THE DELTA INVARIANT AND SIMULTANEOUS NORMALIZATION FOR FAMILIES OF ISOLATED NON-NORMAL SINGULARITIES

GERT-MARTIN GREUEL AND GERHARD PFISTER

Abstract. We consider families of schemes over arbitrary fields resp. analytic varieties with finitely many (not necessarily reduced) isolated non-normal singularities, in particular families of generically reduced curves. We define a modified delta invariant for isolated non-normal singularities of any dimension that takes care of embedded points and prove that it behaves upper semicontinuous in flat families parametrized by an arbitrary principal ideal domain. Moreover, if the fibers contain no isolated points, then the family admits a fiberwise normalization iff the delta invariant is locally constant. The results generalize results by Teissier and Chiang-Hsieh–Lipman for families of reduced curve singularities and provide possible improvements for algorithms to compute the genus of a curve.

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Contents

Introduction

The delta invariant, also called genus defect, is the most basic numerical invariant of a reduced curve singularity. It is used to control the topolgy in a family of algebraic curves over the complex numbers, but it is also used in coding theory for curves over finite fields. Non-reduced curves appear naturally as fibers of the projection of a surface to a curve and can in general not be avoided. In [BG90] a delta invariant was defined for generically reduced complex analytic curves and it was shown that it controls the topology of the fibers in a family, e.g. the Betti numbers, in particular the number of connected components. In [Gr17] the delta invariant was further extended to complex ananlytic isolated non-normal singularities (e.g. isolated singularities) of any dimension and its behavior was studied in connection with simultaneous normalization.

The study of simultaneous normalization of deformations of a reduced curve singularity has been initiated by Teissier in the 1970's in the complex analytic setting. The main result was, that a flat family of reduced curve singularities over a normal base space admits a simultaneous normalization if and only if the delta invariant of the curve singularities is locally constant. This was further carried on by Chiang-Hsieh and Lipman [CL06] in the algebraic setting for families of reduced curves defined over a perfect field, clarifying some points in the proof given in [Te78]. In [CL06] the authors obtain also intermediate results for families of higher dimensional reduced and pure dimensional varieties, but the δ-constant criterion for simultaneous normalization is only proved for families of reduced curves (and for projective morphisms with equidimensional reduced fibers of arbitrary dimension, replacing the δ-invariant by the Hilbert polynomial, see also [Ko11]).

The results by Chiang-Hsieh and Lipman motivated us to reconsider the δ-constant criterion for families of schemes with finitely many isolated not necessarily reduced non-normal singularities, including the case of generically reduced curves, defined over an arbitrary field. We define and use in the algebraic setting a modified delta invariant for an isolated non-normal singularity (INNS) of any dimension analogous to the complex analytic case. One of our main results are semicontinuity theorems for this new δ-invariant for families of schemes parametrized by the spectrum of a principal ideal domain (PID), see Theorem 20 and its corollaries in arbitrary characteristic and Theorem 33 in characteristic 0. By the same theorems the semicontinuity holds also for what we call the ε-invariant, i.e., the length of the embedded components of the non-normal points.

We apply the semicontinuity to prove a δ-constant criterion for simultaneous normalization of a family of INNSs. This means that a family of affine Noetherian schemes over the spectrum of an arbitrary PID with fibers having only finitely many isolated non-normal singularities (but no isolated points) and with singular locus finite over the base admits a fiberwise normalization [1] if and only if the total δ-invariant of the fibers is constant (for a precise formulation see Theorem 31 and Corollary 32).

Although we use ideas from [CL06], the proofs of our main results are quite different. We make substantial use of the fact that our base ring is a (not

[1] The notion of "fiberwise normalization", introduced in this paper, is useful for schemes over an arbitray base field and coincides with "simultaneous normalization" if the base field is perfect.

necessarily local) PID in contrast to [CL06], where the base ring may be a local normal ring of any dimension. Thus, in our setting, the semicontinuity of δ holds for fibers over closed and non-closed points in a neighbourhood of a given point (while in [CL06, Proposition 3.3] the semicontinuity holds only for generalizations).

Moreover, to prove the δ-constant criterion for simultaneous normalization of reduced curves, it is assumed in [CL06, Theorem 4.1] that the fibers are pure dimensional[2] and that the base scheme is the spectrum of a complete, or Henselian, or analytic normal local ring. The pure-dimensionality assumption is however not necessary, even in the more general situation of families of not necessarily reduced INNSs, but we have to take special care of isolated points in the fibers (cf. Theorem 20 and Theorem 31).

Since our base rings include $\mathbb{Z}$ and $\mathbb{k}[t]$, $\mathbb{k}$ any field, we just mention in passing that our results have interesting computational applications. E.g., if an isolated non-normal singularity is defined over $\mathbb{Z}$ resp. over $\mathbb{k}[t]$, the computation of the δ-invariant over $\mathbb{Q}$ resp. $\mathbb{k}(t)$ can be estimated and speeded up by the (much cheaper) computation modulo any (not only lucky) prime $p \in \mathbb{Z}$ resp. modulo $\langle t-a\rangle$, a any element in $\mathbb{k}$, provided that δ of the special fiber is finite. This applies of course also to δ for reduced curves (if the special fiber is reduced, the nearby fibers are reduced too) and hence can be used to improve algorithms to compute the genus of a curve. We refer to [GP21, Remark 24], where we considered δ for families of parametrized curve singularities, and where this is discussed in more detail.

The case of families of parametrized curves shows that allowing non-reduced singularities in the fibers is not an artificial assumption but occurs naturally. Consider e.g. a morphism $\phi : T \times S \to Y \times S$ over S (where T can be $\operatorname{Spec} \mathbb{k}[t]$ in the algebraic case or $\operatorname{Spec} \mathbb{k}[[t]]$ in the formal case or $(\mathbb{C}, 0)$ in the analytic case, with S in the corresponding category) such $C_s = \phi(\mathbb{C} \times \{s\})$ is a reduced curve in Y ($Y = \operatorname{Spec} \mathbb{k}[y_1, ..., y_n]$ resp. $\operatorname{Spec} \mathbb{k}[[y_1, ..., y_n]]$ resp. $(\mathbb{C}^n, 0)$) for $s \in S$. Let $X = \phi(T \times S)$ be closed in $Y \times S$ and flat over S. Then the fibers X_s of $X \to S$ have in general non reduced singularities (the reduction of X_s coincides with C_s) and our results apply to $\delta(X_s)$.

All rings in this paper are associative, commutative and with 1, ring maps map 1 to 1, and ring maps of local rings map the maximal ideal to the maximal ideal. Moreover, we assume all rings, modules and schemes to be Noetherian. A, R denote rings, $\mathbb{k}$ an arbitrary field, dim the Krull dimension and $\dim_{\mathbb{k}}$ the $\mathbb{k}$-vector space dimension. If R is an A-algebra, $\mathfrak{p}$ a prime ideal of A and $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = Q(A/\mathfrak{p})$ the residue field of A at $\mathfrak{p}$, we set

$$M(\mathfrak{p}) := M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} k(\mathfrak{p}) = M \otimes_A k(\mathfrak{p})$$

and call it the *fiber of an R-module M over* $\mathfrak{p}$.

[2] Pure dimensional is also assumed for projectice families of any dimension in [CL06] and [Ko11].

1. Delta for an Isolated Non-Normal Singularity

Let R be a reduced ring. Then $Q(R)$, the *total quotient ring of* R, is a direct product of fields. If $P_1, ..., P_r$ are the minimal primes of R then $Q(R)$ is the direct product of the fields $Q(R/P_i)$. $\overline{R}$ denotes the *integral closure* of R in $Q(R)$. $\overline{R}$ or, more precisely, the natural inclusion $R \hookrightarrow \overline{R}$ is called the *normalization of* R. $\overline{R}$ is the direct product of the integral closures $\overline{R_j}$ of $R_j := R/P_j$ in $Q(R_j)$ (cf. [Stack, tag 035P, Lemma 28.52.3]).

If R is not reduced, let R^{red} be the *reduction* of R, $\pi : R \twoheadrightarrow R^{red}$ the natural projection and $\mathrm{nil}(R) := \ker(\pi)$ the ideal of nilpotent elements of R. We denote by $\nu^{red} : R^{red} \hookrightarrow \overline{R}$ the normalization of R^{red} and call $\overline{R}$ or the composition

$$\nu := \nu^{red} \circ \pi : R \twoheadrightarrow R^{red} \hookrightarrow \overline{R}$$

the *normalization* of R. R is called *normal* if $\nu : R \to \overline{R}$ is an isomorphism. This is equivalent to $R_{\mathfrak{p}}$ being a normal domain for every prime ideal $\mathfrak{p} \subset R$. We often write $\overline{R}/R^{red}$ in place of $\overline{R}/\nu(R)$.

For an arbitrary R-module let $\mathrm{Ann}_R(M) = \{g \in R \,|\, gM = 0\}$ be the annihilator ideal of M in R.

Definition 1. *Let R be a ring. We define*

(1) $\mathcal{C}_R := \mathrm{Ann}_R(\overline{R}/R^{red}) \subset R$, *the* conductor ideal *of* R,
$\widetilde{\mathcal{C}}_R := \mathcal{C}_R \cap \mathrm{Ann}_R(\mathrm{nil}(R)) \subset R$, *the* extended conductor ideal *of* R,
$C_R := \mathrm{Spec}\, R/\mathcal{C}_R$ *the* conductor scheme *of* R *and*
$\widetilde{C}_R := \mathrm{Spec}\, R/\widetilde{\mathcal{C}}_R$ *the* extended conductor scheme *of* R.

(2) *The* non-normal locus of R *is denoted as*
$NNor(R) := \{\mathfrak{p} \in \mathrm{Spec}\, R \,|\, R_{\mathfrak{p}}$ *is not normal*$\}$.

Remark 2. (1) $\mathcal{C}_{R^{red}} = \mathrm{Ann}_{R^{red}}(\overline{R}/R^{red}) = \pi(\mathcal{C}_R)$ is the conductor ideal of R^{red}. We have $\mathrm{nil}(R) \subset \mathcal{C}_R$ since $\mathrm{nil}(R) = \ker(\nu)$, and π induces an isomorphism $R/\mathcal{C}_R \xrightarrow{\cong} R^{red}/\mathcal{C}_{R^{red}}$.

(2) The non-normal locus of R coincides with the union of the non-reduced locus of R and the non-normal locus of R^{red}, i.e., $NNor(R) = \mathrm{Supp}_R(\mathrm{nil}(R)) \cup \mathrm{Supp}_R(\overline{R}/R^{red})$.

(3) We have always $NNor(R) \subset V(\widetilde{\mathcal{C}}_R)$ but equality may not hold and $NNor(R)$ may not be closed in $\mathrm{Spec}\, R$. However, if $\overline{R}/R^{red}$ is (module-)finite over R (equivalently, $\overline{R}$ is finite over R), then $\mathrm{Supp}_R(\overline{R}/R^{red})$ coincides with $V(\mathcal{C}_R)$ and is therefore closed in $\mathrm{Spec}\, R$. We get:

If $\overline{R}$ is finite over R (for examples see Remark 15), then the non-normal locus of R

$$NNor(R) = V(\widetilde{\mathcal{C}}_R) = V(\mathrm{Ann}_R(\mathrm{nil}(R))) \cup V(\mathcal{C}_R).$$

is closed in $\mathrm{Spec}\, R$ and is the zero-locus of the extended conductor ideal $\widetilde{\mathcal{C}}_R$.

Definition 3. *We say that* $\mathfrak{p} \in \mathrm{Spec}\, R$ *is an* isolated non-normal point *of* R*, or that* R has an isolated non-normal singularity *at* $\mathfrak{p}$*, if there is an open* *neighbourhood U of $\mathfrak{p}$ in $\operatorname{Spec} R$ such that $R_{\mathfrak{q}}$ is normal for all $\mathfrak{q} \in U \smallsetminus \{\mathfrak{p}\}$. We also say that $\mathfrak{p}$ is an* INNS *(of R or of $\operatorname{Spec} R$).*

Let $\overline{R}$ be finite over R and $\mathfrak{p}$ an INNS of R. Then $\mathfrak{p}$ is either a normal point or $\mathfrak{p}$ is an isolated point of $V(\widetilde{\mathcal{C}}_R)$. Hence, if $\mathfrak{p}$ is non-normal then it is a maximal ideal, i.e., a closed point of $\operatorname{Spec} R$.

