_i(x_1, x_2, x_3) = \sum_{k+l+m\geq 1} H^{(i)}_{k,l,m} x_1^k x_2^l x_3^m$, where φ_i, $H^{(i)}_{k,l,m} \in K$, $H^{(i)}_{k,0,0} = 0$ for all k if $i \geq 2$ such that

(1) $\varphi(t)^2 + \tilde{a}_3\varphi(t)^3 = H_1(t^2 + a_3t^3, b_4t^4 + t^5 + \sum b_it^i, t^6)$,

(2) $b_4\varphi(t)^4 + \varphi(t)^5 + \sum \tilde{b}_i\varphi(t)^i = H_2(t^2 + a_3t^3, b_4t^4 + t^5 + \sum b_it^i, t^6)$,

(3) $\varphi(t)^6 = H_3(t^2, t^2 + a_3t^3, b_4t^4 + t^5 + \sum b_it^i, t^6)$.

(1) implies that $\varphi_1^2 = H^{(1)}_{1,0}$ and $a_3H^{(1)}_{1,0} = \tilde{a}_3\varphi_1^3$. Using (2) we obtain $H^{(2)}_{1,0} = \varphi_1^4$ and $H^{(2)}_{1,0} = \varphi_1^5$. This implies $\varphi_1 = 1$ and $a_3 = \tilde{a}_3$.

□

Proposition 3.5. *Assume that the characteristic $p = 2$. The following curves are not simple:*

(1) $(t^8 + a_{10}t^{10} + a_{13}t^{13}, t^3)$
(2) $(0, t^3, t^8 + a_{10}t^{10} + a_{13}t^{13})$

Proof. The second case is a consequence of Proposition 21 in [19]. For the first statement let

```
int ch=2;
poly q1=jet(p^8+p^10+u*p^13-subst(H,x,t8+t10+v*t^13,y,t3,z,0,w,0),13);
poly q2=jet(p^3-subst(L,x,t8+t10+v*t^13,y,t3,z,0,w,0),13);

J[1]=n^10+n^8
J[2]=u*n^8+v*n^9
```

This implies that for generic a_{10} and a_{13} we obtain $u = v$. □

Proposition 3.6. *Assume that the characteristic $p = 3$. The following curves are not simple:*

(1) $(t^8, t^3 + b_4t^4 + b_5t^5 + b_8t^8, t^7 + c_8t^8)$
(2) $(t^7 + a_8t^8, t^3 + b_4t^4 + b_5t^5 + b_7t^7 + b_8t^8, t^{11})$
(3) $(0, t^3 + b_4t^4 + b_5t^5 + b_8t^8, t^7 + c_8t^8, t^{11})$

Proof. For the first statement let

```
int ch=3;
poly q1=jet(p^8 -subst(H,x,t8,y,t3+o*t4+v*t5,z,t7,w,0),8);
poly q2=jet(p^3+o*p^4+u*p^5-subst(L,x,t8,y,t3+o*t4+v*t5,z,t7,w,0),8);
poly q3=jet(p^7-subst(M,x,t8,y,t3+o*t4+v*t5,z,t7,w,0),8);

J[1]=n^4*o-n^3*o
J[2]=u*n^3*o-v*n^3*o
J[3]=u*n^5-v*n^3
```

This implies that for generic b_4 and b_5 we obtain $u = v$.
The second statement can be proved similarly.
The third statement is a consequence of Lemma 15 of [19]. □

Proposition 3.7. *Assume that the characteristic $p = 5$. The following curves are not simple:*

$$(t^3, t^5 + b_7t^7, t^6 + c_7t^7)$$

Proof.
```
int ch=5;
poly q1=jet(p^3 -subst(H,x,t3,y,t5+o*t7,z,t6+v*t7,w,0),7);
poly q2=jet(p^5+o*p^7-subst(L,x,t3,y,t5+o*t7,z,t6+v*t7,w,0),7);
poly q3=jet(p^6+u*p^7-subst(M,x,t3,y,t5+o*t7,z,t6+v*t7,w,0),7);

J[1]=n^7*o-n^5*o
J[2]=u*n^5*o-v*n^6*o
J[3]=u*n^7-v*n^6
```
This implies that for generic b_7 and c_7 we obtain $n^2 = 1$ and $u = nv$. □

Proposition 3.8. *Assume that the characteristic $p = 7$. The following curves are not simple:*

$$(t^3, t^5 + b_6t^6, t^7 + c_9t^9)$$

Proof.
```
int ch=7;
poly q1=jet(p^3 -subst(H,x,t3,y,t5+o*t6,z,t7+v*t9,w,0),9);
poly q2=jet(p^5+o*p^6-subst(L,x,t3,y,t5+o*t6,z,t7+v*t9,w,0),9);
poly q3=jet(p^7+u*p^9-subst(M,x,t3,y,t5+o*t6,z,t7+v*t9,w,0),9);

J[1]=n^6*o-n^5*o
J[2]=u*n^9-v*n^7
```
This implies that for generic b_6 and c_9 we obtain $n = 1$ and $u = v$. □

4. Classification in Characteristic $p \neq 2$

Proposition 4.1. *Let $(x_1(t), \dots, x_n(t))$ be a curve of multiplicity 1 then it is $\mathcal{A}_r$-equivalent $(0, t)$ or (t, t^k) for some $k \geq 2$.*

Remark 4.2. This proposition is also true in characteristic $p = 2$.

Proof. If $ord_t(x_1(t)) = 1$ and $ord_t(x_i(t)) \geq 2$ for $i \geq 2$ then $\Gamma = \langle 1 \rangle$ and $\Gamma_r = \{0, k, \dots\}$ for some k. If $ord_t(x_1(t)) \geq 2$ then $ord_t(x_i(t)) = 1$ for some $i \geq 2$ and $\Gamma = \Gamma_r = \langle 1 \rangle$. The result follows from the Theorem 2.4. □

Proposition 4.3. *Let $(x_1(t), \dots, x_n(t))$ be a curve with $ord_t(x_1(t)) > 2$ and semigroup $\langle 2, a \rangle$ then it is $\mathcal{A}_r$-equivalent to one of the following normal forms.*

(1) $(0, t^2, t^a)$,
(2) $(t^a, t^2, 0)$.

Proof. Proposition 2.7 implies that $(x_1(t), \dots, x_n(t))$ is $\mathcal{A}_r$-equivalent to $(0, t^2 + \sum_{i>2} a_it^i, t^a)$ if $\Gamma = \Gamma_r$ and $(t^a, t^2 + \sum_{i>2} a_it^i)$ if $\Gamma \neq \Gamma_r$. Since the characteristic $p \neq 2$, we can find an automorphism $\varphi : K[[t]] \to K[[t]]$ such that $\varphi(t^2 + \sum_{i>2} a_it^i) = t^2$. □

Lemma 4.4. *Let* $(t^2, t^a + t^b + \sum_{\substack{i>max\{a,b\} \\ i\notin\Gamma_r}} b_i t^i, t^c)$ *be a curve with semigroup* $\langle 2, a\rangle$ *in weak normal form* W_{abc} *with*

1. $b = \infty$ *or* b *is even,* $2 < b < 2a$,
2. $c = \infty$ *or* $c \equiv \lfloor \frac{b}{a} \rfloor + 1$ *mod* 2*. If* $c < \infty$ *and* $b < \infty$ *then* $max(a, b) \le c < a + min(a, b)$.

Assume that $p \nmid a - b$ *if* $b > 4$*. Then the curve is* $\mathcal{A}_r$*-equivalent to the normal form* $(t^2, t^a + t^b, t^c)$*, denoted by* $N_{a,b,c}$*. Note that in case* $a = c$*, we have* $(t^2, t^a + t^b, t^a) \sim_{\mathcal{A}_r} (t^2, t^b, t^a)$.

Proof. Let us first consider the case $b < a$. In this case we have $\Gamma_r = \{0, b, b + 2, \ldots, c - 1, \ldots\}$. Assume that $k > a$ is minimal such that $b_k \neq 0$. Since $k \notin \Gamma_r$, k must be odd.

Consider the automorphism $\varphi : K[[t]] \to K[[t]]$ such that $\varphi(t) = t + \alpha t^{k-a+1}$. We obtain

$$\varphi(t^2) = t^2 + 2\alpha t^{k-a+2} + (\alpha)^2 t^{2(k-a+1)},$$

$$\varphi(t^b + t^a + \sum_{\substack{i>a \\ i\notin\Gamma_r}} b_i t^i) = t^b + b\alpha t^{k-a+b} + \cdots + t^a + (a\alpha + b_k)t^k + \sum_{i>k} \overline{b_i} t^i,$$

$$\varphi(t^c) = t^c - c\alpha t^{k-a+c} + \ldots.$$

Since $k - a$ is even we are allowed to substract $b\alpha t^{k-a}\varphi(t^b + t^a + \sum_{\substack{i>a \\ i\notin\Gamma_r}} b_i t^i)$ from the equation above and obtain

$$t^b + t^a + (a\alpha - b\alpha + b_k)t^k + \ldots$$

The exponents of $\varphi(t^2)$ and $\varphi(t^b)$ are even while the exponents of $\varphi(t^c)$ are are odd. This implies, choosing $\alpha = \frac{b_k}{b-a}$, that

$$(\varphi(t^2), (1 - b\alpha t^{k-a})\varphi(t^b + t^a + \sum_{\substack{i>a \\ i\notin\Gamma_r}} b_i t^i), \varphi(t^c)) \sim_{\mathcal{A}_r} (t^2, t^a + t^b + \sum_{i>k} \tilde{b_i} t^i, t^c).$$

Using induction, we obtain the result.
The case $a < b$ can be treated similarly.
It remains to consider the case $(t^2, t^4 + t^a + \sum_{\substack{i>a \\ i\notin\Gamma_r}} b_i t^i, t^c)$ in case that $a - 4$ is divisible by p. In this case we have $\Gamma_r = \{0, 4, 6, \ldots, c-1, \ldots\}$. Assume that $k > a$ is minimal such that $b_k \neq 0$. Consider the automorphism $\varphi : K[[t]] \to K[[t]]$ such that $\varphi(t) = t - \frac{b_k}{4} t^{k-3}$. We obtain the $\mathcal{A}_r$-equivalent curve $(t^2 - \frac{b_k}{2} t^{k-2} + \ldots, t^4 + t^a + \sum_{\substack{i>k \\ i\notin\Gamma_r}} \tilde{b}_i t^i, t^c)$.

Since $k - 2 \in \Gamma$ this curve is equivalent to $(t^2, t^4 + t^a + \sum_{\substack{i>k \\ i\notin\Gamma_r}} \tilde{b}_i t^i, t^c)$. Using induction we obtain the result. □

Proposition 4.5. *Let* $(x_1(t), \ldots, x_n(t))$ *be a curve with* 2*-jet* $(t^2, 0, \ldots, 0)$ *and semigroup* $\Gamma = \langle 2, a\rangle$*. Assume that* $= (t^2, t^a + t^b + \sum_{\substack{i>max\{a,b\} \\ i\notin\Gamma_r}} a_i t^i, t^c)$ *is a weak normal form of* $(x_1(t), \ldots, x_n(t))$*. Then* $(x_1(t), \ldots, x_n(t))$ *is not simple if and only if* $a \ge p + 6$*,* $b \ge 6$ *and* $c \ge p + 10$.

Proof. Corollary 2.8 implies that $b = \infty$ or b is even and $2 < b < 2a$. $c = \infty$ or $c \equiv \lfloor \frac{b}{a} \rfloor + 1 \bmod 2$. If $c < \infty$ and $b < \infty$ then $max(a,b) \le c < a + min(a,b)$. Assume that $a \ge p+6$, $b \ge 6$, $c \ge p+10$ and consider the deformation

$$(t^2, \alpha t^6 + \beta t^{p+6} + \gamma t^{p+8} + t^a + t^b + \sum_{\substack{i > max\{a,b\} \\ i \notin \Gamma_r}} a_i t^i, t^c).$$