Lemma 4. *Let $\overline{R}$ be finite over R. Then R has finitely many non-normal points if and only if $R/\widetilde{\mathcal{C}}_R$ is Artinian.*

Proof. By Remark 2 we have $NNor(R) = V(\widetilde{\mathcal{C}}_R) = \operatorname{Supp}_R(R/\widetilde{\mathcal{C}}_R)$. The result follows follows since $R/\widetilde{\mathcal{C}}_R$ is Artinian $\Leftrightarrow$ $\operatorname{Spec} R/\widetilde{\mathcal{C}}_R$ is a finite set ([AM69, Prop. 8.3]). □

We are now going to define the delta invariant of an INNS. For this we introduce the following

Notation 5. *(1) Let R be a ring and $\mathfrak{p}_1, ..., \mathfrak{p}_r$ the minimal prime ideals of R. We denote by $\mathfrak{p}^i$ resp. $\mathfrak{p}^{>i}$ the intersection*[3] *of the $\mathfrak{p}_j$ with $\dim R/\mathfrak{p}_j = i$ resp. $\dim R/\mathfrak{p}_j > i$. With $X = \operatorname{Spec} R$ and $X^{red} = \operatorname{Spec} R^{red}$ we define for $i \geq 0$:*

$$\begin{array}{rclcrcl} R^i & := & R/\mathfrak{p}^i, & & X^i & := & \operatorname{Spec} R^i, \\ R^{>i} & := & R/\mathfrak{p}^{>i}, & & X^{>i} & := & \operatorname{Spec} R^{>i}. \end{array}$$

Note that R^i and $R^{>i}$ are reduced and thus X^i and $X^{>i}$ are reduced subschemes of X. In particular, X^0 is a finite set of reduced, isolated points of X^{red} and $X^{>0} = X^{red}$ iff X has no isolated points.

We set

$$r_i(X) := \sharp\{\text{irreducible componentes of } X^i\},$$

which is the number of i-dimensional irreducible components of X[4]*.*

(2) Let $\mathfrak{m} \in \operatorname{Spec} R$ be a maximal ideal of R and M an R-module. The 0-th local cohomology group *of M is the submodule*

$$H^0_{\mathfrak{m}}(M) = \{x \in M \mid \mathfrak{m}^k x = 0 \text{ for some } k \geq 0\}.$$

Since M is Noetherian, $H^0_{\mathfrak{m}}(M)$ is Noetherian too and is annihilated by some power of $\mathfrak{m}$; hence $H^0_{\mathfrak{m}}(M)$ has finite length, i.e. is Artinian.

Definition 6. *Let $(R, \mathfrak{m})$ be a local ring with normalization $\overline{R}$, $\Bbbk$ a field, and $\Bbbk \to R$ a ring map. We define:*

(i) *the* epsilon invariant *of R (w.r.t. $\Bbbk$),*

$$\varepsilon_{\Bbbk}(R) := \dim_{\Bbbk} H^0_{\mathfrak{m}}(R),$$

(ii) *the* delta invariant *of R (w.r.t. $\Bbbk$),*

$$\delta_{\Bbbk}(R) := \dim_{\Bbbk} \overline{R}/R^{red} - \varepsilon_{\Bbbk}(R),$$

[3] The empty intersection is the whole ring R. E.g., if no minimal primes $\mathfrak{p}_j$ with $\dim R/\mathfrak{p}_j = i$ exist then $R^i = 0$ and $X^i = \emptyset$.

[4] By definition, the irreducible components of X are the schemes $\operatorname{Spec} R/\mathfrak{p}_j, j = 1, ..., r$.

(iii) *the* (multiplicity of the) conductor *of* R *(w.r.t.* $\Bbbk$*)*

$$c_{\Bbbk}(R) := \dim_{\Bbbk} \overline{R}/\mathcal{C}_{R^{red}} - \varepsilon_{\Bbbk}(R).$$

Hence, if R is reduced and $\dim R > 0$ then $\varepsilon_{\Bbbk}(R) = 0$ and $\delta_{\Bbbk}(R) = \dim_{\Bbbk} \overline{R}/R$, the usual definition of $\delta_{\Bbbk}$.

Remark 7. Let $K = R/\mathfrak{m}$ denote the residue field of the local ring $(R, \mathfrak{m})$ and assume that $\dim_{\Bbbk} K < \infty$.

(1) $\varepsilon_{\Bbbk}(R)$ is finite while $\delta_{\Bbbk}(R)$ and $c_{\Bbbk}(R)$ may be infinite. If R is an INNS with $\overline{R}$ finite over R, then $\delta_{\Bbbk}(R)$ and $c_{\Bbbk}(R)$ are also finite (Lemma 10).

(2) If $\dim R = 0$, then $\overline{R} = R^{red} = K$, $\mathrm{nil}(R) = \mathfrak{m}$ and $H^0_{\mathfrak{m}}(R) = R$. We get $\delta_{\Bbbk}(R) = c_{\Bbbk}(R) = -\varepsilon_{\Bbbk}(R) = -\dim_{\Bbbk} R = -\dim_{\Bbbk} \mathrm{nil}(R) - \dim_{\Bbbk} K < 0$.

(3) Let $\dim R > 0$ and let R be an INNS with $\overline{R}$ finite over R. Since $R_{\mathfrak{p}}$ is reduced for $\mathfrak{p} \in U \smallsetminus \mathfrak{m}$, U some open neighbourhood of $\mathfrak{m}$ in $\operatorname{Spec} R$, we have $\mathrm{nil}(R) = H^0_{\mathfrak{m}}(R)$ and $\varepsilon_{\Bbbk}(R) = \dim_{\Bbbk} \mathrm{nil}(R)$.

If R is normal and $\dim R > 0$ then $\delta_{\Bbbk}(R) = c_{\Bbbk}(R) = \varepsilon_{\Bbbk}(R) = 0$. If R is reduced, then $\delta_{\Bbbk}(R) = 0$ if and only if R is normal. But if R is not reduced, then $\delta_{\Bbbk}(R) = 0$ may happen for non-normal R (see Example 8 (3)).

Example 8. (1) The ideal $I = \langle x \rangle \cap \langle x^2, y^2, xy \rangle = \langle x^2, xy \rangle \subset \Bbbk[[x, y]]$ defines a line with embedded component. With $R = \Bbbk[[x, y]]/I$ we get $\delta_{\Bbbk}(R^{red}) = 0$ and $\varepsilon_{\Bbbk}(R) = 1$, hence $\delta_{\Bbbk}(R) = -1$ and $c_{\Bbbk}(R) = -1$.

(2) The ideal

$$I = \langle x^3y + x^2y^2, x^2y^2 + xy^3 \rangle = \langle x + y \rangle \cap \langle x \rangle \cap \langle y \rangle \cap \langle x^2, y^3 \rangle \subset \Bbbk[[x, y]],$$

defines 3 lines with an embedded component at 0. For $R = \Bbbk[[x, y]]/I$ we have $\delta_{\Bbbk}(R^{red}) = 3$ and $\varepsilon_{\Bbbk}(R) = \dim_{\Bbbk} \sqrt{I}/I = 1$ [5] and hence $\delta_{\Bbbk}(R) = 2$. Since R^{red} is a reduced plane curve singularity, we get $c_{\Bbbk}(R^{red}) = 2\delta_{\Bbbk}(R^{red}) = 6$ and $c_{\Bbbk}(R) = c_{\Bbbk}(R^{red}) - \varepsilon_{\Bbbk}(R) = 5$.

(3) $I = \langle z, x^2 - y^3 \rangle \cap \langle x, y, z^2 \rangle$, $R = \Bbbk[[x, y]]/I$, defines a cusp in the (x, y)-plane and an embedded point in the z-direction. Then $\delta_{\Bbbk}(R^{red}) = 1$ and $\varepsilon_{\Bbbk}(R) = 1$ and hence $\delta_{\Bbbk}(R) = 0$.

Lemma 9. *Let* $(R, \mathfrak{m})$ *be a local ring,* $K = R/\mathfrak{m}$, $\Bbbk \to R$ *a ring map, and* $M \neq 0$ *a finitely generated* R*-module. Then* $\dim_{\Bbbk} M < \infty \Leftrightarrow M$ *is Artinian and* $\dim_{\Bbbk} K < \infty$*.*

Proof. If $\dim_{\Bbbk} M < \infty$ then M is Artinian since it satisfies obviously the descending chain condition. By Nakayama's lemma, $M/\mathfrak{m}M$ is a finite dimensional K-vector space $\neq 0$. We have $\dim_{\Bbbk} K \leq \dim_{\Bbbk} M/\mathfrak{m}M \leq \dim_{\Bbbk} M < \infty$. Conversely, if M Artinian then $\mathfrak{m}^n M = 0$ for some n. The K-vector space $\mathfrak{m}^k M/\mathfrak{m}^{k+1} M$ has finite K-dimension, hence finite $\Bbbk$-dimension since $\dim_{\Bbbk} K < \infty$. Thus $\dim_{\Bbbk} M < \infty$. □

Lemma 10. *Let* $(R, \mathfrak{m})$ *be a local ring with normalization* $\overline{R}$ *finite over* R*.*

[5] We compute ε and δ with SINGULAR [DGPS]: `codim` computes $\dim_{\Bbbk} \mathrm{nil}(R) = \dim_{\Bbbk} \sqrt{I}/I$ and the procedure `normal(..,"wd")` computes $\delta_{\Bbbk}(R^{red})$; the number of isolated points of $\operatorname{Spec} R$ can be determined with a primary decomposition of I.

(1) *The following are equivalent:*
   (i) $\mathfrak{m}$ *is an isolated non-normal point of* R*;*
   (ii) $R/\widetilde{\mathcal{C}}_R$ *is an Artinian ring;*
   (iii) $\overline{R}/\mathcal{C}_{R^{red}}$ *and* $\mathrm{nil}(R)$ *are Artinian* R*-modules;*
   (iv) $\overline{R}/R^{red}$ *and* $\mathrm{nil}(R)$ *are Artinan* R*-modules.*

(2) *Let* $\Bbbk$ *be a field and* $\Bbbk \to R$ *a ring map. Then the following are equivalent:*
   (i) $\mathfrak{m}$ *is an isolated non-normal point of* R *and* $\dim_{\Bbbk} R/\mathfrak{m} < \infty$*;*
   (ii) $\dim_{\Bbbk}(R/\widetilde{\mathcal{C}}_R)$ *is finite;*
   (iii) $c_{\Bbbk}(R^{red})$ *and* $\varepsilon_{\Bbbk}(R)$ *are finite.*
   (iv) $\delta_{\Bbbk}(R^{red})$ *and* $\varepsilon_{\Bbbk}(R)$ *are finite;*

   If any of these conditions hold, $c_{\Bbbk}(R)$ *and* $\delta_{\Bbbk}(R)$ *are finite and satisfy* $c_{\Bbbk}(R) = \delta_{\Bbbk}(R) + \dim_{\Bbbk}(R/\mathcal{C}_R)$.

Proof. We may assume that $(R, \mathfrak{m})$ is not normal. By Remark 2 we have $NNor(R) = V(\widetilde{\mathcal{C}}_R) = V(\mathrm{Ann}(\mathrm{nil}(R))) \cup V(\mathcal{C}_R)$.

(1) It is well known that a finitely generated R-module M is Artinian $\Leftrightarrow$ $\mathfrak{m}^k M = 0$ for some $k > 0 \Leftrightarrow \dim M = 0 \Leftrightarrow \mathrm{Supp}_R(M) = \{\mathfrak{m}\}$. Now (1) follows from $NNor(R) = \mathrm{Supp}_R(R/\widetilde{\mathcal{C}}_R) = \mathrm{Supp}_R(\mathrm{nil}(R)) \cup \mathrm{Supp}_R(\overline{R}/R^{red})$.

(2) The equivalence of (i) - (iv) follows from (1) for $\Bbbk = K := R/\mathfrak{m}$, noting that $\dim_{\Bbbk} \mathrm{nil}(R) = \varepsilon_{\Bbbk}(R)$ if $\dim(R) > 0$ and $\dim_{\Bbbk} \mathrm{nil}(R) = \varepsilon_{\Bbbk}(R) - 1$ if $\dim(R) = 0$. Together with Lemma 9 the equivalence follows for $\Bbbk$ with $\dim_{\Bbbk} K < \infty$. The exact sequence

$$0 \to R^{red}/\mathcal{C}_{R^{red}} \to \overline{R}/\mathcal{C}_{R^{red}} \to \overline{R}/R^{red} \to 0$$

implies $c_{\Bbbk}(R^{red}) = \delta_{\Bbbk}(R^{red}) + \dim_{\Bbbk}(R^{red}/\mathcal{C}_{R^{red}})$ and hence $c_{\Bbbk}(R) = \delta_{\Bbbk}(R) + \dim_{\Bbbk}(R/\mathcal{C}_R)$ by definition of $c_{\Bbbk}$ and $\delta_{\Bbbk}$. □

Now let R be any (not necessarily local) ring and let $\overline{R}$ be finite over R. It follows from Lemma 4 and 10 that R has only finitely many non-normal points $\Longleftrightarrow$ $R/\widetilde{\mathcal{C}}_R$ is Artinian. If R is a $\Bbbk$-algebra then: $\dim_{\Bbbk}(R/\widetilde{\mathcal{C}}_R) < \infty$ $\Longleftrightarrow$ $NNor(R)$ is finite and $\dim_{\Bbbk} k(\mathfrak{p}) < \infty$ for all $\mathfrak{p} \in NNor(R)$. The following definition generalizes δ, ε and c to non-local rings.

Definition 11. *Let* R *be a* $\Bbbk$*-algebra with normalization* $\overline{R}$ *finite over* R*. Assume that* R *has only finitely many isolated non-normal points* $\{\mathfrak{p}_1, ..., \mathfrak{p}_r\}$ *and that* $\delta_{\Bbbk}(R_{\mathfrak{p}_i}) < \infty$ *for* $i = 1, ..., r$*. We define*

$$\delta_{\Bbbk}(R) := \sum_1^r \delta_{\Bbbk}(R_{\mathfrak{p}_i}),$$

$\varepsilon_{\Bbbk}(R) := \sum_1^r \varepsilon_{\Bbbk}(R_{\mathfrak{p}_i})$ *and* $c_{\Bbbk}(R) := \sum_1^r c_{\Bbbk}(R_{\mathfrak{p}_i})$*, which are all finite.*

Example 12. Let $R_{\mathbb{C}} = \mathbb{C}[x, y]/I$, $I = \langle y^2 - 2x^2 \rangle \cap \langle y - x^2 \rangle$. $V(I)$ consists of two straight lines and a parabola meeting in $(0, 0)$ and in $(\pm\sqrt{2}, 2)$. The three INNS correspond to the maximal ideals $\mathfrak{p}, \pm\mathfrak{q}$. $\mathfrak{p}$ is a triple point with

$\delta_{\mathbb{C}}(R_{\mathbb{C},\mathfrak{p}}) = 3$, while $\pm\mathfrak{q}$ are ordinary nodes with $\delta_{\mathbb{C}}(R_{\mathbb{C},\pm\mathfrak{q}}) = 1$ each, hence $\delta_{\mathbb{C}}(R_{\mathbb{C}}) = 5$.

Let $R_{\mathbb{Q}} = \mathbb{Q}[x,y]/I$, with I as above. Then has $R_{\mathbb{Q}}$ has (in $\operatorname{Spec} R_{\mathbb{Q}}$) two INNS, at the maximal ideals $\mathfrak{p} = \langle x, y\rangle$ and $\mathfrak{q} = \langle x^2-2, y-2\rangle$, with $k(\mathfrak{p}) = \mathbb{Q}$ and $k(\mathfrak{q}) = \mathbb{Q}(\sqrt{2})$. We get $\delta_{\mathbb{Q}}(R_{\mathbb{Q},\mathfrak{p}}) = 3$ and $\delta_{\mathbb{Q}}(R_{\mathbb{Q},\mathfrak{q}}) = 2$, hence $\delta_{\mathbb{Q}}(R_{\mathbb{Q}}) = 5$. The equality $\delta_{\mathbb{Q}}(R_{\mathbb{Q}}) = \delta_{\mathbb{C}}(R_{\mathbb{C}})$ is a general fact, since[6] $R_{\mathbb{C}} = R_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$.

Using that $R^{red} = R^0 \oplus R^{>0}$, $\overline{R} = \overline{R}^0 \oplus \overline{R}^{>0}$ and $R^0 = \overline{R}^0$ we get

$$\begin{aligned}
\varepsilon_{\Bbbk}(R) &= \dim_{\Bbbk} Ker(R \to \overline{R}) + \#\{\text{isolated points of } \operatorname{Spec} R\},\\
&= \dim_{\Bbbk} Ker(R \to \overline{R}^{>0})\\
\delta_{\Bbbk}(R) &= \dim_{\Bbbk} \overline{R}^{>0}/R^{>0} - \varepsilon_{\Bbbk}(R)\\
&= \dim_{\Bbbk} Coker(R \to \overline{R}^{>0}) - \dim_{\Bbbk} Ker(R \to \overline{R}^{>0}).
\end{aligned}$$

It follows that δ can be expressed as an Euler characteristic.

Lemma 13. *With the assumptions of Definition 11 consider the 2–term complex with R in degree 0, $R^\bullet : 0 \to R \to \overline{R}^{>0} \to 0$. Then*

$$\delta_{\Bbbk}(R) = -\chi_{\Bbbk}(R^\bullet),$$

where $\chi_{\Bbbk}(L^\bullet) := \sum_i (-1)^i \dim_{\Bbbk} H^i(L^\bullet)$ for a complex $L^\bullet$ of $\Bbbk$-modules with finite dimensional cohomology.

The following technical lemma compares δ and ε of R with that of a finite modification of R whose positive dimensional part is a partial normalization of $R^{>0}$. It is a key lemma for the semicontinuity of δ.

Lemma 14. *Let R be a $\Bbbk$-algebra with $\overline{R}$ finite over R, having only finitely many isolated non-normal singularities, with residue fields finite over $\Bbbk$. Consider a finite morphism of $\Bbbk$-algebras $\mu : R \to \widetilde{R}$. Let $N \subset \operatorname{Spec} R$ be a finite set of closed points with residue fields finite over $\Bbbk$, such that $\operatorname{Spec} \mu$ is an isomorphism over $\operatorname{Spec} R \smallsetminus N$.*

Then the positive dimensional parts $R^{>0}$ and $\widetilde{R}^{>0}$ have the same normalization and μ satisfies

$$\begin{aligned}
\dim_{\Bbbk} Coker(\mu) - \dim_{\Bbbk} Ker(\mu) &= \delta_{\Bbbk}(R) - \delta_{\Bbbk}(\widetilde{R})\\
&= \varepsilon_{\Bbbk}(\widetilde{R}) - \varepsilon_{\Bbbk}(R) + \dim_{\Bbbk} Coker(\mu^{>0})
\end{aligned}$$

with $\mu^{>0} : R^{>0} \to \widetilde{R}^{>0}$ the induced map. $\mu^{>0}$ is finite and injective and a partial normalization[7] of the reduced positive dimensional part of R[8] and all numbers are finite.

[6] Let B be a $\Bbbk$-algebra and K a separable field extension of $\Bbbk$ then $\overline{B \otimes_{\Bbbk} K} = \overline{B} \otimes_{\Bbbk} K$ ([Stack], Lemma 32.27.4) and hence $\delta_K(B \otimes_{\Bbbk} K) = \delta_{\Bbbk}(B)$.

[7] Let $\nu : R \to \overline{R}$ be the normalization of R. A *partial normalization* of R is a birational morphism $\mu : R \to \widetilde{R}$ such that $\nu = \tilde{\nu} \circ \mu : R \to \widetilde{R} \to \overline{R}$, with $\tilde{\nu}$ the normalization of $\widetilde{R}$.

[8] In the case $R^{>0} = 0$, i.e. $X^{>0} = \emptyset$, the statements here and further on have to be interpreted approriately, e.g. with $\delta_{\Bbbk}(R^{>0}) = 0$ and $\varepsilon_{\Bbbk}(R^{>0}) = 0$.

Proof. By assumption R has only finitely many non-normal singularities with $\delta_{\Bbbk}(R)$ and $\varepsilon_{\Bbbk}(R)$ finite (Lemma 10). Since μ is an isomorphism outside finitely many closed points, $Ker(\mu)$ and $Coker(\mu)$ are Artinian. Then the kernel and cokernel of μ and of $\mu^{>0}$ are finite over $\Bbbk$ (Lemma 9).

We have $\operatorname{Spec}\widetilde{R}^{red} = \operatorname{Spec}\widetilde{R}^{>0} \cup \{\text{finitely many isolated points}\}$ and the restriction of μ induces a birational morphism $R^{>0} \to \widetilde{R}^{>0}$, since it is an isomorphism outside finitely many closed points. Let $\nu^{>0} : R^{>0} \to \overline{R}^{>0}$ be the normalization of $R^{>0}$. By [Stack, Lemma 28.52.5 (3), 035Q] $\nu^{>0}$ factors as $\nu^{>0} = \tilde{\nu} \circ \mu^{>0} : R^{>0} \to \widetilde{R}^{>0} \to \overline{R}^{>0}$ with $\tilde{\nu} : \widetilde{R}^{>0} \to \overline{R}^{>0}$ the normalization of $\widetilde{R}^{>0}$ and $\tilde{\nu}$ finite. Hence $\mu^{>0}$ is a partial normalization. It is finite since μ is finite and injective since $R^{>0}$ is reduced.

Now consider the 2–term complexes (with R resp. $\widetilde{R}$ in degree 0)

$$R^{\bullet} : 0 \to R \to \overline{R}^{>0} \to 0\,,$$
$$\widetilde{R}^{\bullet} : 0 \to \widetilde{R} \to \overline{R}^{>0} \to 0$$

and the morphism of complexes $\mu^{\bullet} : R^{\bullet} \to \widetilde{R}^{\bullet}$ with $\mu^0 = \mu$ and the identity in degree 1. Let $K^{\bullet}$ resp. $C^{\bullet}$ be the 1–term complexes $Ker(\mu)$ resp. $Coker(\mu)$, concentrated in degree 0. Then we have the exact sequence of complexes

$$0 \to K^{\bullet} \to R^{\bullet} \to \widetilde{R}^{\bullet} \to C^{\bullet} \to 0\,.$$

Taking Euler characteristics we get (by Lemma 13)

$$\dim_{\Bbbk} Coker(\mu) - \dim_{\Bbbk} Ker(\mu) = \chi_{_{\Bbbk}}(\widetilde{R}^{\bullet}) - \chi_{_{\Bbbk}}(R^{\bullet}) = \delta_{\Bbbk}(R) - \delta_{\Bbbk}(\widetilde{R})$$

showing the first equality. Since $\delta_{\Bbbk}(R) = \delta_{\Bbbk}(R^{>0}) - \varepsilon_{\Bbbk}(R)$ we get

$$\delta_{\Bbbk}(R) - \delta_{\Bbbk}(\widetilde{R}) = \varepsilon_{\Bbbk}(\widetilde{R}) - \varepsilon_{\Bbbk}(R) + \delta_{\Bbbk}(R^{>0}) - \delta_{\Bbbk}(\widetilde{R}^{>0}).$$

From this and from the inclusions $R^{>0} \hookrightarrow \widetilde{R}^{>0} \hookrightarrow \overline{R}^{>0}$ the second equality follows. □

Remark 15. (1) For all results of this paper we have to assume that $\overline{R}$ is (module-) finite over R. Integral domains that satisfy this conditions are called *N-1 rings*. An *N-2 ring* (or *Japanese ring*) is an integral domain R such for every finite field extension L of $Q(R)$ the integral closure of R in L is finite over R. R is a *Nagata ring* if R is Noetherian and for every prime ideal $\mathfrak{p}$ the ring $R/\mathfrak{p}$ is N-2 (see [Stack, Lemma 10.157.2, tag 03GH]).

(2) R is Nagata iff (cf. [CL06, 1.4.3])

- (a) for every maximal ideal $\mathfrak{n}$ of R the canonical map $R_{\mathfrak{n}} \to \widehat{R}_{\mathfrak{n}}$ from the local ring $R_{\mathfrak{n}}$ to its completion is reduced (flat with reduced fibers) and
- (b) for every reduced finitely generated R-algebra R' the set of normal points is open and dense in $\operatorname{Spec} R'$.
  Condition (b) is implied by (a) if R is semi-local. For further properties of Nagata rings we refer to [Stack, Section 10.157, tag 032E].

(3) Examples of Nagata rings are:

(1) fields, $\mathbb{Z}$, Noetherian complete local rings,
(2) Dedekind domains with perfect[9] fraction field[10],
(3) finite type ring extensions of any of the above,
(for (1) (2) (3) see [Stack, Proposition 10.157.16, tag 0335]),
(4) quasi-excellent, in particular excellent rings (e.g. analytic local rings), ([Stack, Lemma 15.51.5, tag 07QV]),
(5) localizations of a Nagata ring ([Stack, Lemma 10.157.6, tag 032U]),
(6) A-algebras (essentially) of finite type over a Nagata ring A ([Stack, Proposition 10.157.16, tag 0335]),
(7) $A[[x_1, ..., x_n]]$ is Nagata if A is Nagata, ([KS19, Appendix A, Property PSEP]).

A scheme X is called Nagata if for every $x \in X$ there exists an affine open neighbourhood $U \subset X$ of x such that the ring $\mathcal{O}_X(U)$ is Nagata. Note that there are discrete valuation rings that are not Nagata ([Stack, Example 10.157.17, tag 09E1]).

2. Semicontinuity of the Delta Invariant

We consider now families of isolated non-normal singularities. The following Proposition 16 is fundamental for the semicontinuity results of this paper.