Proposition 3.4 implies that this curve is not simple for generic α, β and γ. This implies that $(x_1(t), \dots, x_n(t))$ is not simple. Assume that $a < p+6$ then $p \nmid a - b$ or $b = 4$. Proposition 4.4 implies that the curve is $\mathcal{A}_r$-equivalent to $(t^2, t^b + t^a, t^c)$. If $b < 6$ then as before we obtain that the curve is $\mathcal{A}_r$-equivalent to $(t^2, t^4 + t^a, t^c)$. If $a \ge p+6$, $b \ge 6$ but $c < p+10$ then $c < a+4$. Since $c \equiv \lfloor \frac{b}{a} \rfloor + 1 \bmod 2$, i.e. c is odd if $b < a$ and even if $b > a$ we obtain $c \le a+2$ if $b < a$ or $c \le a+3$ if $b > a$. In both cases the curve is $\mathcal{A}_r$-equivalent to $(t^2, t^b + t^a, t^c)$. A deformation of the curve has the same properties. This implies that the curve is simple. □

Lemma 4.6. *Assume that the characteristic $p \ne 3$. Let $(x_1(t), \dots, x_n(t))$ be a curve with semigroup $\Gamma = \langle 3, k, r \rangle$, $r = \infty$ and $3 \nmid k$ or $k < r < 2k-2$ and $k.r \equiv 2$ mod 3. Assume that 3-jet is of multiplicity 3 and $ord_t(x_1(t)) > 3$. Then the curve is $\mathcal{A}_r$-equivalent to one of the following parametrizations.*

1. $(0, t^3, t^k + t^b + \sum_{\substack{i>b \\ i \notin \Gamma}} b_i t^i, t^r)$,
2. $(t^k + t^b + \sum_{\substack{i>b \\ i \notin \Gamma}} b_i t^i, t^3, t^r)$,
3. $(t^r, t^3, t^k + t^b + \sum_{\substack{i>b \\ i \notin \Gamma_r}} b_i t^i)$, $\Gamma_r = \langle 3, k, r+3 \rangle$ *with* $r = \infty$ *and* $3 \nmid k$ *or* $k < r < 2k-2$ *and* $k.r \equiv 2$ *mod* 3, *with* $b = \infty$ *or* $k < b$, $k.b \equiv 2$ *mod* 3.

Proof. The statement follows immediately from Proposition 2.9, since the characteristic $p \ne 3$. □

Corollary 4.7. *Assume that the characteristic $p \ne 3$. With the notations of Lemma 4.6, $(x_1(t), \dots, x_n(t))$ is $\mathcal{A}_r$-equivalent to one of the following normal forms*

1. $(0, t^3, t^k + t^b, t^r)$,
2. $(t^k + t^b, t^3, t^r)$,
3. $(t^r, t^3, t^k + t^b)$.

The curve $(x_1(t), \dots, x_n(t))$ is not simple if and only if one of the following conditions hold

1. $k \ge 2p + 9$
2. $k \ge p + 9$ *and* $b \ge p + k$ *and in cases (1) and (2)* $r > p + k$, *in case (3)* $r + 3 > p + k$.

Proof. The Corollary is a consequence of Theorem 9 of [19]. □

Remark 4.8. Lemma 4.6 and Corollary 4.7 hold also in characteristic $p = 2$.

Proposition 4.9. *Assume that the characteristic $p = 3$. Let $(x_1(t), \dots, x_n(t))$ be a curve with semigroup $\Gamma = \langle 3, k, r \rangle$. Assume that $ord_t x_1(t) \ge 4$. Then the curve is simple if and only if it is $\mathcal{A}_r$-equivalent to one of the following normal forms*

1. $(0, t^3 + \alpha t^4 + \beta t^5, t^k, t^r)$,
2. $(t^k, t^3 + \alpha t^4 + \beta t^5, t^r)$,

(3) $(t^r, t^3+\alpha t^4, t^5)$.

with $(k=5,\ r\in\{7,\infty\},\ \beta=0,\ \alpha\in\{0,1\})$ *or* $(k=7,\ r=8,\ \alpha\beta=0,\ \alpha,\beta\in\{0,1\})$.

Proof. The Proposition is a consequence of Theorem 9 of [19] and Proposition 3.6. $\square$

Lemma 4.10. *Let* $(x_1(t),\dots,x_n(t))$ *be a curve with semigroup* $\Gamma=\langle 3,k,r\rangle$, $r=\infty$ *and* $3\nmid k$ *or* $k<r<2k-2$ *and* $k.r\equiv 2$ *mod* 3. *Assume that* 3*-jet is* $(t^3,0,\dots,0)$. *If* $\Gamma\geq\langle 3,7,11\rangle$ [4] *then* $(x_1(t),\dots,x_n(t))$ *is not simple. If* $\Gamma=\langle 3,7,8\rangle$ *and the order sequence of a weak normal form is different from* $(3,6,7,8)$ *then* $(x_1(t),\dots,x_n(t))$ *is not simple.*

Proof. Proposition 3.1 implies that a curve with semigroup $\Gamma=\langle 3,7,11\rangle$ and the order sequence $(3,6,7,11)$ is not simple. It is not difficult to see that a curve with semigroup $\langle 3,k,r\rangle$, $k\geq 8$ deforms into a curve with semigroup $\langle 3,7,11\rangle$ and the order sequence $(3,6,7,11)$.

Now consider a curve with semigroup $\langle 3,7,8\rangle$. Proposition 2.13 implies that we have the following possibilities for the order sequence:

(1) $(3,7,8),(3,7,8,9),(3,7,8,12),(3,6),(3,6,7)$.
(2) $(3,6,7,8)$.

Curves with order sequence from (1) deform to a curve with semigroup $\langle 3,7,11\rangle$ and the order sequence $(3,6,7,11)$. This is obvious for the order sequences $(3,7,8)$, $(3,7,8,12)$, $(3,5)$ and $(3,6,7)$. Now consider a curve with order sequence $(3,7,8,9)$ given by the parametrization

$$(t^3, t^7+\sum_{i>7}a_it^i, t^8+\sum_{i>8}b_it^i, t^9+\sum_{i>9}c_it^i)$$

and its deformation

$$(t^3, \alpha t^6+t^7+\sum_{i>7}a_it^i, \beta t^7+t^8+\sum_{i>8}b_it^i, t^9+\sum_{i>9}c_it^i).$$

For generic α,β this curve is $\mathcal{A}_r$-equivalent to

$$(t^3, \alpha t^6+t^7+\sum_{i>7}a_it^i, \beta t^7+t^8+\sum_{i>8}b_it^i, t^{11}).$$

$\square$

Proposition 4.11. *Assume that the characteristic* $p\neq 3$. *Let* $(x_1(t),\dots,x_n(t))$ *be a curve with semigroup* $\Gamma=\langle 3,4,r\rangle$ *and* 3*-jet* $(t^3,0,\dots,0)$. *Then the curve* $(x_1(t),\dots,x_n(t))$ *is simple and* $\mathcal{A}_r$*-equivalent to one of the following parametrizations:*

If $r=\infty$

(1) (t^3,t^4)
(2) (t^3,t^4+t^6)
(3) (t^3,t^4,t^6)
(4) (t^3,t^4,t^9)
(5) (t^3,t^4+t^6,t^9)

[4] Let $\Gamma=\langle a_1,\dots,a_l\rangle$, $\widetilde{\Gamma}=\langle b_1,\dots,b_s\rangle$ be semigroups given by minimal generators. If $l<s$ (resp. $l>s$) then we extend the minimal generators $a_1,\dots,a_l$ to $a_1,\dots,a_l,\infty,\dots,\infty$ (resp. $b_1,\dots,b_s$ to $b_1,\dots,b_s,\infty,\dots,\infty$. $\Gamma<\widetilde{\Gamma}$ if $\Gamma\neq\widetilde{\Gamma}$ and there exist i such that $a_j=b_j$ for $j<i$ and $a_i<b_i$.

If $r = 5$

(6) (t^3, t^4, t^5)
(7) (t^3, t^4, t^5, t^6).

Proof. For the semigroup $\langle 3, 4, r\rangle$, we have two possibilities for r, $r \in \{5, \infty\}$. If $r = \infty$, we have according to the Remark 2.11 the order sequences $(3, 4)$, $(3, 4, 6)$ and $(3, 4, 9)$. Since the characteristic $p \neq 3$, we obtain as corresponding weak normal forms $(t^3, t^4 + b_5t^5 + b_6t^6 + b_9t^9)$, $(t^3, t^4 + b_5t^5, t^6)$ and $(t^3, t^4 + b_5t^5 + b_6t^6, t^9)$. Consider the first normal form and use the automorphism $\varphi : K[[t]] \to K[[t]]$ defined by $\varphi(t) = t - \frac{b_5}{4}t^2$, we obtain

$$(t^3, t^4 + b_5t^5 + b_6t^6 + b_9t^9) \sim_{\mathcal{A}_r} (t^3 + \overline{b_5}t^5, t^4 + \overline{b_6}t^6 + \overline{b_9}t^9).$$

Using the automorphism defined by $\varphi(t) = t - \frac{\overline{a_5}}{3}t^3$, we obtain an $\mathcal{A}_r$-equivalent parametrization $(t^3, t^4 + \overline{\overline{b_6}}t^6 + \overline{\overline{b_9}}t^9)$. Now we use the automorphism defined by $\varphi(t) = t - \frac{\overline{\overline{b_9}}}{4}t^6$ to obtain (t^3, t^4) or $(t^3, t^4 + t^6)$.
Similarly, we obtain from the second weak normal form $(t^3, t^4 + b_5t^5, t^6)$, using the automorphisms defined by $\varphi(t) = t - \frac{b_5}{4}t^2$ and then using $\varphi(t) = t - \frac{c}{3}t^3$ (for suitable c), the normal form (t^3, t^4, t^6). From the third weak normal form, we obtain similarly (t^3, t^4, t^9) and $(t^3, t^4 + t^6, t^9)$.
If $r = 5$, we have according to Remark 2.11, the order sequences $(3, 4, 5)$, $(3, 4, 5, 6)$ with corresponding weak normal forms $(t^3, t^4 + b_6t^6, t^5 + c_6t^6)$ resp. (t^3, t^4, t^5, t^6). Using similar automorphisms as above, we obtain from the third weak normal form (t^3, t^4, t^5). The parametrizations are simple, since there are no parameters in the normal form and the deformation keep the semigroup or have multiplicity 2. □

Proposition 4.12. *Assume that the characteristic* $p = 3$*. Let* $(x_1(t), \ldots, x_n(t))$ *be a curve with semigroup* $\Gamma = \langle 3, 4, r\rangle$ *and 3-jet* $(t^3, 0, \ldots, 0)$*. Then the curve* $(x_1(t), \ldots, x_n(t))$ *is simple and* $\mathcal{A}_r$*-equivalent to one of the following parametrizations:*
If $r = \infty$

(1) (t^3, t^4)
(2) $(t^3 + t^5, t^4)$
(3) (t^3, t^4, t^6)
(4) $(t^3 + t^5, t^4, t^6)$
(5) (t^3, t^4, t^9)
(6) $(t^3 + t^5, t^4, t^9)$

If $r = 5$

(6) (t^3, t^4, t^5)
(7) (t^3, t^4, t^5, t^6).

Proof. For the semigroup $\langle 3, 4, r\rangle$, we have two possibilities for r, $r \in \{5, \infty\}$. If $r = \infty$, we have according to the Remark 2.11 the order sequences $(3, 4)$, $(3, 4, 6)$ and $(3, 4, 9)$. Since the characteristic $p \neq 2$, we obtain as corresponding weak normal forms $(t^3 + a_5t^5, t^4)$, $(t^3 + a_5t^5, t^4, t^6)$ and $(t^3 + a_5t^5, t^4, t^9)$.
If $r = 5$, we have according to Remark 2.11, the order sequences $(3, 4, 5)$, $(3, 4, 5, 6)$ with corresponding weak normal forms $(t^3, t^4 + b_6t^6, t^5)$ resp. (t^3, t^4, t^5, t^6). Using the automorphisms defined by $\varphi(t) = t - \frac{b_6}{4}t^3$ we obtain that the first case is $\mathcal{A}_r$-equivalent to (t^3, t^4, t^5). The parametrizations are simple, since there are no

parameters in the normal form and the deformation keep the semigroup or have multiplicity 2. □

Proposition 4.13. *Assume that the characteristic $p \neq 3,5$. Let $(x_1(t),\dots,x_n(t))$ be a curve with semigroup $\Gamma = \langle 3,5,r\rangle$ and 3-jet $(t^3,0,\dots,0)$. Then the curve $(x_1(t),\dots,x_n(t))$ is simple and $\mathcal{A}_r$-equivalent to one of the following parametrizations:*
If $r=\infty$

(1) (t^3,t^5)
(2) (t^3,t^5+t^6)
(3) (t^3,t^5+t^9)
(4) (t^3,t^5,t^6)
(5) (t^3,t^5,t^6+t^7)
(6) (t^3,t^5,t^9)
(7) (t^3,t^5+t^6,t^9)
(8) (t^3,t^5,t^{12})
(9) (t^3,t^5+t^6,t^{12})
(10) (t^3,t^5+t^9,t^{12}).