Proposition 16. *Let $\varphi : A \to R$ be a flat morphism of rings with A a principal ideal domain and $\mu : R \to \widetilde{R}$ a finite morphism of A-algebras. Assume that*

(1) *the composition $\widetilde{\varphi} := \mu \circ \varphi : A \to \widetilde{R}$ is flat,*
(2) *$Ker(\mu)$ and $Coker(\mu)$ are finite over A,*
(3) *the normalization $\overline{R(\mathfrak{q})}$ is finite over $R(\mathfrak{q})$ and the residue fields at the non-normal points of $R(\mathfrak{q})$ are finite over $k(\mathfrak{q})$ for $\mathfrak{q} \in Im(\operatorname{Spec} \varphi)$.*

[9] A field $\Bbbk$ is perfect if $\Bbbk$ is of characteristic 0 or of characteristic $p > 0$ and every element has a p-th root (e.g. if $\Bbbk$ is finite).

[10] This statement is formulated in [Stack] only for Dedekind domains with fraction fields of characteristic 0, but the proof works for perfect fields of positive characteristic as well.

Then, for $\mathfrak{p} \in Im(\operatorname{Spec}\varphi)$ *there exists an open neighborhood* $U \subset \operatorname{Spec} A$ *of* $\mathfrak{p}$ *such that for* $\mathfrak{q} \in U \cap Im(\operatorname{Spec}\varphi)$ *the following holds:*

(i) $\delta_{k(\mathfrak{p})}(R(\mathfrak{p})) - \delta_{k(\mathfrak{q})}(R(\mathfrak{q})) = \delta_{k(\mathfrak{p})}(\widetilde{R}(\mathfrak{p})) - \delta_{k(\mathfrak{q})}(\widetilde{R}(\mathfrak{q}))$.

(ii) $\varepsilon_{k(\mathfrak{p})}(R(\mathfrak{p})) - \varepsilon_{k(\mathfrak{q})}(R(\mathfrak{q})) = \varepsilon_{k(\mathfrak{p})}(\widetilde{R}(\mathfrak{p})) - \varepsilon_{k(\mathfrak{q})}(\widetilde{R}(\mathfrak{q}))$
$+ \dim_{k(\mathfrak{p})} Coker(\mu(\mathfrak{p})^{>0}) - \dim_{k(\mathfrak{q})} Coker(\mu(\mathfrak{q})^{>0})$.

(iii) *If* $\operatorname{Spec}\widetilde{R}^1 \cap \operatorname{Spec}\widetilde{R}^{>1} = \emptyset$ *(e.g. if* $\widetilde{R} = \overline{R}$ *or* $\widetilde{R}^1 = 0$*), then*
$\varepsilon_{k(\mathfrak{p})}(R(\mathfrak{p})) - \varepsilon_{k(\mathfrak{q})}(R(\mathfrak{q})) = \varepsilon_{k(\mathfrak{p})}(\widetilde{R}(\mathfrak{p})) - \varepsilon_{k(\mathfrak{q})}(\widetilde{R}(\mathfrak{q}))$
$+ \dim_{k(\mathfrak{p})}(\widetilde{R}^{>1}/\mu(R^{>1})) \otimes_A k(\mathfrak{p}) - \dim_{k(\mathfrak{q})}(\widetilde{R}^{>1}/\mu(R^{>1})) \otimes_A k(\mathfrak{q})$
$\geq \varepsilon_{k(\mathfrak{p})}(\widetilde{R}(\mathfrak{p})) - \varepsilon_{k(\mathfrak{q})}(\widetilde{R}(\mathfrak{q}))$.

(iv) *If* $Ker(\mu) = 0$ *then* $Ker(\mu(\mathfrak{q}) : R(\mathfrak{q}) \to \widetilde{R}(\mathfrak{q})) = 0$ *for* $\mathfrak{q} \neq \mathfrak{p}$.

Here $\widetilde{R}(\mathfrak{q}) = \widetilde{R} \otimes_A k(\mathfrak{q})$ *and* $\mu(\mathfrak{q})^{>0} : R(\mathfrak{q})^{>0} \to \widetilde{R}(\mathfrak{p})^{>0}$ *is the induced map of positive dimensional parts, which is a partial normalization of* $R(\mathfrak{q})^{>0}$.

Proof. We set

$$\begin{aligned} \mathcal{N} : &= Ker(\mu : R \to \widetilde{R}), \\ \mathcal{M} : &= Coker(\mu : R \to \widetilde{R}). \end{aligned}$$

Both R-modules are finitely generated A-modules by assumption and hence $\mathcal{N}(\mathfrak{q}) = \mathcal{N} \otimes_A k(\mathfrak{q})$ and $\mathcal{M}(\mathfrak{q}) = \mathcal{M} \otimes_A k(\mathfrak{q})$ are finite dimensional vector spaces over $k(\mathfrak{q})$ for $\mathfrak{q} \in \operatorname{Spec} A$. Then they are Artinian $R(\mathfrak{q})$-modules with

$$N(\mathfrak{q}) := \operatorname{Supp}_{R(\mathfrak{q})} \mathcal{N}(\mathfrak{q}) \cup \operatorname{Supp}_{R(\mathfrak{q})} \mathcal{M}(\mathfrak{q})$$

a finite set of closed points of $R(\mathfrak{q})$. The set $N := \operatorname{Supp}_R \mathcal{N} \cup \operatorname{Supp}_R \mathcal{M}$ is closed in $\operatorname{Spec} R$ with $N(\mathfrak{q}) = \{\mathfrak{n} \in N | \mathfrak{n} \cap A = \mathfrak{q}\} = N \cap \operatorname{Spec} R(\mathfrak{q})$. Since $\operatorname{Spec}\mu : \operatorname{Spec}\widetilde{R} \smallsetminus \mu^{-1}(N) \to \operatorname{Spec} R \smallsetminus N$ is an isomorphism, the fiber map $\operatorname{Spec}\mu(\mathfrak{q}) : \operatorname{Spec}\widetilde{R}(\mathfrak{q}) \to \operatorname{Spec} R(\mathfrak{q})$ is an isomorphism over $\operatorname{Spec} R(\mathfrak{q}) \smallsetminus N(\mathfrak{q})$ with $\mu(\mathfrak{q})^{-1}(N(\mathfrak{q}))$ a finite set of closed points [Stack, tag 02LS, Lemma 36.39.1.] (since μ and hence $\mu(\mathfrak{q})$ is finite). It follows that the assumptions of Lemma 14 are satisfied for $\mu(\mathfrak{q}) : R(\mathfrak{q}) \to \widetilde{R}(\mathfrak{q})$ and $\Bbbk = k(\mathfrak{q})$. In particular, $\mu(\mathfrak{q}) : R(\mathfrak{q})^{>0} \to \widetilde{R}(\mathfrak{q})^{>0}$ is a partial normalization of $R(\mathfrak{q})^{>0}$.

Let $\mathfrak{q} \in Im(\operatorname{Spec}\varphi)$ be non-zero. Since A is principal, $\mathfrak{q}$ is a maximal ideal, generated by one element $t_\mathfrak{q} \in A$. We denote the image of $t_\mathfrak{q}$ in R resp. $\widetilde{R}$ by $f_\mathfrak{q}$ resp. $\tilde{f}_\mathfrak{q}$, which are non-zero divisors since R and $\widetilde{R}$ are flat over A. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \xrightarrow{f_\mathfrak{q}} & R & \longrightarrow & R(\mathfrak{q}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu(\mathfrak{q})} & & \\ 0 & \longrightarrow & \widetilde{R} & \xrightarrow{\tilde{f}_\mathfrak{q}} & \widetilde{R} & \longrightarrow & \widetilde{R}(\mathfrak{q}) & \longrightarrow & 0, \end{array}$$

with exact rows. Since A is a principal ideal domain we have a decomposition

$$\mathcal{M} = \mathcal{F} \oplus \mathcal{T}$$

with $\mathcal{F}$ a free A–module and $\mathcal{T}$ an A–torsion submodule concentrated on finitely many maximal ideals in A. $\mathcal{N}$ is a free A-module (since it is torsion free as a submodule of the flat, hence torsion free A-module R). Since $\mathcal{F}$ and $\mathcal{N}$ are free, they are of constant rank m and n respectively and we get for every $\mathfrak{q} \in Im(\operatorname{Spec}\varphi)$,

$$m := \dim_{k(q)} \mathcal{F}(\mathfrak{q}),\ n := \dim_{k(q)} \mathcal{N}(\mathfrak{q}).$$

Now fix a $\langle 0\rangle \neq \mathfrak{p} \in Im(\operatorname{Spec}\varphi)$. There exists an open neighbourhood U of $\mathfrak{p}$ in $\operatorname{Spec} A$ such that $\mathcal{T}_\mathfrak{q} = 0$ for $\mathfrak{q} \in U \smallsetminus \{\mathfrak{p}\}$ and hence $\dim_{k(\mathfrak{p})} \mathcal{T}_\mathfrak{p} < \infty$. The snake lemma, applied to the diagram above, gives the exact sequence

$$0 \to \mathcal{N} \xrightarrow{f_\mathfrak{q}} \mathcal{N} \to Ker(\mu(\mathfrak{q})) \to \mathcal{M} \xrightarrow{\tilde{f}_\mathfrak{q}} \mathcal{M} \to Coker(\mu(\mathfrak{q})) \to 0,$$

and from this we get

$$0 \to \mathcal{N}(\mathfrak{q}) \to Ker(\mu(\mathfrak{q})) \to Ker(\tilde{f}_\mathfrak{q}) \to 0,$$
$$0 \to Ker(\tilde{f}_\mathfrak{q}) \to \mathcal{F} \oplus \mathcal{T} \xrightarrow{\tilde{f}_\mathfrak{q}} \mathcal{F} \oplus \mathcal{T} \to Coker(\mu(\mathfrak{q})) \to 0\ .$$

$\tilde{f}_\mathfrak{q}$ respects the decomposition into free and torsion part, with $Ker(\tilde{f}_\mathfrak{q}|\mathcal{F}) = 0$ and $Coker(\tilde{f}_\mathfrak{q}|\mathcal{F}) = \mathcal{F}(\mathfrak{q})$. Since $\mathcal{T}$ is finite dimensional, kernel and cokernel of $\tilde{f}_\mathfrak{q} : \mathcal{T} \to \mathcal{T}$ have the same dimension for each $\mathfrak{q} \in U$ (being 0 for $\mathfrak{q} \neq \mathfrak{p}$).

If $\mathcal{N} = 0$ then $Ker(\mu(\mathfrak{q})) = Ker(\tilde{f}_\mathfrak{q})$ and $= 0$ for $\mathfrak{q} \neq \mathfrak{p}$ since $\mathcal{T}_\mathfrak{q} = 0$ and statement (iv) follows.

By Lemma 14 $Coker(\mu(\mathfrak{q}))$ and $Ker(\mu(\mathfrak{q}))$ are finite dimensional over $k(\mathfrak{q})$ and we get

$$\begin{aligned} m &= \dim_{k(\mathfrak{q})} Coker(\mu(\mathfrak{q})) - \dim_{k(\mathfrak{q})} Ker(\tilde{f}_\mathfrak{q}) \\ &= \dim_{k(\mathfrak{q})} Coker(\mu(\mathfrak{q})) - \dim_{k(\mathfrak{q})} Ker(\mu(\mathfrak{q})) + n. \end{aligned}$$

It follows that $\dim_{k(\mathfrak{q})} Coker(\mu(\mathfrak{q})) - \dim_{k(\mathfrak{q})} Ker(\mu(\mathfrak{q})) = m - n$ is independent $\mathfrak{q} \in U \smallsetminus \langle 0\rangle$. The same holds for $\mathfrak{q} = \langle 0\rangle$ since $\mathcal{T}(\mathfrak{q}) = 0$ and hence $Coker(\mu(\mathfrak{q})) = \mathcal{F}(\mathfrak{q})$ and $Ker(\mu(\mathfrak{q})) = \mathcal{N}(\mathfrak{q})$. Lemma 14 implies now statement (i) and (ii).