If $r=7$

(11) (t^3,t^5,t^7) *and* $p\neq 7$
(12) (t^3,t^5+t^6,t^7) *and* $p\neq 7$
(13) (t^3,t^5,t^6,t^7)
(14) (t^3,t^5,t^7,t^9)
(15) (t^3,t^5+t^6,t^7,t^9)

Proof. Proposition 3.1 implies that the curves with order sequence $(3,6,7,11)$ are not simple. This implies that for simple singularities with semigroup $\Gamma=\langle 3,5,r\rangle$ and 3-jet $(t^3,0,\dots,0)$, we have as possible semigroups $\langle 3,5\rangle$ and $\langle 3,5,7\rangle$.
Let us consider first the curves with semigroup $\langle 3,5\rangle$. According to the Remark 2.11, we have the following possible order sequences $(3,5)$, $(3,5,6)$, $(3,5,9)$ and $(3,5,12)$.
Assume that the order sequence is $(3,5)$, since the characteristic $p\neq 3$ we have the following weak normal form

$$(t^3,t^5+b_6t^6+b_7t^7+b_9t^9+b_{12}t^{12}).$$

We have to consider the following cases:

(1) $b_6\neq 0$.
(2) $b_6=0$.

(1) : In this case, we may assume that $b_6=1$. We consider the following automorphism

$$\varphi(t)=t-\frac{b_6}{5}t^3$$

and obtain that the parametrization is equivalent to

$$(t^3+\overline{b_7}t^7,t^5+t^6+\overline{b_9}t^9+\overline{b_{12}}t^{12}).$$

Now we consider the automorphism $\varphi(t)=t-\frac{\overline{a_7}}{3}t^5$ and obtain that the parametrization is equivalent to

$$(t^3,t^5+t^6+\overline{\overline{b_9}}t^9+\overline{\overline{b_{12}}}t^{12}).$$

Now we consider the automorphism $\varphi(t) = t - \frac{\overline{\overline{b_9}}}{6}t^4$ and obtain that the parametrization is equivalent to

$$(t^3, t^5 + t^6 + \widetilde{b_{12}}t^{12}).$$

Now we consider the automorphism $\varphi(t) = t - \frac{\widetilde{b_{12}}}{5}t^8$ to obtain the equivalent parametrization $(t^3, t^5 + t^6)$.
(2) : We use the automorphism $\varphi(t) = t - \frac{b_7}{5}t^3$ to obtain the equivalent parametrization

$$(t^3 + \overline{a_7}t^7, t^5 + \overline{b_9}t^9 + \overline{b_{12}}t^{12}).$$

Next we use the automorphism $\varphi(t) = t - \frac{\overline{a_7}}{3}t^5$ to obtain

$$(t^3, t^5 + \overline{\overline{b_9}}t^9 + \overline{\overline{b_{12}}}t^{12}).$$

Now we use the automorphism $\varphi(t) = t - \frac{\overline{\overline{b_{12}}}}{5}t^8$ to obtain

$$(t^3, t^5 + \overline{\overline{b_9}}t^9).$$

If $\overline{\overline{b_9}} \neq 0$ then the parametrization is equivalent to $(t^3, t^5 + t^9)$. If $\overline{\overline{b_9}} = 0$ then we obtain (t^3, t^5). All together, we obtained the cases $(1), (2), (3)$ of the proposition.
Similarly we can prove that the order sequence $(3, 5, 6)$ leads to the cases (4) and (5). The order sequence $(3, 5, 9)$ leads to the cases (6) and (7) and the order sequence $(3, 5, 12)$ leads to the cases (8), (9) and (10).
Now we consider the semigroup $\langle 3, 5, 7\rangle$. According to Remark 2.11, we have the following possible order sequences $(3, 5, 7)$, $(3, 5, 6, 7)$ and $(3, 5, 7, 9)$. As before it can be proved that $(3, 5, 7)$ leads to the cases (11) and (12). If the characteristic $p = 7$ these curves are not simple (Proposition 3.8). The sequence $(3, 5, 6, 7)$ leads to the case (13) and the sequence $(3, 5, 7, 9)$ leads to the cases (14) and (15).
The parametrizations are simple, since there are no parameters in the normal form and the deformation keep the semigroup, have semigroup $\langle 3, 4, r\rangle$ or have multiplicity 2. ∎

Remark 4.14. The curve (t^3, t^5, t^6, t^7) is also simple in characteristic $p = 5$.

Proposition 4.15. *Assume that the characteristic $p = 3$. Let $(x_1(t), \dots, x_n(t))$ be a curve with semigroup $\Gamma = \langle 3, 5, r\rangle$ and 3-jet $(t^3, 0, \dots, 0)$. Then the curve $(x_1(t), \dots, x_n(t))$ is simple and $\mathcal{A}_r$-equivalent to one of the following parametrizations:*
If $r = \infty$

(1) $(t^3 + \alpha t^4 + \beta t^7, t^5, t^s)$
with $\alpha\beta = 0$, $\alpha, \beta \in \{0, 1\}$ and $s \in \{6, 9, 12, \infty\}$.

If $r = 7$

(2) $(t^3 + \alpha t^4, t^5, t^7, t^s)$
with $\alpha \in \{0, 1\}$ and $s \in \{6, 9, \infty\}$.