To prove (iii) assume that $\operatorname{Spec}\widetilde{R}^1 \cap \operatorname{Spec}\widetilde{R}^{>1} = \emptyset$. Then $\widetilde{R}(\mathfrak{p})^0 = (\widetilde{R}^1)(\mathfrak{p})$ and $\widetilde{R}(\mathfrak{p})^{>0} = (\widetilde{R}^{>1})(\mathfrak{p})$ for $\mathfrak{p} \in Im(\operatorname{Spec}\varphi)$ (φ is flat) and $Coker(\mu(\mathfrak{p})^{>0}) = (\widetilde{R}^{>1}/\mu(R^{>1})) \otimes_A k(\mathfrak{p})$. By assumption $\widetilde{R}/\mu(R)$ is a finite A-module and hence also $\widetilde{R}^{>1}/\mu(R^{>1})$. Thus $\dim_{k(\mathfrak{p})}(\widetilde{R}^{>1}/\mu(R^{>1})) \otimes k(\mathfrak{p})$ is semicontinuous on $\operatorname{Spec} A$ ([GP20, Lemma 1]), which proves $\dim_{k(\mathfrak{p})} Coker(\mu(\mathfrak{p})^{>0}) \geq \dim_{k(\mathfrak{q})} Coker(\mu(\mathfrak{q})^{>0})$ and hence (iii). We notice, that if $\widetilde{R}^{>1} = 0$ the proof gives $\varepsilon_{k(\mathfrak{p})}(R(\mathfrak{p})) - \varepsilon_{k(\mathfrak{q})}(R(\mathfrak{q})) = \varepsilon_{k(\mathfrak{p})}(\widetilde{R}(\mathfrak{p})) - \varepsilon_{k(\mathfrak{q})}(\widetilde{R}(\mathfrak{q}))$. □

Lemma 17. *Let $\varphi : A \to R$ be a morphism of rings with A a principal ideal domain. Let the normalization $\nu : R \to \overline{R}$ be finite and $\mu : R \to \widetilde{R}$ be a finite morphism, which is a partial normalization of R.*

(1) *Let φ be flat.*

(i) *Let Q be the (non-empty) intersection of some associated primes of R and set $R' := R/Q$. Then the induced map $\varphi' : A \to R'$ is flat. In particular $\varphi^{red} : A \to R^{red}$ is flat.*
(ii) *The map $\tilde{\varphi} = \mu \circ \varphi : A \to \widetilde{R}$ is flat if $\widetilde{R}$ is reduced. In particular, $\overline{\varphi} = \nu \circ \varphi : A \to \overline{R}$ is flat.*

(2) *Let R and $\widetilde{R}$ be reduced. Then φ is flat $\iff$ $\tilde{\varphi}$ is flat.*

(3) *Let $\mathfrak{n} \in \operatorname{Spec} R$, $\mathfrak{p} = \mathfrak{n} \cap A$ and $\varphi : A_{\mathfrak{p}} \to R_{\mathfrak{n}}$ flat.*
(i) *If $R_{\mathfrak{n}}(\mathfrak{p}) = R_{\mathfrak{n}} \otimes_{A_{\mathfrak{p}}} k(\mathfrak{p})$ is reduced, then $R_{\mathfrak{n}}$ is reduced.*
(ii) *If $\dim R_{\mathfrak{n}} \geq 2$ and $R_{\mathfrak{n}}(\mathfrak{p})$ reduced, then $\operatorname{depth}(R_{\mathfrak{n}}) \geq 2$ and $r_1(R_{\mathfrak{n}}) = 0$ (i.e., no 1-dimensional irreducible component of $\operatorname{Spec} R$ passes through $\mathfrak{n}$).*
(iii) *If $\dim R_{\mathfrak{n}} \geq 2$ and $R_{\mathfrak{n}}(\mathfrak{p})$ an INNS then: $R_{\mathfrak{n}}(\mathfrak{p})$ is reduced $\iff$ $\operatorname{depth}(R_{\mathfrak{n}}) \geq 2$.*

(4) *Let φ be flat and $\dim R/\mathfrak{n} \geq 2$ for every minimal prime $\mathfrak{n}$ of R. Assume that $R(\mathfrak{p})$ has only isolated non-normal singularities for $\mathfrak{p} \in Im(\operatorname{Spec} \varphi)$. Then the following are equivalent:*
(i) *R is reduced,*
(ii) *for each $\mathfrak{p} \in Im(\operatorname{Spec} \varphi)$, $R(\mathfrak{p})$ is reduced at all normal closed points of R.*

Proof. Since A is a PID, φ is flat $\iff$ $\varphi(a)$ is a non-zero divisor (n.z.d.) in R for each $a \neq 0$ in A $\iff$ $\varphi(a)$ is not contained in any associated prime of R.

(1) The above characterization implies (i). To see (ii), let $\nu = \tilde{\nu} \circ \mu : R \to \widetilde{R} \to \overline{R}$, with $\tilde{\nu}$ the normalization of $\widetilde{R}$. Consider first the case that $\mu = \nu : R \to \overline{R}$ is the normalization of R. Let $\mathfrak{p}_1, ..., \mathfrak{p}_r$ be the minimal primes of R. Then $\overline{R} = \oplus_i \overline{R/\mathfrak{p}_i}$ and $\bar{\varphi}(a) = (b_1, ..., b_r)$, with $b_i = \bar{\varphi}(a \; mod \; \mathfrak{p}_i)$. Since φ^{red} is flat by (i), $\varphi(a) \notin \mathfrak{p}_i$ for all i and hence $b_i \neq 0$ for all i, showing that $\bar{\varphi}(a)$ is a n.z.d. in $\overline{R}$, i.e., $\bar{\varphi}$ is flat. If $\widetilde{R}$ is a partial normalization of R and reduced, then $\widetilde{R} \subset \overline{R}$ and hence $\tilde{\varphi}$ is flat.

(2) By (1) (ii) the flatness of φ implies that of $\tilde{\varphi}$. The flatness of $\tilde{\varphi}$ implies that of φ since R is reduced and $R \subset \widetilde{R}$.

(3) (i) By [Mat86, Corollary to Theorem 23.9] $R_{\mathfrak{n}}$ is reduced (normal), if $R_{\mathfrak{n}} \otimes_{A_{\mathfrak{p}}} k(\mathfrak{q})$ is reduced (normal) for all $\mathfrak{q} \in \operatorname{Spec} A_{\mathfrak{p}}$. Since $A_{\mathfrak{p}}$ has only two prime ideals $R_{\mathfrak{n}}$ is reduced by the following Lemma 19.

(ii) $R_{\mathfrak{n}}(\mathfrak{p})$ is reduced iff it satisfies Serre's condition (R_0) and (S_1).
If $\dim R_{\mathfrak{n}} \geq 2$ and $R_{\mathfrak{n}}(\mathfrak{p})$ reduced, then $\dim R_{\mathfrak{n}}(\mathfrak{p}) \geq 1$ and depth $R_{\mathfrak{n}}(\mathfrak{p}) \geq 1$ (from (S_1)). Hence $\operatorname{depth}(R_{\mathfrak{n}}) \geq 2$ since φ is flat ([Mat86, Corollary to Theorem 23.3]) and then $r_1(R_{\mathfrak{n}}) = 0$ ([BH98, Proposition 1.2.13]).

(iii) Implication $\Rightarrow$ follows direclty from (ii). For the converse direction we note that $\operatorname{nil}(R) = H^0_{\mathfrak{m}}(R)$ for $(R, \mathfrak{m})$ a local INNS of dimension ≥ 1 (Remark 7) and that $H^0_{\mathfrak{m}}(R) = 0$ iff $\operatorname{depth}(R) \geq 1$ ([BH98, Proposition 3.5.4.]). Hence $R_{\mathfrak{n}}(\mathfrak{p})$ is reduced iff $\operatorname{depth}(R_{\mathfrak{n}}(\mathfrak{p})) \geq 1$, which holds since $\mathfrak{p}$ is generated by a non-zerodivisor and therefore $\operatorname{depth}(R_{\mathfrak{n}}(\mathfrak{p})) = \operatorname{depth}(R_{\mathfrak{n}}) - 1$ ([Stack] Lemma 10.71.7).

(4) Let R be reduced and $\mathfrak{n} \in \operatorname{Spec} R$ a normal closed point of R. By assumption $\dim R_{\mathfrak{n}} \geq 2$ and by Serre's condition (S2) $\operatorname{depth}(R_{\mathfrak{n}}) \geq 2$. Hence $R(\mathfrak{p})_{\mathfrak{n}} = R_{\mathfrak{n}}(\mathfrak{p})$ is reduced by (3) (iii), and this proves (i) $\Rightarrow$ (ii). Conversely, as R is reduced at all normal points we consider a non-normal point $\mathfrak{n}$ of R ($\mathfrak{n}$ is then a closed point). Let $\mathfrak{p} = \mathfrak{n} \cap A$, t a generator of $\mathfrak{p}$ and f the image of t in R. Since $R_{\mathfrak{n}}$ is an INNS, $\operatorname{nil}(R_{\mathfrak{n}})$ is concentrated on $\mathfrak{n}$ and hence killed by a power of f. Since each power of f is a non-zero divisor of $R_{\mathfrak{n}}$, $\operatorname{nil}(R_{\mathfrak{n}}) = 0$ and (ii) $\Rightarrow$ (i) follows. □

Example 18. The condition in Lemma 17 (iii) that $R_{\mathfrak{n}}(\mathfrak{p})$ is an *INNS* is necessary:
Let $A = \Bbbk[z], \Bbbk$ algebraically closed and $\operatorname{char}(\Bbbk) = p > 0$ and $R = A[x,y]/\langle f\rangle$, $f = y^p - x^p - z$. Then R is regular of depth 2 at every closed point, and the canonical map $\varphi : A \to R$ is flat (z is a non-zero divisor in R). For every closed point $s \in \Bbbk$ the fiber $R(s) = \Bbbk[x,y]/y^p - x^p - s$ is not an INNS and not reduced ($s^{1/p} \in \Bbbk$) (the generic fiber is however regular).

Such an example is not possible in characteristic 0. E.g. if $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is flat then $f^{-1}(t)$ is smooth for $t \neq 0$ close to 0.

Lemma 19. *Let $\varphi : (A, \mathfrak{m}) \to (R, \mathfrak{n})$ be a flat morphism of local rings.*

1. $\overline{R} \otimes_A Q(A) = \overline{R \otimes_A Q(A)}$*; in particular, if R is normal, then $R \otimes_A Q(A)$ is normal.*
2. *If $(A, \mathfrak{m})$ is a discrete valuation ring and $R \otimes_A A/\mathfrak{m}$ reduced, then $R \otimes_A Q(A)$ is reduced.*

Proof. $A \subset R$ since R is flat, hence faithfully flat over A.

(1) We have $R \otimes_A Q(A) = \{\frac{r}{a} | r \in R, a \in A$ a non-zerodivisor$\}$, hence $Q(R \otimes_A Q(A)) = Q(R)$ and $Q(\overline{R} \otimes_A Q(A)) = Q(\overline{R}) = Q(R)$. Thus $\overline{R} \otimes_A Q(A))$ is normal and $\overline{R \otimes_A Q(A)} \subset \overline{R} \otimes_A Q(A))$. Since the last inclusion is birational it is an equality.

(2) Since A is a DVR, $\mathfrak{m} = \langle t\rangle$ for some t. Then $R \otimes_A Q(A) = R_t = \{\frac{r}{t^\nu} | r \in R, \nu \geq 0\}$. Assume $(\frac{r}{t^\nu})^n = 0$ for some n. Then $r^n = 0$ since t is a non-zero divisor of R and $\bar{r}^n = 0$, $\bar{r}$ the image of r in $R/\mathfrak{m}R = R/tR$. Since R/tR is reduced, $r = tr'$ for some $r' \in R$. By induction $r \in \cap t^\nu R$, and $\cap t^\nu R = 0$ by Krull's intersection theorem. Hence $r = 0$. □

We can now prove the semicontinuity of δ and ε in families over the spectrum of principal ideal domain.

Theorem 20. *Let $\varphi : A \to R$ be a flat morphism of rings, A a principal ideal domain, and let the normalization $\nu : R \to \overline{R}$ be finite. Let $X = \operatorname{Spec} R$, $\overline{X} = \operatorname{Spec} \overline{R}$, $n = \operatorname{Spec} \nu : \overline{X} \to X$ and $f = \operatorname{Spec} \varphi : X \to S = \operatorname{Spec} A$. Assume that $Coker(\nu)$ and $Ker(\nu^{>1} : R \to \overline{R^{>1}})$ are finite over A and, moreover, that for each $s \in f(X)$ the normalization of $X_s = f^{-1}(s)$ is finite over X_s and that X_s is normal outside finitely many isolated non-normal singularities at which the residue fields are finite over $k(s)$.*

Then $\overline{f} := f \circ n : \overline{X} \to S$ *is flat,* $\delta_{k(s)}(X_s) < \infty$, $\varepsilon_{k(s)}(X_s) < \infty$, *and for each* $s \in f(X)$ *there exists an open neighbourhood* $V \subset S$ *of* s *such that the following holds for* $U = V \cap f(X)$*:*

(1) $\delta_{k(s)}(X_s) - \delta_{k(t)}(X_t)$
$= \delta_{k(s)}((X^{red})_s) - \delta_{k(t)}((X^{red})_t)$
$= \delta_{k(s)}((X^{>1})_s) - \delta_{k(t)}((X^{>1})_t)$
$= \delta_{k(s)}((\overline{X})_s) - \delta_{k(t)}((\overline{X})_t)$
$= \delta_{k(s)}((\overline{X}^{>1})_s) - \delta_{k(t)}((\overline{X}^{>1})_t)$ *for* $t \in U$*.*

(2) *If* $(\overline{X^{>1}})_t$ *is normal for* $t \in U \setminus \{s\}$*, then*
$\delta_{k(s)}(X_s) - \delta_{k(t)}(X_t) = \delta_{k(s)}((\overline{X^{>1}})_s) \geq 0$.