Proof. Let us consider first the curves with semigroup $\langle 3, 5\rangle$. According to the Remark 2.11, we have the following possible order sequences $(3, 5)$, $(3, 5, 6)$, $(3, 5, 9)$ and $(3, 5, 12)$.
Assume that the order sequence is $(3, 5)$, since the characteristic $p \neq 5$ we have the following weak normal form

$$(t^3 + a_4t^4 + a_7t^7, t^5).$$

If $a_4 \neq 0$ we may assume that $a_4 = 1$ and consider the automorphism $\varphi(t) = t - \frac{a_7}{4}t^4$ to obtain the $\mathcal{A}_r$-equivalent parametrization $(t^3 + t^4, t^5)$. If $a_4 = 0$ and $a_7 \neq 0$ we may assume that $a_7 = 1$ and obtain $(t^3 + t^7, t^5)$. All together we obtain the case (1) with $s = \infty$.
Similarly we obtain for the other order sequences case (1) for a suitable choice of s.
Now we consider the semigroup $\langle 3, 5, 7\rangle$. According to Remark 2.11, we have the following possible order sequences $(3, 5, 7)$, $(3, 5, 6, 7)$ and $(3, 5, 7, 9)$. As before it can be proved that we obtain (2).
The parametrizations are simple, since there are no parameters in the normal form and the deformation keep the semigroup, have semigroup $\langle 3, 4, r\rangle$ or have multiplicity 2. □

Proposition 4.16. *Assume that the characteristic $p \neq 3, 5$. Let $(x_1(t), \dots, x_n(t))$ be a curve with semigroup $\Gamma = \langle 3, 7, 8\rangle$ and 3-jet $(t^3, 0, \dots, 0)$.*
The curve $(x_1(t), \dots, x_n(t))$ is simple if it is $\mathcal{A}_r$-equivalent to (t^3, t^6, t^7, t^8).

Proof. A curve $(x_1(t), \dots, x_n(t))$ is not simple if the order sequence is $(3, 7, 8, \dots)$ (Proposition 3.1). The curve can only be simple if the order sequence is $(3, 6, 7, 8)$. It is an immediate consequence of Remark 2.11 that this curve is simple, since the characteristic $p \neq 3$. In characteristic $p = 5$ the curve is not simple since the curve $(t^3, \alpha t^5 + t^6, \beta t^6 + t^7, t^8)$ is in a suitable deformation (Proposition 3.7). □

Proposition 4.17. *Assume that the characteristic $p = 3$. Let $(x_1(t), \dots, x_n(t))$ be a curve with semigroup $\Gamma = \langle 3, 7, 8\rangle$ and 3-jet $(t^3, 0, \dots, 0)$.*
The curve $(x_1(t), \dots, x_n(t))$ is simple if it is $\mathcal{A}_r$-equivalent to $(t^3+\alpha t^4+\beta t^5, t^6, t^7, t^8)$ with $\alpha\beta = 0$, $\alpha, \beta \in \{0, 1\}$.

Proof. A curve $(x_1(t), \dots, x_n(t))$ is not simple if the order sequence is $(3, 7, 8, \dots)$ (Proposition 3.1). The curve can only be simple if the order sequence is $(3, 6, 7, 8)$. We obtain using Remark 2.11

$$(t^3 + a_4t^4 + a_5t^5, t^6, t^7, t^8).$$

As in the proof of Proposition 4.15 we can show that this leads to the normal form of the Proposition.
The parametrizations are simple, since there are no parameters in the normal form and the deformation keep the semigroup, have semigroup $\langle 3, 5, r\rangle$, have semigroup $\langle 3, 4, r\rangle$ or have multiplicity 2.

□

Proposition 4.18. *Let $(x_1(t), \dots, x_n(t))$ be a curve with 4-jet $(t^4, 0, \dots, 0)$. Then the curve $(x_1(t), \dots, x_n(t))$ is simple if and only if the semigroup $\Gamma = \langle 4, 5, 6, 7\rangle$ with one of the following normal forms.*

(1) (t^4, t^5, t^6, t^7)
(2) $(t^4, t^5, t^6, t^7, t^8)$.
we obtain additionally if $p = 3$
(3) $(t^4, t^5, t^6 + t^8, t^7)$
we obtain additionally if $p = 5$
(4) $(t^4, t^5 + t^8, t^6, t^7)$
we obtain additionally if $p = 7$
(5) $(t^4, t^5, t^6, t^7 + t^8)$

Proof. Corollary 3.2 implies that curves with 4-jet $(t^4, 0, \ldots, 0)$ and semigroup $\langle 4,5,6\rangle$ and order sequence $(4,5,6,8)$ are not simple. This implies that the only possibility for a simple curve is a curve with semigroup $\langle 4,5,6,7\rangle$. From Proposition 2.12, we obtain the curve $(t^4, t^5, t^6, t^7, t^8)$ or a curve with weak normal form $(t^4, t^5 + a_8 t^8, t^6 + b_8 t^8, t^7 + c_8 t^8)$.
If $p \neq 7$ we use the automorphism $\varphi(t) = t - \frac{c_8}{7} t^2$, and obtain an equivalent curve

$$(t^4, t^5 + \overline{a_8} t^8, t^6 + \overline{b_8} t^8, t^7).$$

Now using the automorphism $\varphi(t) = t - \frac{\overline{b_8}}{6} t^3$, we obtain an equivalent curve

$$(t^4, t^5 + \overline{\overline{a_8}} t^8, t^6, t^7).$$

Finally, if $p \neq 5$ we use the automorphism $\varphi(t) = t - \frac{\overline{\overline{a_8}}}{5} t^4$ and obtain the curve (t^4, t^5, t^6, t^7). If $p = 5$ or $p = 7$ we obtain the additional cases. ☐

Proposition 4.19. *Let $(x_1(t), \ldots, x_n(t))$ be a curve with 4-jet $(0, t^4, 0, \ldots, 0)$. Then the curve $(x_1(t), \ldots, x_n(t))$ is simple if and only if it has one of the following semigroups $\langle 4,5,6\rangle$, $\langle 4,5,7\rangle$, $\langle 4,6,7\rangle$, $\langle 4,5,6,7\rangle$ and $\langle 4,6,7,9\rangle$. We obtain the following normal forms:*