(3) $\delta_{k(s)}(X_s) - \delta_{k(\eta)}(X_\eta) = \delta_{k(s)}((\overline{X^{>1}})_s) \geq 0$, *$\eta$ the generic point of* S*.*

(4) $\varepsilon_{k(s)}(X_s) - \varepsilon_{k(t)}(X_t) = \varepsilon_{k(s)}((X^{>1})_s) \geq 0$ *for* $t \in U \setminus \{s\}$*.*

(5) $(\overline{X^{>1}})_t = (\overline{X}^{>1})_t := (\overline{f}|\overline{X}^{>1})^{-1}(t)$ *is reduced for every* $t \in U$*. If* X *is reduced then* X_t *is reduced for* $t \in U \setminus \{s\}$*.*

Before giving the proof, we'd like to comment on the result. A semicontinuity theorem in the algebraic setting was proved by Chiang-Hsieh and Lipman in [CL06, Proposition 3.3] with $(A, \mathfrak{m})$ a normal local ring of any dimension (satisfying $A/\mathfrak{m}$ perfect and $A \to \hat{A}$ normal) under the assumption that (in our notation) the fibers X_t and $(\overline{X})_t$, $t \in \operatorname{Spec} A$, are equidimensional and reduced. The authors prove that $\delta_{k(t)}(X_t) - \delta_{k(\eta)}(X_\eta) = \delta_{k(t)}((\overline{X})_t)$ for $t \in \operatorname{Spec} A$ (since $\operatorname{Spec} A$ has only one closed point, the semicontinuity holds only for generalizations to the generic point η). Apart from the fact that there is no restriction for dim A in [CL06], our result is stronger in two ways. (a): we do not assume that the fibers are reduced or equidimensional and (b): A doesn't have to be local, it can be e.g. $\mathbb{Z}$ or $\Bbbk[t]$, $\Bbbk$ an arbitrary field. Thus our semicontinuity holds also for closed points in a neighbourhood of the given point s in $\operatorname{Spec} A$.

For complex analytic map germs $f : (X, x) \to (S, 0)$, with $(S, 0) = (\mathbb{C}, 0)$ and (X_0, x) a *reduced* curve singularity, the result is classical and due to Teissier [Te78].

Remark 21. (1) We have $\operatorname{nil}(R) = Ker(R \to R^{red}) = Ker(\nu) \subset Ker(\nu^{>1}) = \mathfrak{p}^1$ since $\overline{R} \to \overline{R}^{>1} = \overline{R^{>1}} = \overline{R/\mathfrak{p}^1}$ is surjective. Hence, if $Ker(\nu^{>1})$ is finite over A then $Ker(\nu)$ is also finite over A. On the other hand, since $Coker(\nu)$ surjects onto $Coker(\nu^{>1})$, if $Coker(\nu)$ is finite over A, then $Coker(\nu^{>1})$ is finite over A.

(2) We remark that every irreducible component of $\overline{X}$ has dimension ≥ 1 ($\overline{f}$ is flat) and that $\overline{X} = \overline{X}^{>1} \sqcup \overline{X}^1$ and $(\overline{X})_t = (\overline{X}^{>1})_t \sqcup (\overline{X}^1)_t$ with $(\overline{X}^{>1})_t = (\overline{X})_t^{>0}$ and $(\overline{X}^1)_t = (\overline{X})_t^0$. In particular,
$(\overline{X^{>1}})_t = (\overline{X})_t \iff (\overline{X})_t$ has no isolated points $\iff$ X has no 1-dimensional components that meet X_t.

(3) Since φ is injective, the generic point $\eta = \langle 0 \rangle$ is contained in $f(X)$ ([Stack, tag 00FJ, Lemma 29.4.]). It may however happen that $f(X)$ does not contain any open subset of S. E.g. $f(X) = \{\eta\}$ for $A = \mathbb{Z} \to R = \mathbb{Q}[x]$ since $A \cap \mathfrak{p} = \langle 0 \rangle$ for every prime ideal $\mathfrak{p} \in R$ (see also Example 23).

(4) The statement of the theorem is especially interesting if f is surjective or if f is open (which holds if X is of finite presentation over S by [Stack, tag 00I1, Proposition 10.40.8.], or for analytic maps).

Proof. (of Theorem 20) Let $\mu : R \to \widetilde{R}$ be one of the maps $\nu^{red} : R \to R^{red}$, $\nu' : R \to R^{>1}$, $\nu : R \to \overline{R}$, and $\nu^{>1} : R \to \overline{R}^{>1}$. Since $Ker(\nu^{red}) = Ker(\nu) \subset Ker(\nu^{>1}) = Ker(\nu')$, it follows that $Ker(\mu)$ and $Coker(\mu)$ are finite over A, since $Ker(\nu^{>1})$ and $Coker(\nu)$ are finite over A by assumption.

If $R^{>1} = 0$, i.e. X is of pure dimension 1, then $R = Ker(\nu^{>1})$ and hence R is finite over A. Since R is A-flat, it is torsion free, hence free and $\dim_{k(\mathfrak{p})}(R(\mathfrak{p})) = \varepsilon_{k(\mathfrak{p})}(R(\mathfrak{p})) = -\delta_{k(\mathfrak{p})}(R(\mathfrak{p}))$ is constant on $\operatorname{Spec} A$. Thus the theorem holds in this case trivially and we will assume in the following the $R^{>1} \neq 0$.

By Lemma 17 the maps $\tilde{f} = \operatorname{Spec}(\mu \circ \varphi) : \widetilde{X} = \operatorname{Spec} \widetilde{R} \to S$ are flat. We use the notation $(\widetilde{X})_t = \tilde{f}^{-1}(t)$.

(1) We can apply Proposition 16 (i) to each map μ and we get the equalities in statement (1).

Now consider $\overline{f}^{>1} := \overline{f}|\overline{X}^{>1} : \overline{X}^{>1} \to S$. From Proposition 16 (i) and (iii) we get an open neighbourhood U of s such that for $t \in U \cap f(X)$

(a) $\delta_{k(s)}(X_s) - \delta_{k(t)}(X_t) = \delta_{k(s)}((\overline{X}^{>1})_s) - \delta_{k(t)}((\overline{X}^{>1})_t)$.

Altogether this proves (1).

(2) $(\overline{X}^{>1})_t$ normal implies $\delta_{k(t)}((\overline{X}^{>1})_t) = 0$, $t \neq s$. Since $\overline{X}^{>1} = \overline{X^{>1}}$ is normal and of dimension ≥ 2 at each closed point x, $\operatorname{depth}(\overline{X}^{>1}) \geq 2$ at x. By Lemma 17 (3)(iii), $(\overline{X}^{>1})_t, t \in f(X)$, is reduced at every closed point, hence at every point. Thus $\delta_{k(t)}((\overline{X}^{>1})_s) \geq 0$ and (a) proves (2).

(3) By Lemma 19 the ring $\overline{R}^{>1}(\eta)$ and thus the generic fiber $\overline{X}^{>1}_\eta$ is normal, and (3) follows from (2).

To get estimates for ε we apply Proposition 16 (iii) to $f' = \operatorname{Spec}(\nu' \circ \varphi) : X^{>1} \to S$ and get

(b) $\varepsilon_{k(s)}(X_s) - \varepsilon_{k(t)}(X_t) = \varepsilon_{k(s)}((X^{>1})_s) - \varepsilon_{k(t)}((X^{>1})_t)$
$+ \dim_{k(s)} Coker(\nu'^{>1}) \otimes_A k(s) - \dim_{k(t)} Coker(\nu'^{>1}) \otimes_A k(t)$
$= \varepsilon_{k(s)}((X^{>1})_s) - \varepsilon_{k(t)}((X^{>1})_t)$,
since $Coker(\nu'^{>1}) = R^{>1}/\nu'(R^{>1}) = 0$.

(4) Since $X^{>1}$ is reduced, $(X^{>1})_t$ is reduced for $t \in U \setminus \{s\}$ by (5). Hence $\varepsilon_{k(t)}((X^{>1})_t) = 0$ and (4) follows from (b).

(5) If X is reduced, i.e. if R is reduced, then $\nu : R \to \overline{R}$ is injective. By Proposition 16 (iv) $R(\mathfrak{q}) \to \overline{R}(\mathfrak{q})$ is injective for $\mathfrak{q} \in U \setminus \{\mathfrak{p}\}$, U some

neighbourhood of $\mathfrak{p}$. Hence X_t is reduced for $t \in U \smallsetminus \{s\}$. By the proof of (2) $(\overline{X}^{>1})_t$ is reduced for all $t \in U$. □

We illustrate Theorem 20 with some examples.

Example 22. Let $\varphi : A = \Bbbk[z] \to R = A[x,y]/I$, $I = \langle x^2 + y^2 - z^2 \rangle \cap \langle x - y, y^2 - z \rangle$. Then $X = V(I) = X^2 \cup X^1$, with X^2 the normal surface singularity defined by $x^2 + y^2 - z^2 = 0$ and X^1 the smooth curve defined by $x - y = y^2 - z = 0$, meeting X^2 at $(0,0,0)$ and $(x,y,z) = (x,y,2)$ with $x = y = \pm\sqrt{2}$ (if $\sqrt{2} \in \Bbbk$).

For $z = 0$ the fiber X_0 is the nodal curve $x^2 + y^2 = 0$ with an embedded point, and we compute $\varepsilon_{\Bbbk}(X_0) = 2$ and thus $\delta_{\Bbbk}(X_0) = -1$. For $z = t \in \Bbbk, t \neq 0, 2$ the fiber X_t is a smooth curve and two extra reduced points not on the curve. Hence X_t is normal with $\varepsilon_{\Bbbk}(X_t) = 2$. We get $\delta_{\Bbbk}(X_t) = -2$ and $\delta_{\Bbbk}(X_0) - \delta_{\Bbbk}(X_t) = 1$.

The normalization $\overline{X}$ is the disjoint union of $X^{>1} = X^2$ and X^1. $(\overline{X})_0$ is the disjoint union of a nodal curve and a double point. We get $\delta_{\Bbbk}((\overline{X}^{>1})_0) = 1$, confirming statement (1) of Theorem 20.

Example 23. Consider $\varphi : A = \Bbbk[z] \to R = \Bbbk[z]_{\langle z \rangle}[x,y,z]/I$, with I as in Example 22. Let $\mathfrak{m}$ a maximal ideal in R. If $z \in \mathfrak{m}$ then $\mathfrak{m} \cap A = \langle z \rangle$. If $z \notin \mathfrak{m}$ then $\mathfrak{m} \cap A = \langle 0 \rangle$ (otherwise $\mathfrak{m} \cap A = \langle p \rangle$ for some irreducible polynomial $p(z) \notin \langle z \rangle$ and since p is a unit in R, $p \notin \mathfrak{m}$). It follows that $f = \operatorname{Spec}\varphi : X = \operatorname{Spec} R \to S = \operatorname{Spec} A$ is flat and $f(X)$ consists of two points $\langle z \rangle$ and $\eta := \langle 0 \rangle$ (every point of $f(X)$ is the image of a closed point). As in Example 22 we get $\delta_{\Bbbk}(X_0) = -1$ and $\delta_{\Bbbk}(X_\eta) = -2$.

Example 24. We provide two examples showing the necessity of the assumptions in Theorem 20.

(1) The condition that $(\overline{X}^{>1})_t, t \neq s$, is normal in Theorem 20 (1) is necessary in positive characteristic: Let $A = \Bbbk[z], \Bbbk$ algebraically closed, $\operatorname{char}(\Bbbk) = p > 2$, and $R = A[x,y]/\langle f \rangle, f = y^2 - x^p - z$. Then R is regular, hence normal, of dimension 2 and the canonical map $\varphi : A \to R$ is flat (z is a non-zero divisor in R). For every closed point $s \in \Bbbk$ the fiber $R(s) = \Bbbk[x,y]/y^2 - x^p - s$ has an isolated non-normal point, which is not normal since $y^2 = (x + s^{1/p})^p$.

We have $\delta_{\Bbbk}(R(s)) = (p-1)/2$ is the same for every closed point $s \in \Bbbk$. Since $\overline{R}^{>1}(s) = R(s)$ the equality in Theorem 20 (1) does not hold, while the equality in (2) holds for the generic fiber. Note that the generic fiber $\Bbbk(z)[x,y]/y^2 - x^p - z$ is regular (hence normal) but not geometrically normal: it is not-normal in $\Bbbk(z^{1/p})[x,y]/y^2 - x^p - z$.

We remark that such an example is not possible in characteristic 0 (cf. Section 4).

(2) The assumption that $Ker(\nu^{>1} : R \to \overline{R^{>1}})$ is finite over A is necessary for the upper semicontinuity of ε (the finiteness over A of $Ker(\nu)$ is not sufficient). For $A = \Bbbk[x]$, $R = A[y]/\langle y(xy-1) \rangle$, we have $\varepsilon_{\Bbbk}(R(0)) = 1$ but

$\varepsilon_{\Bbbk}(R(s)) = 2$ for $s \in \Bbbk - \{0\}$. In this case $R^{>1} = 0$ and $X^{>1} = \emptyset$. Hence $Ker(\nu^{>1}) = R$, which quasi-finite but not finite over A (the class of x in R is not integral over A), while $Ker(\nu) = 0$.

For Nagata rings R the assumptions simplify and we get with the notations of Theorem 20:

Corollary 25. *Let $\varphi : A \to R$ be a flat morphism of rings with A a PID and R Nagata. Let $\nu : R \to \overline{R}$ be the normalization and assume that $Coker(\nu)$ and $Ker(\nu^{>1})$ are finite over A and for each $s \in f(X)$ the fiber X_s has finitely many isolated non-normal singularities with residue fields finite over $k(s)$. Then the conclusions of Theorem 20 hold.*

Proof. Since R is Nagata, $\overline{R}$ is finite over R ([Stack, Lemma 10.157.2 (03GH)]) and since $R/\mathfrak{p}R, \mathfrak{p} \in \operatorname{Spec} A$, is Nagata ([Stack, Lemma 10.160.16]), the normalization of X_s is finite over X_s. Now apply Theorem 20. □

Let R have no 1-dimensional components. Then the formulation of Theorem 20 becomes less cumbersum and the assumption that $Coker(\nu)$ and $Ker(\nu^{>1}) = Ker(\nu)$ are finite over A is implied by $R/\widetilde{\mathfrak{C}}_R$ being finite over A, where $\widetilde{\mathfrak{C}}_R$ is the extended conductor ideal from Definition 1, with $V(\widetilde{\mathfrak{C}}_R)$ the non-normal locus of $\operatorname{Spec} R$. In fact, any finitely generated R-module M is finite over $R/\operatorname{Ann}_R(M)$ and if $R/\operatorname{Ann}_R(M)$ if finite over A then M is finite over A. Hence, we get

Corollary 26. *Let X be a Noetherian Nagata scheme with all irreducible components of dimension ≥ 2 and $n : \overline{X} \to X$ the normalization. Let S be the spectrum of a PID and $f : X \to S$ a flat morphism such that for each $s \in f(X)$ the fiber X_s has only finitely many isolated non-normal singularities with residue fields finite over $k(s)$. Assume that the extended conductor scheme $\widetilde{C}_X$[11] of X is finite over S. Then the conclusions of Theorem 20 hold with $(X^{>1})_t = X_t$ and $(\overline{X^{>1}})_t = (\overline{X})_t$ for $t \in U$.*

Theorem 20 and its corollaries say that over the spectrum of a PID the the delta invariant of the generic fiber X_η is minimal among all fibers $X_s, s \in f(X)$. It does not say, however, that the delta invariant of a special fiber X_s is bigger or equal than the delta invariant of the fibers X_t over closed points t in a neighbourhood of s, except $(\overline{X^{>1}})_t$ is normal for $t \neq s$. By Example 24 the equality in Theorem 20 (2) does not hold in positive characteristic (nevertheless, semicontinuity may hold in general but we do not know this). For characterisitc 0 see section 4.

3. Fiberwise and Simultaneous Normalization

While the notion of simultaneous normalization is well known, the following (weaker) definition of fiberwise normalization is new. It is useful if the residue fields of the base scheme are not perfect.

[11] If X is covered by open affine sets $X_i = \operatorname{Spec} R_i$, then $\widetilde{C}_X|X_i = \widetilde{C}_{R_i}$ as defined in Definition 1.

Definition 27. *Let $m : \widetilde{X} \to X$ and $f : X \to S$ be morphisms of schemes.*

(1) *We call $m : \widetilde{X} \to X$ a* fiberwise normalization *of f if*
  (a) *m is finite,*
  (b) *the composition $\widetilde{f} := f \circ m : \widetilde{X} \to S$ is flat,*
  (c) *the non-empty fibers of $\widetilde{f}$ are normal, and*
  (d) *the induced map $m_t : \widetilde{f}^{-1}(t) \to f^{-1}(t)$ is birational*[12] *for every $t \in f(X)$*
(2) *A fiberwise normalization of f if is called a* simultaneous normalization *the non-empty fibers of $\widetilde{f}$ are geometrically normal.*

Recall that a $\Bbbk$-algebra R is called *geometrically normal* (resp. *geometrically reduced*) if $R \otimes_{\Bbbk} \Bbbk'$ is normal (resp. reduced) for every field extension $\Bbbk \subset \Bbbk'$ (equivalently, for every finite field extension). If $\Bbbk$ is a perfect field, then a $\Bbbk$-algebra is normal (resp. reduced) iff it is geometrically normal (resp. reduced), see [Stack, tag 030V, Lemma 10.43.3. and tag 037Z, Lemma 10.160.1.]. A morphism of rings $\varphi : A \to R$ is called *normal* (resp. *reduced*) if it is flat and if the non-empty fibers $R(\mathfrak{p}) := R \otimes_A k(\mathfrak{p})$, are geometrically normal (resp. geometrically reduced) as $k(\mathfrak{p})$-algebras, where $k(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} = \mathrm{Quot}(A/\mathfrak{p})$ is the residue field of $A_\mathfrak{p}$, $\mathfrak{p} \in \operatorname{Spec} A$.

A morphism of schemes $f : X \to S$ is geometrically normal (resp. geometrically reduced), if this holds for the induced morphisms of local rings. Hence, if the residue fields of all local rings of S are perfect (e.g. of characteristic 0), then the notions of fiberwise normalization and simultaneous normalization coincide.

Note that simultaneous normalization is preserved under base change, while this is in general not the case for fiberwise normalization of schemes over non-perfect fields. On the other hand, the following results show that the weaker assumption of a fiberwise normalization is often sufficient and useful.

Lemma 28. *Let $f : X \to S$ be flat and assume that f admits a fiberwise normalization $m : \widetilde{X} \to X$. Let $\widetilde{f} = f \circ m : \widetilde{X} \to S$. Then the following holds:*

(1) *m is birational.*
(2) *$\widetilde{X}$ is reduced (resp. normal) at $\tilde{x}$ if and only if S is reduced (resp. normal) at $\widetilde{f}(\tilde{x})$. If $\widetilde{X}$ is normal, then $\widetilde{X} \cong \overline{X}$ and m is the normalization map.*
(3) *The induced fiber map $m_t : \widetilde{X}_t = \widetilde{f}^{-1}(t) \to f^{-1}(t) = X_t$ is the normalization of X_t for every $t \in f(X)$.*
(4) *Let S be normal, $t \in S$ and $x \in X_t$. Denote by $X_t^i, i = 1, ..., r$, the irreducible components of X_t passing through x and by $X^j, j = 1, ..., s$, the irreducible components of X passing through x. Then*

[12] A morphism of schemes is *birational* if it is a bijection between the generic points and an isomorphism of the corresponding local rings.

$r = s$ and for each j there exists a unique $i = i(j)$ such that $X^i_t \subset X^j$. The corresponding components satisfy the dimension formula

$$\dim(X^j, x) = \dim(X^i_t, x) + \dim(S, t).^{13}$$

In particular, if $\dim(X_t, x) > 0$ then each irreducible component of X passing through x has dimension $> \dim(S, t)$.

The dimension formula in (4) is not a direct consequence of the flatness of f, since the restriction of a flat map to an irreducible component need not be flat; it is a consequence of the existence of a fiberwise normalization.

Proof. (1) A flat morphism f maps a generic point $x \in X$ to a generic point $f(x) \in S$, since flat mappings have the going down property ([Stack] Lemma 10.38.19, tag 00HS). Since $m_t : \widetilde{f}^{-1}(t) \to f^{-1}(t), t \in f(X)$, induces a bijection of generic points of the fibers and an isomorphism of their local rings, this holds also for m by the following Lemma 29 (see also [CL06, Theorem 2.3]), since the minimal primes of a ring correspond uniquely to the generic points of its spectrum.

(2) The first statement follows from [Mat86, Corollary of Theorem 23.9]. The second statement follows since m is finite by definition and birational by (1).

(3) Since m_t is finite and birational it is the normalization map.

(4) Since S is normal, m is the normalization map by (2) and thus the number r of irreducible components X passing through x equals $\#m^{-1}(x)$. In the same way $s = \#m_t^{-1}(x)$ holds. Since $\#m^{-1}(x) = \#m_t^{-1}(x)$ we get the first statement of (2). For each X^j there exists a unique point $\tilde{x}^j \in m^{-1}(x) \cap m^{-1}(X^j)$ and we get for $i = i(j)$

$$\dim(X^j, x) = \dim(\widetilde{X}, \tilde{x}^j) = \dim(\widetilde{X}_t, \tilde{x}^j) + \dim(S, t) = \dim(X^i_t, x) + \dim(S, t),$$

since $\widetilde{f}$ is flat. Finally, $\dim(X_t, x) > 0$ means of course that all irreducible components X^i_t have dimension > 0 at x and the last statement follows. □

Lemma 29. *Let A be a ring and R a flat A-algebra. Then the total ring of fractions satisfies $Q(R) = Q(R \otimes_A Q(A))$. Let $\widetilde{R}$ be a flat A-algebra and $\mu : R \to \widetilde{R}$ an A-algebra homomorphism inducing an isomorphism $R \otimes_A Q(A) \cong \widetilde{R} \otimes_A Q(A)$. Then μ induces an isomorphism $Q(R) \cong Q(\widetilde{R})$ and $\mu : R \to \widetilde{R}$ is birational.*

Proof. Let R be an A-algebra via $\varphi : A \to R$. $R \otimes_A Q(A) = \{\frac{r}{\varphi(a)} | r \in R$ and $a \in A$ a non-zerodivisor$\}$. Since φ is flat, it maps non-zero divisors to non-zero divisors. Hence $R \otimes_A Q(A) \subset Q(R)$ and $Q(R) = Q(R \otimes_A Q(A))$. Similarly we obtain $Q(\widetilde{R}) = Q(\widetilde{R} \otimes_A Q(A))$.

By assumption μ induces an isomorphism $R \otimes_A Q(A) \cong \widetilde{R} \otimes_A Q(A)$ and it follows that μ induces an isomorphism $Q(R) \cong Q(\widetilde{R})$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the associated prime ideals of R. Then $\mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_r$ is the set of zero

[13] $\dim(X, x)$ denotes the dimension of the local ring $\mathcal{O}_{X,x}$ of the scheme X at x.

divisors of R and $\mathfrak{p}_1 Q(R), \ldots, \mathfrak{p}_r Q(R)$ are the prime ideals of $Q(R)$ ([Mat86, Theorem 4.1]). Here the minimal associated primes of R correspond to the minimal primes of $Q(R)$ (also the embedded associated primes of R correspond to the embedded primes of $Q(R)$, but this is not relevant for us), and similarly for $Q(\widetilde{R})$. Since $Q(R) \cong Q(\widetilde{R})$ the minimal prime ideals of $Q(R)$ and $Q(\widetilde{R})$ are in 1-1 correspondence. Since for a minimal prime $\mathfrak{p}$ of R we have $R_{\mathfrak{p}} = Q(R)_{\mathfrak{p}Q(R)}$ we obtain all together a bijection between the minimal primes of R and $\widetilde{R}$ and an isomorphism of the corresponding local rings. This implies that $\mu : R \longrightarrow \tilde{R}$ is birational. □

We want to characterize fiberwise resp. simultaneous normalization of a family of INNS numerically by the δ-invariant of the fibers. The following example shows that we have to be careful with families of affine fibers.

Example 30. Consider Example 24 with $R = A[y]/\langle y(xy-1)\rangle$ and $A = \mathbb{k}[x]$. The canonical map $A \to R$ is flat, with normal fibers, and hence the identity $R \to R$ is a fiberwise normalization (resp. simultaneous normalization if $\mathrm{char}(\mathbb{k}) = 0$). But the δ-invariant is not constant ($\delta_{\mathbb{k}}(R(0)) = -1$ and $\delta_{\mathbb{k}}(R(s)) = -2$ for $s \in \mathbb{k} - \{0\}$). The same holds for $A = \mathbb{k}[x]_{\langle x\rangle}$ and s the generic point of $\operatorname{Spec} A$. This does not contradict the following Theorem 31 since $Ker(\nu^{>1})$ is not finite over A.

The following theorem is a numerical characterization for the existence of a simultaneous normalization of a family of isolated normal singularities over the spectrum of a PID.