(1) $(0, t^4, t^5, t^6)$
(2) $(0, t^4, t^5, t^7)$
(3) $(0, t^4, t^6, t^7)$
(4) $(0, t^4, t^5, t^6, t^7)$
(5) (t^5, t^4, t^6, t^7)
(6) (t^6, t^4, t^5, t^7)
(7) (t^7, t^4, t^5, t^6)
(8) $(0, t^4, t^6, t^7, t^9)$
(9) (t^9, t^4, t^6, t^7)
and additionally if $p = 3$
(10) $(0, t^4, t^5, t^6 + t^7)$
(11) $(0, t^4, t^6 + t^9, t^7)$
(12) $(t^9, t^4, t^6 + t^9, t^7)$
(13) $(t^7, t^4, t^5, t^6 + t^7)$
and additionally if $p = 5$
(14) $(0, t^4, t^5 + t^7, t^6)$
(15) $(0, t^4, t^5 + t^6, t^7)$
(16) $(t^6, t^4, t^5 + t^6, t^7)$
(17) $(t^7, t^4, t^5 + t^7, t^6)$
and additionally if $p = 7$
(18) $(0, t^4, t^6, t^7 + t^9)$
(19) $(t^9, t^4, t^6, t^7 + t^9)$

Proof. Proposition 3.1 implies that a curve with order sequence $(6,4,5)$ is not simple. Corollary 3.2 implies that curves with order sequence $(5,4,10)$ and $(7,4,6,9)$ are not simple. This implies that the only candidates for simple curves with 4-jet $(0, t^4, 0, \ldots, 0)$ must have one of the following semigroups $\langle 4,5,6\rangle$, $\langle 4,5,7\rangle$, $\langle 4,6,7\rangle$, $\langle 4,5,6,7\rangle$ and $\langle 4,6,7,9\rangle$. Proposition 2.13 gives the weak normal forms of the curves. We will prove here only one case, the other cases are similar.
We consider the case $\Gamma = \langle 4,6,7,9\rangle$. According to the two order sequences $(\infty, 4, 6, 7, 9)$

and $(9,4,6,7)$, we have in characteristic $p \neq 2$ two weak normal forms (Proposition 2.13):

$$(0, t^4, t^6, t^7, t^9)$$

and

$$(t^9, t^4, t^6 + b_9 t^9, t^7 + c_9 t^9).$$

For the second weak normal form, we use in characteristic $p \neq 7$ the automorphism $\varphi(t) = t - \frac{c_9}{7} t^3$ and obtain an equivalent parametrization

$$(t^9, t^4, t^6 + \overline{b_9} t^9, t^7).$$

If the characteristic $p \neq 3$ we use the automorphism $\varphi(t) = t - \frac{\overline{b_9}}{6} t^4$ to obtain (t^9, t^4, t^6, t^7). If the characteristic $p = 3$ we obtain additionally $(t^9, t^4, t^6 + t^9, t^7)$. If the characteristic $p = 7$ we obtain additionally $(t^9, t^4, t^6, t^7 + t^9)$.
It is not difficult to see that deformations with fixed 4-jet $(0, t^4, 0, \ldots, 0)$ of (1) to (19) are again in the class of (1) to (19). If the multiplicity drops by 1, we obtain only simple parametrizations. The same holds if multiplicity drops by 2. □

Proposition 4.20. *Let $(x_1(t), \ldots, x_n(t))$ be a curve of multiplicity 5. The curve $(x_1(t), \ldots, x_n(t))$ is simple if and only if $ord_t(x_1(t)) > 5$ and the semigroup is $\langle 5,6,7,8 \rangle$, $\langle 5,6,7,8,9 \rangle$ or $\langle 5,6,7,9 \rangle$ with the following normal forms:*

(1) $(0, t^5, t^6, t^7, t^8)$
(2) $(t^9, t^5, t^6, t^7, t^8)$
(3) $(0, t^5, t^6, t^7, t^8, t^9)$
(4) $(0, t^5, t^6, t^7, t^9)$
(5) $(t^8, t^5, t^6, t^7, t^9)$
additionally if $p = 3$
(6) $(0, t^5, t^6 + t^9, t^7, t^8)$
(7) $(t^9, t^5, t^6 + t^9, t^7, t^8)$
(8) $(0, t^5, t^6 + t^8, t^7, t^9)$
(9) $(t^8, t^5, t^6 + t^8, t^7, t^9)$
additionally if $p = 5$
(10) $(0, t^5 + t^9, t^6, t^7, t^8)$
(11) $(t^9, t^5 + t^9, t^6, t^7, t^8)$
(12) $(0, t^5 + t^8, t^6, t^7, t^9)$
(13) $(t^8, t^5 + t^8, t^6, t^7, t^9)$
additionally if $p = 7$
(14) $(0, t^5, t^6, t^7 + t^9, t^8)$
(15) $(t^9, t^5, t^6, t^7 + t^9, t^8)$
(16) $(0, t^5, t^6, t^7 + t^8, t^9)$
(17) $(t^8, t^5, t^6, t^7 + t^8, t^9)$

Proof. Corollary 3.2 (5) implies that curves with 5-jet $(t^5, 0, \ldots, 0)$ or $(0, \ldots, 0)$ are not simple.
We may now assume that the 5-jet is $(0, t^5, 0, \ldots, 0)$. Proposition 3.1 implies that curves with order sequence $(8,5,6,7)$ are not simple. This implies that we have the following candidates for simple curves with weak normal form (see Proposition 2.14):

(1) $(0, t^5 + a_9 t^9, t^6 + b_9 t^9, t^7 + c_9 t^9, t^8 + d_9 t^9)$
(2) $(t^9, t^5 + a_9 t^9, t^6 + b_9 t^9, t^7 + c_9 t^9, t^8 + d_9 t^9)$
(3) $(0, t^5, t^6, t^7, t^8, t^9)$

(4) $(0, t^5 + a_8 t^8, t^6 + b_8 t^8, t^7 + c_8 t^8, t^9)$
(5) $(t^8, t^5 + a_8 t^8, t^6 + b_8 t^8, t^7 + c_8 t^8, t^9)$.