Theorem 31. *Let $\varphi : A \to R$ be a flat morphism of rings with A a principal ideal domain and R Nagata. Let $\nu : R \to \overline{R}$ be the normalization. Set $S = \operatorname{Spec} A$, $X = \operatorname{Spec} R$, $\overline{X} = \operatorname{Spec} \overline{R}$, $f = \operatorname{Spec} \varphi : X \to S$, $n = \operatorname{Spec} \nu : \overline{X} \to X$. Assume that $Coker(\nu)$ and $Ker(\nu^{>1} : R \to \overline{R^{>1}})$ are finite over A and that for each $t \in f(X)$ the fiber X_t has finitely many isolated non-normal singularities with residue fields finite over $k(t)$.*

1. *Assume that f admits a fiberwise normalization. Then $\delta_{k(t)}(X_t)$ is constant on S.*
2. *$\delta_{k(t)}(X_t)$ is constant on S if and only if $f^{>1} : X^{>1} \to S$ admits a fiberwise normalization.*
3. *If X has no 1-dimensional irreducible components then f admits a fiberwise normalization if and only if $\delta_{k(t)}(X_t)$ is constant on S.*

Proof. (1) Since A is regular, hence normal, any simultaneous normalization is the normalization by Lemma 28. Since the fibers of $\overline{f} = f \circ n$ are normal by assumption, $(\overline{X}^{>1})_t$ is normal and has positive dimension. Hence $\delta_{k(t)}((\overline{X}^{>1})_t) = 0$ for $t \in S$. The constancy of $\delta_{k(t)}(X_t)$ follows from Theorem 20 (2) (resp. Corollary 25).

(2) If $\delta_{k(t)}(X_t)$ is constant then $\delta_{k(t)}(X_t) = \delta_{k(\eta)}(X_\eta)$ for $t \in \operatorname{Spec} A$ and η the generic point of $\operatorname{Spec} A$. By Theorem 20 (3) (applied to any $t \in U$)

$\delta_{k(t)}((\overline{X^{>1}})_t) = 0$ and $(\overline{X^{>1}})_t$ is reduced. Hence $(\overline{X^{>1}})_t$ is normal and equal to the normalization of $(X^{>1})_t$ (since $(\overline{X^{>1}})_t \to (X^{>1})_t$ is birational and finite). Since $\overline{f}$ is flat by Lemma 17, $\overline{f}^{>1} = f|\overline{X}^{>1}$ is also flat and $n^{>1} : \overline{X}^{>1} \to X^{>1}$ is a fiberwise normalization of $f^{>1}$.

By Theorem 20 (1) $\delta_{k(t)}(X_t)$ is constant $\iff$ $\delta_{k(t)}((X^{red})_t)$ is constant $\iff$ $\delta_{k(t)}((X^{>1})_t)$ is constant. If $f^{>1}$ admits a fiberwise normalization, then $\delta_{k(t)}((X^{>1})_t)$ is constant by (1) and thus $\delta_{k(t)}(X_t)$ is constant.

(3) The "only if " direction follows from (1). $X^1 = \emptyset$ means $r_1(X) = 0$ and $f^{>1} = f^{red} : X^{red} \to S$. By (2) $\delta_{k(t)}(X_t) =$ constant implies that $\overline{X} \to X^{red}$ is a fiberwise normalization of f^{red}. But then $\overline{X} \to X$ is a fiberwise normalization of f (the flatness of f implies the flatness of f^{red} and $\overline{f} : \overline{X} \to X \to S$ by Lemma 17). □

Corollary 32. *In addition to the assumptions of Theorem 31 assume that all residue fields $k(\mathfrak{p})$, $\mathfrak{p} \in \operatorname{Spec} A$, are perfect (e.g., $A = \mathbb{Z}$ or A a $\Bbbk$-algebra with char$(\Bbbk) = 0$). Then the statements (2) and (3) of Theorem 31 hold with 'fiberwise normalization' replaced by 'simultaneous normalization'.*

4. Charcteristic 0 and the Analytic Case

If the characteristic is zero then the assumptions of the semicontinuity theorem (Theorem 20) can be weakened and the statement is stronger for morphisms of finite type and for analytic morphisms (using theorems of Bertini and Sard type). The main property is that reduced spaces are regular (hence normal) on an open dense subset.

Theorem 33. *Let $\Bbbk$ be a field of characteristic zero and let $\varphi : A \to R$ be a flat morphism of $\Bbbk$-algebras. Assume that A is a PID and that R is of finite type over $\Bbbk$. Let $\nu : R \to \overline{R}$ be the normalization and assume that $Ker(\nu^{>1})$ and $Coker(\nu)$ are finite over A. Let $X = \operatorname{Spec} R$, $\overline{X} = \operatorname{Spec} \overline{R}$, $n = \operatorname{Spec} \nu : \overline{X} \to X$, $f = \operatorname{Spec} \varphi : X \to S = \operatorname{Spec} A$ and $\overline{f} = f \circ n : \overline{X} \to S$. Let $X_s = f^{-1}(s)$, $s \in f(X)$, have only finitely many non-normal points.*

Then $\varepsilon_{k(s)}(X_s)$ and $\delta_{k(s)}(X_s)$ are finite for $s \in S$ and each $s \in S$ has an open neighbourhood U such that for $t \in U \setminus \{s\}$ the following holds:

1. *$(\overline{X}^{>1})_s$ is reduced and $(\overline{X})_t$ is regular, hence normal.*
2. *$\delta_{k(s)}(X_s) - \delta_{k(t)}(X_t) = \delta_{k(s)}((\overline{X^{>1}})_s) \geq 0$,*
3. *$\varepsilon_{k(s)}(X_s) - \varepsilon_{k(t)}(X_t) = \varepsilon_{k(s)}((X^{>1})_s) \geq 0$.*

Proof. Since X is of finite type over $\Bbbk$, any fiber X_s, $s \in f(X)$, is of finite type over $k(s)$. Since the non-normal points x of X_s are closed in X_s, the field extension $k(s) \subset k(x)$ is finite (by Hilbert's Nullstellensatz). Moreover, X is Nagata by Remark 15 and hence $\overline{X}$ is finite over X, as well as the normalization $\overline{(X_s)}$ over X_s. By Theorem 20 statements (2) and (3) hold and $(\overline{X}^{>1})_s$ is reduced for $s \in f(X)$.

It remains to show that $(\overline{X})_t$ is regular for $t \neq s$ in some neighbourhood of s. The map $\overline{f} : \overline{X} \to S$ and its restriction to any irreducible component of $\overline{X}$ is flat by Lemma 17 and dominant ($A \to \overline{R}$ is injective). Since $\overline{X}$ is reduced, there exists an open dense subset $V \subset \overline{X}$ such that V is smooth ([Va17, Theorem 21.3.5]) and, since $\Bbbk$ is perfect, a point x is smooth iff its local ring is regular ([Stack, tag 0B8X, Lemma 32.25.8.]). Since $\overline{f}$ is flat and of finite presentation, it follows, that $\overline{f}(V)$ is an open subset of S ([Stack, tag 00I1, Proposition 10.40.8.]). Since it is not empty, it is dense.

By ([Va17, Theorem 25.3.3]) there exists an open dense subset $W \subset S$ such that the restriction of $\overline{f}$ to the non-empty open set $V \cap \overline{f}^{-1}(W)$ is a smooth morphism. Then X_t is smooth, hence regular for $t \in W$. Since S is 1-dimensional, W is the complement of finitely many points, this implies (1). □

Let us consider the real or complex analytic case. $\mathbb{K}$ denotes $\mathbb{R}$ or $\mathbb{C}$ and δ resp. ε denotes $\delta_{\mathbb{K}}$ resp. $\varepsilon_{\mathbb{K}}$. The following theorems were proved in [Gr17] for $\mathbb{K} = \mathbb{C}$. The proofs follow similar arguments as in Section 2 and 3. The proofs for $\mathbb{K} = \mathbb{R}$ are analogous.

The analytic case is in some sense technically easier than the general algebraic case: complex spaces are Nagata, the characteristic is 0, and all points are closed. Moreover, the non-normal locus $NNor(f)$ of an analytic morphism $f : X \to S$ is a closed analytic subset of X and $f(NNor(f))$ is neglectible [14] in S, provided there is an open dense subset $V \subset S$ consisting of smooth points of S such that $f^{-1}(V)$ consists of normal points of X. If the restriction of f to $NNor(f)$ is proper, then "neglectible" can be replaced by "a nowhere dense closed analytic subset" (see [BF93, Theorem 2.1(3)] for a proof of these statements).

Theorem 34. ([Gr17, Theorem 7.14]) *Let $f : (X, x) \to (\mathbb{K}, 0)$ be flat morphism of $\mathbb{K}$-analytic germs with fibre (X_0, x) an isolated non–normal singularity. For a good representative $f : X \to T$ the following holds for $t \neq 0$.*

(1) $(\overline{X}^{>1})_0$ *is reduced and* $(\overline{X})_t$ *is smooth.*
(2) $\delta(X_0) - \delta(X_t)$ *is equal to*
 (i) $\delta((X^{red})_0) - \delta((X^{red})_t) = \delta((X^{red})_0) - \delta((X_t)^{red})$,
 (ii) $\delta((\overline{X})_0) - \delta((\overline{X})_t) = \delta((\overline{X})_0) + \dim_{\mathbb{K}} \mathcal{O}_{(X^1)_t}$,
 (iii) $\delta((X^{>1})_0) - \delta((X^{>1})_t) = \delta((\overline{X}^{>1})_0) \geq 0$,
(3) $\varepsilon(X_0) - \varepsilon(X_t) = \varepsilon((X^{>1})_0) \geq 0$.

We use in the analytic case that a quasi-finite morphism has a finite representative (in the Euclidean topology). For $\mathbb{K} = \mathbb{C}$ we have $\dim_{\mathbb{K}} \mathcal{O}_{(X^1)_t} = \#\{\text{isolated points of } X_t\}$. Statement (ii) is in [Gr17, Theorem 7.14] formulated with $r_1(X)$ instead of $\dim_{\mathbb{K}} \mathcal{O}_{(X^1)_t}$, which is wrong in general. We have $r_1(X) \leq \dim_{\mathbb{K}} \mathcal{O}_{(X^1)_t}$ and $r_1(X) = 0$ iff $\dim_{\mathbb{K}} \mathcal{O}_{(X^1)_t} = 0$.

[14] i.e. contained in a countable union of nowhere dense locally closed analytic subsets of S.

Theorem 35. ([Gr17, Theorem 7.17]) *Let $f : (X, x) \to (\mathbb{K}, 0)$ be flat with (X_0, x) an INNS of dimension ≥ 1. Then there exists a small representative $f : X \to S$ of the germ f, such that $f : X \to S$ admits a simultaneous normalization if and only if $\delta(X_t)$ is constant and $r_1(X, x) = 0$.*

The assumption that (X_0, x) an INNS implies that the non-normal locus of f (which is analytic in X) is finite over S (for S sufficiently small), and hence X_t has only isolated non-normal singularities for $t \in S$. We recall that an excellent local ring $(R, \mathfrak{m})$ is normal if R/f is normal for $f \in \mathfrak{m}$ a non-zero divisor of R ([BF93, Lemma 4.4]). Hence, for $f : X \to S$ a flat analytic morphism, with S a smooth 1-dimensional manifold, the non-normal locus of X is contained in the non-normal locus of f. Let $n : \overline{X} \to X$ be the normalization of X. Then the non-normal locus of X is the union of the supports of the sheaves $Coker(n_* : \mathcal{O}_X \to n_*\mathcal{O}_{\overline{X}})$ and $Ker(n_* : \mathcal{O}_X \to n_*\mathcal{O}_{\overline{X}})$ and the assumption that (X_0, x) an INNS implies that these sheaves are finite over S. The necessity of $r_1(X, x) = 0$ follows from the dimension formula of the analytic analog of Lemma 28 (4). We thus see that analogous assumptions as in Theorem 33 hold also in the analytic situation of Theorem 34.

It is not difficult to see that $Coker(n_* : \mathcal{O}_X \to n_*\mathcal{O}_{\overline{X}})$ and $Ker(n_*^{>1} : \mathcal{O}_X \to n_*\mathcal{O}_{\overline{X}^{>1}})$ are finite over S iff the non-normal loci of X and $X^{>1}$, defined by the extended conductor schemes, are finite over S (see Remark 21 (1)). We use this in the following Theorem 36 for (global) morphisms of analytic spaces.

Theorem 36. ([Gr17, Theorem 7.19S]) *Let $f : X \to S$ be a flat morphism of $\mathbb{K}$-analytic spaces with S a 1–dimensional analytic manifold. Assume that the non-normal loci of f and $f^{>1} : X^{>1} \to S$ are finite over S. Then the following are equivalent:*

(i) *f admits a simultaneous normalization.*
(ii) *$\delta(X_t)$ is locally constant on S and the 1–dimensional part X^1 of X is smooth and does not meet the higher dimensional part $X^{>1}$.*

In particular, if X has no 1-dimensional part, then

$$f \text{ admits a simultaneous normalization} \iff \delta(X_t) \text{ is locally constant on } S.$$

The finiteness of $NNor(f^{>1})$ over S was mistakenly omitted in [Gr17, Theorem 7.19S].

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Email address: `greuel@mathematik.uni-kl.de`
Email address: `pfister@mathematik.uni-kl.de`