We will prove that weak normal form (2) is equivalent to $(t^9, t^5, t^6, t^7, t^8)$. The other cases are similar. As before, we consider the automorphism $\varphi(t) = t - \frac{d_9}{8} t^2$ and obtain the equivalent parametrization $(t^9, t^5 + \overline{a_9} t^9, t^6 + \overline{b_9} t^9, t^7 + \overline{c_9} t^9, t^8)$.
If the characteristic $p \neq 7$ we perform the automorphism $\varphi(t) = t - \frac{\overline{c_9}}{7} t^3$ to obtain $(t^9, t^5 + \overline{\overline{a_9}} t^9, t^6 + \overline{\overline{b_9}} t^9, t^7, t^8)$. The automorphism $\varphi(t) = t - \frac{\overline{\overline{b_9}}}{6} t^9$ leads to $(t^9, t^5 + \tilde{a}_9 t^9, t^6, t^7, t^8)$. If the characteristic $p \neq 5$ we obtain using again a suitable automorphism the normal form $(t^9, t^5, t^6, t^7, t^8)$. If the characteristic $p = 7$ we obtain the additional case (15). If the characteristic $p = 5$ we obtain the additional case (11). If the characteristic $p = 3$ we obtain the additional case (7).
To prove that the curves (1) to (5) are simple, we consider first a deformation which keeps the multiplicity. It is easy to see that we obtain again a curve equivalent to (1), (2), (3), (4) or (5). A deformation with multiplicity 4, 3 or 2 is also simple. This is easy to check. □

5. Classification in Characteristic 2

In this section we want to classify the simple curves in characteristic $p = 2$.

Proposition 5.1. *Let $(x_1(t), \ldots, x_n(t))$ be a curve in weak normal form. Assume that*

$$ord_t x_1(t) \geq 3,\ ord_t x_2(t) \geq 4 \text{ and } ord_t x_3(t) \geq 6,$$

Then the curve is not simple.
Especially curves of multiplicity greater or equal to 4 are not simple.

Proof. We consider the deformation

$$(\alpha t^2 + x_1(t), \beta t^4 + \gamma t^5 + x_2(t), \delta t^6 + x_3(t), x_4(t), \ldots, x_n(t))$$

. Proposition 3.4 implies that for generic α, β, γ and δ the curve is not simple. □

Proposition 5.2. *The simple curves with 2-jet $(t^2, 0, \ldots, 0)$ or 3-jet $(t^3, 0, \ldots, 0)$ are in the following list.*

(1) (t^2, t^3)
(2) (t^2, t^4, t^5)
(3) $(t^2 + t^3, t^4, t^5)$
(4) (t^3, t^4, t^5)
(5) $(t^3, t^4 + t^6, t^5)$

Proof. It is a consequence of Proposition 3.4 and Proposition 5.1 that the curves listed above are the only candidates for simple curves. The curves do not contain parameters, deformations of the curves of multiplicity 2 resp. 3 give again one of these curves. □

Proposition 5.3. *Let $(x_1(t), \ldots, x_n(t))$ be a curve with semigroup $\langle 2, a \rangle$ and 2-jet $(0, t^2, 0, \ldots, 0)$. Then the curve is simple and $\mathcal{A}_r$-equivalent to one of the following normal forms*

(1) $(0, t^2 + t^m, t^a)$,
(2) $(t^a, t^2 + t^m)$

with $m = \infty$ or m is odd and in case (1) $2 < m < a$, in case (2) $2 < m < a + 2$.

Proof. The first case is a consequence of Lemma 11 in [26]. The second case can be proved similarly to the proof of Lamma 11 in [26]. □

Proposition 5.4. *Let $(x_1(t),\dots,x_n(t))$ be a curve with semigroup $\langle 3,a,r\rangle$ and 3-jet $(0,t^3,0,\dots,0)$. Then $(x_1(t),\dots,x_n(t))$ is $\mathcal{A}_r$-equivalent to one of the following normal forms*

(1) $(0,t^3,t^k+t^b,t^r)$,
(2) (t^k+t^b,t^3,t^r),
(3) (t^r,t^3,t^k+t^b).

$r=\infty$ and $3\nmid k$ or $k<r<2k-2$ and $k.r\equiv 2 \mod 3$. $b=\infty$ or $k<b<2k-2$ and $k.b\equiv 2 \mod 3$ and $b<r$.
The curve is simple if and only if $k\leq 10$.

Proof. The first part of the proposition is a consequence of Corollary 4.7 and Remark 4.8. This implies also that the curve is simple if $k\leq 10$. In case of $k>10$ one condition for the curve being simple is $k<13$. This implies $k=11$. Further we have $b<k+p=13$ or $r<k+p=13$. This is impossible since $k<b$ and $k<r$. □

6. Proof of the main Theorem

The aim of this section is to prove Theorem 1.5.
Corollary 3.2 implies that curves with 5-jet $(t^5,0\dots,0)$ or $(0,\dots,0)$ are not simple. Proposition 5.1 implies that in characteristic $p=2$ curves of multiplicity greater or equal to 4 are not simple. Table 11 contains the simple curves with 5-jet $(0,t^5,0,\dots,0)$ in characteristic $p\neq 2$. This is a result of Proposition 4.20. Table 10 contains the simple curves with 4-jet $(0,t^4,0,\dots,0)$ in characteristic $p\neq 2$. This is a result of Proposition 4.19. Table 9 contains the simple curves with 4-jet $(t^4,0,\dots,0)$ in characteristic $p\neq 2$. This is a result of Proposition 4.18. Table 6 contains the simple curves with 3-jet $(t^3,0,\dots,0)$ in characteristic $p\neq 2,3$. This is a result of Lemma 4.10, Remark 4.14 and the Propositions 4.11,4.13 and 4.16. Table 7 contains the simple curves with 3-jet $(t^3,0,\dots,0)$ in characteristic $p=3$. This is a result of Lemma 4.10, Remark 4.14 and the Propositions 4.12,4.15 and 4.17. Table 8 contains the simple curves with 3-jet $(t^3,0,\dots,0)$ in characteristic $p=2$. This is a result of Lemma 4.10 and Proposition 5.2. Table 4 contains the simple curves with 3-jet $(0,t^3,0,\dots,0)$ in characteristic $p\neq 3$. This is a result of Corollary 4.7 and Remark 4.8. Table 5 contains the simple curves with 3-jet $(0,t^3,0,\dots,0)$ in characteristic $p=3$. This is a result of Proposition 4.9. Table 2 contains the simple curves of multiplicity 2 in characteristic $p\neq 2$. This is a result of the Propositions 4.3 and 4.5. Table 3 contains the simple curves of multiplicity 2 in characteristic $p=2$. This is a result of Propositions 5.2 and 5.3. Table 1 contains the simple curves of multiplicity 1. This is a result of Proposition 4.1 and Remark 4.2.

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